

Week 8: Multiple Regression

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¹Many of these slides adapted from Matt Blackwell, Adam Glynn, Justin Grimmer, Jens Hainmueller and Erin Hartman.

Where We've Been and Where We're Going...

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 - ▶ statistical theory for regression
 - ▶ regression with two variables

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- Long Run
 - ▶ probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

- 1 Matrix Notation of Regression
- 2 OLS Properties Under Gauss-Markov Assumptions
 - Unbiasedness
 - Classical Standard Errors
- 3 Asymptotics of OLS with Only IID Sampling
 - Clustering
- 4 Standard Hypothesis Tests
 - t -Tests
 - Adjusted R^2
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- Outcome is a **linear combination** of the $\mathbf{x}$, $\mathbf{z}$, and $\mathbf{u}$ vectors

Grouping Things into Matrices

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- We can also write this at the individual level, where $\mathbf{x}'_i$ is the i th row of $\mathbf{X}$:

$$y_i = \mathbf{x}'_i\boldsymbol{\beta} + u_i$$

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- Here we need the equivalent of a matrix 'division' operator.

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Disclaimer: the final equation is exactly true for all non-intercept coefficients if you remove the intercept from $\mathbf{X}$ such that $\hat{\beta}_{-0} = V(\mathbf{X}_{-0})^{-1}\text{Cov}(\mathbf{X}_{-0}, \mathbf{y})$. The numerator and denominator are the variances and covariances if $\mathbf{X}$ and $\mathbf{y}$ are demeaned and normalized by the sample size minus 1.

Classical OLS Assumptions in Matrix Form

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- 1 Linearity: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$
- 2 Random/iid sample: $(y_i, \mathbf{x}'_i)$ are a iid sample from the population.
- 3 No perfect collinearity: $\mathbf{X}$ is an $n \times (k + 1)$ matrix with rank $k + 1$
- 4 Zero conditional mean: $E[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
- 5 Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
- 6 Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I}_n)$

Classical OLS Assumptions in Matrix Form

- 1 Linearity: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$
- 2 Random/iid sample: $(y_i, \mathbf{x}'_i)$ are a iid sample from the population.
- 3 No perfect collinearity: $\mathbf{X}$ is an $n \times (k + 1)$ matrix with rank $k + 1$
- 4 Zero conditional mean: $E[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
- 5 Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
- 6 Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I}_n)$

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- If $\mathbf{X}$ has rank $k + 1$, then $(\mathbf{X}'\mathbf{X})$ is invertible (see essentially any linear algebra book for proof)
- Just like variation in X led us to be able to divide by the variance in simple OLS

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- Using the zero mean conditional error assumptions:

$$E[\mathbf{u}|\mathbf{X}] = \begin{bmatrix} E[u_1|\mathbf{X}] \\ E[u_2|\mathbf{X}] \\ \vdots \\ E[u_n|\mathbf{X}] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

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So, yes!

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- The variance of a vector is actually a matrix:

$$\text{var}[\mathbf{u}] = \Sigma_u = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_n) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_n) \\ \vdots & & \ddots & \\ \text{cov}(u_n, u_1) & \text{cov}(u_n, u_2) & \dots & \text{var}(u_n) \end{bmatrix}$$

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- This matrix is always **symmetric** since $\text{cov}(u_i, u_j) = \text{cov}(u_j, u_i)$ by definition.

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Rule: Variance of Linear Function of Random Vector

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Definition (Variance of Linear Transformation of Random Vector)

Let $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{B}$ be a linear transformation of a random vector $\mathbf{u}$ with non-random vectors or matrices $\mathbf{A}$ and $\mathbf{B}$. Then the variance of the transformation is given by:

$$V[f(\mathbf{u})] = V[\mathbf{A}\mathbf{u} + \mathbf{B}] = \mathbf{A}V[\mathbf{u}]\mathbf{A}'$$

Conditional Variance of $\hat{\beta}$

$\hat{\beta} = \beta + (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}$ and $E[\hat{\beta}|\mathbf{X}] = \beta + E[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}|\mathbf{X}] = \beta$ so the OLS estimator is a linear function of the errors. Thus:

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This gives the $(k+1) \times (k+1)$ **variance-covariance matrix** of $\hat{\beta}$.

Conditional Variance of $\hat{\beta}$

$\hat{\beta} = \beta + (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}$ and $E[\hat{\beta}|\mathbf{X}] = \beta + E[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}|\mathbf{X}] = \beta$ so the OLS estimator is a linear function of the errors. Thus:

$$\begin{aligned} V[\hat{\beta}|\mathbf{X}] &= V[\beta|\mathbf{X}] + V[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}|\mathbf{X}] \\ &= V[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}|\mathbf{X}] \\ &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' V[\mathbf{u}|\mathbf{X}] ((\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}')' \quad (\mathbf{X} \text{ is nonrandom given } \mathbf{X}) \\ &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' V[\mathbf{u}|\mathbf{X}] \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \sigma^2 \mathbf{I} \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \quad (\text{by homoskedasticity}) \\ &= \sigma^2 \mathbf{I} (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2 (\mathbf{X}'\mathbf{X})^{-1} \end{aligned}$$

This gives the $(k+1) \times (k+1)$ **variance-covariance matrix** of $\hat{\beta}$.

To estimate $V[\hat{\beta}|\mathbf{X}]$, we replace σ^2 with its unbiased estimator $\hat{\sigma}^2$, which is now written using matrix notation as:

$$\hat{\sigma}^2 = \frac{\sum_i \hat{u}_i^2}{n - (k+1)} = \frac{\hat{\mathbf{u}}' \hat{\mathbf{u}}}{n - (k+1)}$$

Sampling Variance for $\hat{\beta}$

Under assumptions 1-5, the **variance-covariance matrix** of the OLS estimators is given by:

$$V[\hat{\beta}|\mathbf{X}] = \sigma^2 (\mathbf{X}'\mathbf{X})^{-1} =$$

	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\dots$	$\hat{\beta}_k$
$\hat{\beta}_0$	$V[\hat{\beta}_0]$	$\text{Cov}[\hat{\beta}_0, \hat{\beta}_1]$	$\text{Cov}[\hat{\beta}_0, \hat{\beta}_2]$	$\dots$	$\text{Cov}[\hat{\beta}_0, \hat{\beta}_k]$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\hat{\beta}_k$	$\text{Cov}[\hat{\beta}_0, \hat{\beta}_k]$	$\text{Cov}[\hat{\beta}_k, \hat{\beta}_1]$	$\text{Cov}[\hat{\beta}_k, \hat{\beta}_2]$	$\dots$	$V[\hat{\beta}_k]$

Recall that standard errors are the square root of the diagonals of this matrix.

Overview of Inference in the General Setting

- Under assumption 1-5 in large samples:

$$\frac{\hat{\beta}_j - \beta_j}{\widehat{SE}[\hat{\beta}_j]} \sim N(0, 1)$$

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- Thus, confidence intervals and hypothesis tests proceed in essentially the same way.

Properties of the OLS Estimator: Summary

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Theorem

Under Assumptions 1–6, the $(k + 1) \times 1$ vector of OLS estimators $\hat{\beta}$, conditional on $\mathbf{X}$, follows a **multivariate normal distribution** with mean β and variance-covariance matrix $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$:

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- With a large sample, $\hat{\beta}$ approximately follows the same distribution under Assumptions 1–5 only, i.e., without assuming the normality of $\mathbf{u}$.

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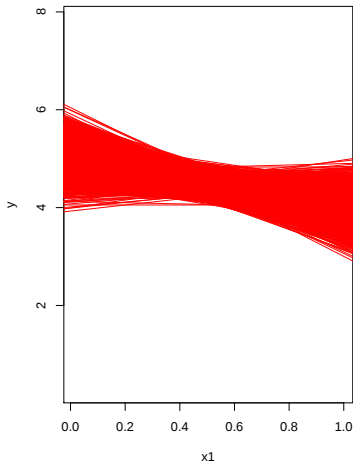
Implications of the Variance-Covariance Matrix

- Note that the sampling distribution is a **joint distribution** because it involves multiple random variables.
- This is because the sampling distribution of the terms in $\hat{\beta}$ are correlated.
- In a practical sense, this means that our uncertainty about coefficients is **correlated** across variables.

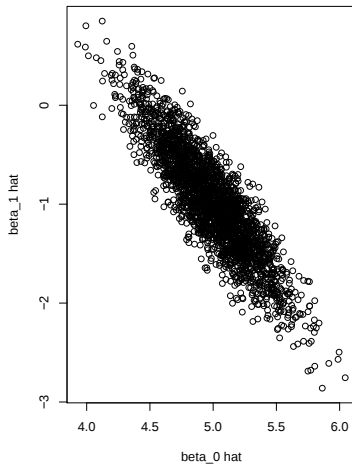
Multivariate Normal: Simulation

$Y = \beta_0 + \beta_1 X_1 + u$ with $u \sim N(0, \sigma_u^2 = 4)$ and $\beta_0 = 5$, $\beta_1 = -1$, and $n = 100$:

Sampling distribution of Regression Lines

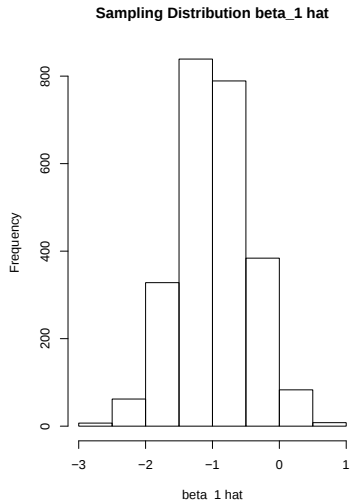
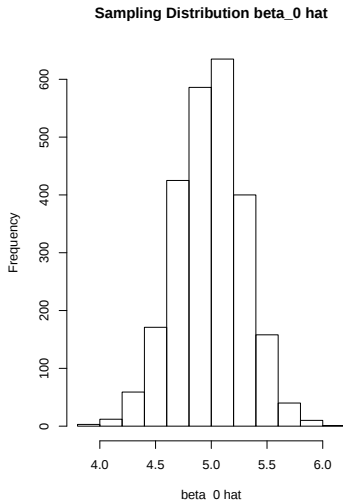


Joint sampling distribution



Marginals of Multivariate Normal RVs are Normal

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- 2 OLS Properties Under Gauss-Markov Assumptions
 - Unbiasedness
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- **Finite-sample** guarantees came at the price of **strong assumption** (CEF is linear for unbiasedness, homoskedastic-normal errors for standard errors)
- **Asymptotic results** will hold more generally under essentially only **IID sampling**, but require more data.
- Define the **Best Linear Projection** (BLP) model as

$$\begin{aligned}\mathbf{y} &= \mathbf{X}\beta + \mathbf{u} \\ E[\mathbf{X}'\mathbf{u}] &= 0 \\ E[\mathbf{X}'\mathbf{X}] &> 0 \text{ (invertability)}\end{aligned}$$

NB: ideally we would rewrite these all in terms of scalar random variable Y and random vectors for X . I've opted to keep them in matrices so the notation is parallel (you can imagine the vectors for Y are length 1 and matrices for X have one row), but the key idea remains that the expectations are over random draws.

Estimator

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- **Plug-in principle** $\rightsquigarrow$ replace population expectation with sample versions:

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$$

Asymptotic OLS inference

- With this representation, we can write the OLS estimator as follows:

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- Note that we have only used IID sampling and no perfect collinearity so far.

Theorem (Asymptotic Normality of OLS Under Random Sampling)

Under the Best Linear Projection model,

$$\sqrt{N}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \Omega)$$

where

$$\Omega = E[\mathbf{X}'\mathbf{X}]^{-1} E[\mathbf{X}' \text{Diag}(\mathbf{u}^2) \mathbf{X}] E[\mathbf{X}'\mathbf{X}]^{-1}$$

and

- $\hat{\beta}$ is approximately normal asymptotically with mean β and variance Ω/n .
- Ω/n is the **asymptotic covariance matrix** of $\hat{\beta}$ and the square root of the diagonal are the standard errors.
- Asymptotically this provides us with all the necessary infrastructure for confidence intervals and tests.

Comparing to the Classical Approach (Homoskedasticity)

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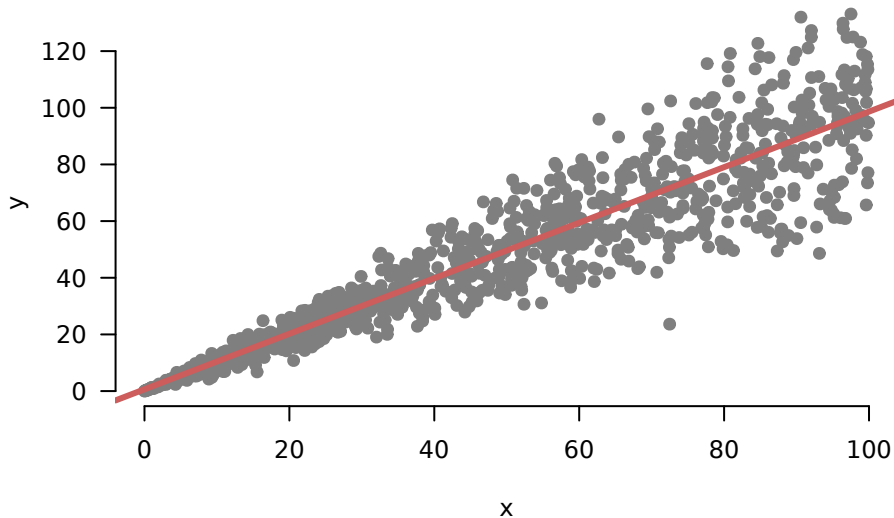
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- Replace σ^2 with estimate $\hat{\sigma}^2$ will give us our estimate of the covariance matrix

What Does This Rule Out?



Non-constant Error Variance

- Homoskedastic:

$$V[\mathbf{u}|\mathbf{X}] = \sigma^2 \mathbf{I} = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

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$$V[\mathbf{u}|\mathbf{X}] = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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- Independent, not identical

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- Idea: If we can consistently estimate the components of Σ , we could directly use this expression by replacing Σ with its estimate, $\hat{\Sigma}$.

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Suppose we have **heteroskedasticity of unknown form** (but zero covariance):

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The estimate based on the above is called the **heteroskedasticity consistent (HC)** or **robust standard errors**. This also coincides with the agnostic standard errors!

Intuition for Robust Standard Errors

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- There are various **small sample corrections** to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

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- Robust SEs converge to same point as the bootstrap.

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- Violation of iid/random sampling called **clustering** or **clustered dependence**.
- Loosely speaking, you are tricking the regression into thinking it has more information than it does.

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- Under this clustered dependence, we can write this as:

$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{g=1}^m \mathbf{X}'_g \Sigma_g \mathbf{X}_g \right) (\mathbf{X}'\mathbf{X})^{-1}$$

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- We can use a **plugin estimator** for the sampling variance on the previous slide with a small sample correction (many variants available, here we give the CR1 variant Stata uses):

$$\hat{V}_{CR1}[\hat{\beta}|\mathbf{X}] = \frac{m}{(m-1)} \frac{n-1}{n-k} (\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{g=1}^m \mathbf{x}'_g \hat{\mathbf{u}}_g \hat{\mathbf{u}}_g' \mathbf{x}_g \right) (\mathbf{X}'\mathbf{X})^{-1}$$

- Another excellent strategy is to use the **block bootstrap** (resample clusters instead of units).

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- NB: this broad strategy of sandwich estimators works for other kinds of settings (time-series etc.). See `sandwich` package in R.

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Running Example: Chilean Referendum on Pinochet

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- We model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

Hypothesis Testing in R

Model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

```
R Code
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284   0.0514034   7.864 6.57e-15 ***
fem           0.1360034   0.0237132   5.735 1.15e-08 ***
educ        -0.0607604   0.0138649  -4.382 1.25e-05 ***
age           0.0037786   0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared:  0.05112,    Adjusted R-squared:  0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

The t-Value for Multiple Linear Regression

- Consider testing a hypothesis about a single regression coefficient β_j :

$$H_0 : \beta_j = c$$

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$$\hat{SE}(\hat{\beta}_j) = \sqrt{\widehat{V}(\hat{\beta}_j)} = \sqrt{\widehat{V}(\hat{\beta})_{(j,j)}} = \sqrt{\hat{\sigma}^2(\mathbf{X}'\mathbf{X})_{(j,j)}^{-1}}$$

where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.

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where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.

That is, take the variance-covariance matrix of $\hat{\beta}$ and square root the diagonal element corresponding to j .

Getting the Standard Errors

R Code

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> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
Coefficients:
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---
```

We can pull classical variance-covariance matrix in R from the `lm()` object:

R Code

```
> V <- vcov(fit)
> V
              (Intercept)              fem              educ              age
(Intercept)  2.642311e-03 -3.455498e-04 -5.270913e-04 -3.357119e-05
fem          -3.455498e-04  5.623170e-04  2.249973e-05  8.285291e-07
educ         -5.270913e-04  2.249973e-05  1.922354e-04  3.411049e-06
age          -3.357119e-05  8.285291e-07  3.411049e-06  6.914098e-07

> sqrt(diag(V))
(Intercept)              fem              educ              age
0.0514034097 0.0237132251 0.0138648980 0.0008315105
```

Using the t-Value as a Test Statistic

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- 2 Compare the value to the **critical value** $t_{\alpha/2}$ for the α level test, which under the null hypothesis satisfies

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$$P(-t_{\alpha/2} \leq T \leq t_{\alpha/2}) = 1 - \alpha$$

- 3 Decide whether the realized value of T in our data is unusual given the distribution of the test statistic under the null hypothesis.
- 4 Finally, either declare that we reject H_0 or not, or report the p-value.

Confidence Intervals

To construct confidence intervals, there is again no difference compared to the case of $k = 1$, except that we need to use t_{n-k-1} instead of t_{n-2}

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So we also know the probability that the value of our test statistics falls into a given interval:

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We rearrange:

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$$\left[\hat{\beta}_j - t_{\alpha/2} \hat{SE}[\hat{\beta}_j], \hat{\beta}_j + t_{\alpha/2} \hat{SE}[\hat{\beta}_j]\right]$$

and thus can construct the confidence intervals as usual using:

$$\hat{\beta}_j \pm t_{\alpha/2} \cdot \hat{SE}[\hat{\beta}_j]$$

Confidence Intervals in R

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	0.4042284	0.0514034	7.864	6.57e-15	***
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R Code

```
> confint(fit)
```

	2.5 %	97.5 %
(Intercept)	0.303407780	0.50504909
fem	0.089493169	0.18251357
educ	-0.087954435	-0.03356629
age	0.002147755	0.00540954

Testing Hypothesis About a Linear Combination of β_j

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R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> summary(fit)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	4452.7	783.4	5.684	2.07e-07	***
RegionAfrica	-2552.8	1204.5	-2.119	0.0372	*
RegionAsia	148.9	1149.8	0.129	0.8973	
RegionLatAmerica	-271.3	1007.0	-0.269	0.7883	
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- $\hat{\beta}_{Asia}$ and $\hat{\beta}_{LAm}$ are close. So we may want to test the null hypothesis:

$$H_0 : \beta_{LAm} = \beta_{Asia} \Leftrightarrow \beta_{LAm} - \beta_{Asia} = 0$$

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against the alternative of

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- What would be an appropriate **test statistic** for this hypothesis?

Testing Hypothesis About a Linear Combination of β_j

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- Let's consider a t-value:

$$T = \frac{\hat{\beta}_{LAm} - \hat{\beta}_{Asia}}{\hat{SE}(\hat{\beta}_{LAm} - \hat{\beta}_{Asia})}$$

We will reject H_0 if T is sufficiently different from zero.

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We will reject H_0 if T is sufficiently different from zero.

- Note that unlike the test of a single hypothesis, both $\hat{\beta}_{LAm}$ and $\hat{\beta}_{Asia}$ are random variables, hence the denominator.

Testing Hypothesis About a Linear Combination of β_j

- Our test statistic:

$$T = \frac{\hat{\beta}_{LAm} - \hat{\beta}_{Asia}}{\hat{SE}(\hat{\beta}_{LAm} - \hat{\beta}_{Asia})} \sim t_{n-k-1}$$

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- How do you find $\hat{SE}(\hat{\beta}_{LAm} - \hat{\beta}_{Asia})$?

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- Is it $\hat{SE}(\hat{\beta}_{LAm}) - \hat{SE}(\hat{\beta}_{Asia})$?

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- Is it $\hat{SE}(\hat{\beta}_{LAm}) - \hat{SE}(\hat{\beta}_{Asia})$? **No!**
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- Recall the following property of the variance:

$$V(X \pm Y) = V(X) + V(Y) \pm 2\text{Cov}(X, Y)$$

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- Recall the following property of the variance:

$$V(X \pm Y) = V(X) + V(Y) \pm 2\text{Cov}(X, Y)$$

Therefore, the standard error for a linear combination of coefficients is:

$$\hat{SE}(\hat{\beta}_1 \pm \hat{\beta}_2) = \sqrt{\hat{V}(\hat{\beta}_1) + \hat{V}(\hat{\beta}_2) \pm 2\widehat{\text{Cov}}[\hat{\beta}_1, \hat{\beta}_2]}$$

which we can calculate from the estimated covariance matrix of $\hat{\beta}$.

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which we can calculate from the estimated covariance matrix of $\hat{\beta}$.

- Since the estimates of the coefficients are correlated, we need the covariance term.

Example: GDP per capita on Regions

Example: GDP per capita on Regions

R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> V <- vcov(fit)
> V
```

	(Intercept)	RegionAfrica	RegionAsia	RegionLatAmerica
(Intercept)	613769.9	-613769.9	-613769.9	-613769.9
RegionAfrica	-613769.9	1450728.8	613769.9	613769.9
RegionAsia	-613769.9	613769.9	1321965.9	613769.9
RegionLatAmerica	-613769.9	613769.9	613769.9	1014054.6
RegionOecd	-613769.9	613769.9	613769.9	613769.9

	RegionOecd
(Intercept)	-613769.9
RegionAfrica	613769.9
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RegionOecd	1014054.6

Example: GDP per capita on Regions

We can then compute the test statistic for the hypothesis of interest:

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```
R Code
> se <- sqrt(V[4,4] + V[3,3] - 2*V[3,4])
> se
[1] 1052.844
>
> tstat <- (coef(fit)[4] - coef(fit)[3])/se
> tstat
RegionLatAmerica
-0.3990977
```

$$t = \frac{\hat{\beta}_{LAm} - \hat{\beta}_{Asia}}{\hat{SE}(\hat{\beta}_{LAm} - \hat{\beta}_{Asia})} \quad \text{where}$$
$$\hat{SE}(\hat{\beta}_{LAm} - \hat{\beta}_{Asia}) = \sqrt{\hat{V}(\hat{\beta}_{LAm}) + \hat{V}(\hat{\beta}_{Asia}) - 2\widehat{\text{Cov}}[\hat{\beta}_{LAm}, \hat{\beta}_{Asia}]}$$

Plugging in we get $t \approx -0.40$. So what do we conclude?

Example: GDP per capita on Regions

We can then compute the test statistic for the hypothesis of interest:

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R Code
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NB: this same strategy can be used to get standard errors for interactions at any value of the other variables.

Adjusted R^2

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```
_____ R Code _____
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284   0.0514034   7.864 6.57e-15 ***
fem           0.1360034   0.0237132   5.735 1.15e-08 ***
educ        -0.0607604   0.0138649  -4.382 1.25e-05 ***
age           0.0037786   0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared:  0.05112,      Adjusted R-squared:  0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

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- Still since people report it, the next slide derives adjusted R^2 .

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- Adjusted R^2 will always be smaller than R^2 and can sometimes be negative!

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- Regardless the intuition here is useful: as we add more variables R^2 must stay constant or go up mechanically and that creates complications.

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 - Unbiasedness
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- **F tests** allows us to test **joint hypothesis**

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The χ^2 Distribution

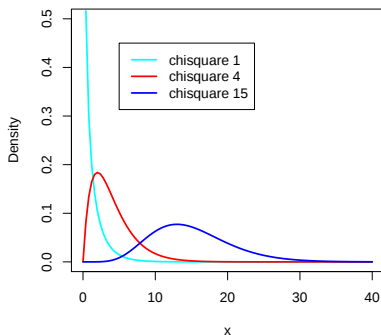
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Properties: $X > 0$, $E[X] = n$ and $V[X] = 2n$. In R: `dchisq()`, `pchisq()`, `rchisq()`

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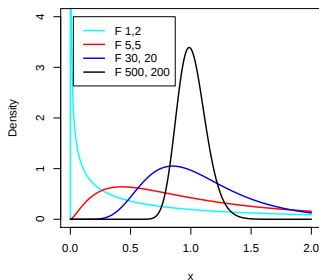
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In R: `df()`, `pf()`, `rf()`

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Unrestricted Model (UR)

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> summary(fit.UR)
~~~~~
Coefficients:
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educ        -0.038571   0.019578  -1.970 0.048988 *
age           0.005482   0.001114   4.921 9.44e-07 ***
fem:age      -0.003779   0.001673  -2.259 0.024010 *
fem:educ     -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared:  0.05451,    Adjusted R-squared:  0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Restricted Model (R)

```
_____ R Code _____  
> fit.R <- lm(vote1 ~ educ + age, data = Chile)  
> summary(fit.R)  
Coefficients:  
                Estimate Std. Error t value Pr(>|t|)  
(Intercept)  0.4878039   0.0497550   9.804  < 2e-16 ***  
educ         -0.0662022   0.0139615  -4.742 2.30e-06 ***  
age          0.0035783   0.0008385   4.267 2.09e-05 ***  
---  
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
  
Residual standard error: 0.4921 on 1700 degrees of freedom  
Multiple R-squared:  0.03275,          Adjusted R-squared:  0.03161  
F-statistic: 28.78 on 2 and 1700 DF,  p-value: 5.097e-13
```

F Test in R

```
R Code
> SSR.UR <- sum(resid(fit.UR)^2) # = 402
> SSR.R <- sum(resid(fit.R)^2)   # = 411

> DFdenom <- df.residual(fit.UR) # = 1703
> DFnum <- 3

> F <- ((SSR.R - SSR.UR)/DFnum) / (SSR.UR/DFdenom)
> F
[1] 13.01581

> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

F Test in R

```
R Code
> SSR.UR <- sum(resid(fit.UR)^2) # = 402
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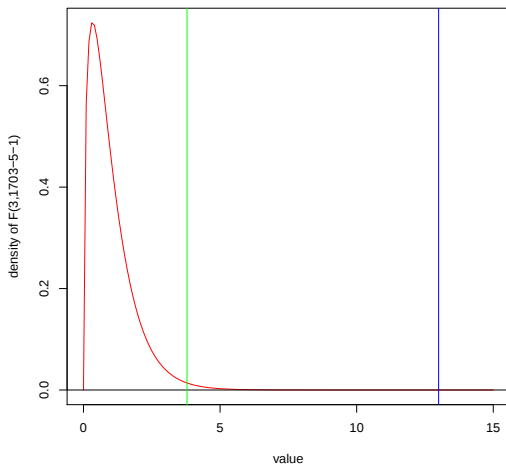
> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

$F_0 = 13$ is greater than the **critical value** for a .01 level test. So we *reject* the null hypothesis.

Null Distribution, Critical Value, and Test Statistic

Note that the F statistic is always positive, so we only look at the right tail of the reference F (or χ^2 in a large sample) distribution.



F Test Examples I

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The F test can be used to test various joint hypotheses which involve multiple linear restrictions.

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- Have any of you used an F-test like this in your research?
- This is called the **omnibus test** and is routinely reported by statistical software.

Omnibus Test in R

R Code

```
> summary(fit.UR)
~~~~~

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.293130   0.069242   4.233 2.42e-05 ***
fem             0.368975   0.098883   3.731 0.000197 ***
educ           -0.038571   0.019578  -1.970 0.048988 *
age             0.005482   0.001114   4.921 9.44e-07 ***
fem:age        -0.003779   0.001673  -2.259 0.024010 *
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Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,      Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Omnibus Test in R with Random Noise

R Code

```
> set.seed(08540)
> p <- 10; x <- matrix(rnorm(p*1000), nrow=1000)
> y <- rnorm(1000); summary(lm(y~x))
~~~~~
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-0.0115475	0.0320874	-0.360	0.7190
x1	-0.0019803	0.0333524	-0.059	0.9527
x2	0.0666275	0.0314087	2.121	0.0341 *
x3	-0.0008594	0.0321270	-0.027	0.9787
x4	0.0051185	0.0333678	0.153	0.8781
x5	0.0136656	0.0322592	0.424	0.6719
x6	0.0102115	0.0332045	0.308	0.7585
x7	-0.0103903	0.0307639	-0.338	0.7356
x8	-0.0401722	0.0318317	-1.262	0.2072
x9	0.0553019	0.0315548	1.753	0.0800 .
x10	0.0410906	0.0319742	1.285	0.1991

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.011 on 989 degrees of freedom
Multiple R-squared: 0.01129, Adjusted R-squared: 0.001294
F-statistic: 1.129 on 10 and 989 DF, p-value: 0.3364

F Test Examples II

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \dots + \beta_k X_k + u$$

Next, let's consider:

$$H_0 : \beta_1 = \beta_2 = \beta_3$$

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- What question are we asking?

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→ Are the coefficients X_1 , X_2 and X_3 different from each other?

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Next, let's consider:

$$H_0 : \beta_1 = \beta_2 = \beta_3$$

- What question are we asking?
→ Are the coefficients X_1 , X_2 and X_3 different from each other?
- How many restrictions?
→ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)
- How do we fit the restricted model?
→ The null hypothesis implies that the model can be written as:

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3) + \dots + \beta_k X_k + u$$

So we create a new variable $X^* = X_1 + X_2 + X_3$ and fit:

$$Y = \beta_0 + \beta_1 X^* + \dots + \beta_k X_k + u$$

Testing Equality of Coefficients in R

R Code

```
> fit.UR2 <- lm(REALGDP CAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> summary(fit.UR2)
```

```
~~~~~
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	1899.9	914.9	2.077	0.0410	*
Asia	2701.7	1243.0	2.173	0.0327	*
LatAmerica	2281.5	1112.3	2.051	0.0435	*
Transit	2552.8	1204.5	2.119	0.0372	*
Oecd	12224.2	1112.3	10.990	<2e-16	***

```
---
```

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3034 on 80 degrees of freedom

Multiple R-squared: 0.7096, Adjusted R-squared: 0.6951

F-statistic: 48.88 on 4 and 80 DF, p-value: < 2.2e-16

Are the coefficients on *Asia*, *LatAmerica* and *Transit* statistically significantly different?

Testing Equality of Coefficients in R

```
_____ R Code _____  
> D$Xstar <- D$Asia + D$LatAmerica + D$Transit  
> fit.R2 <- lm(REALGDPCAP ~ Xstar + Oecd, data = D)  
  
> SSR.UR2 <- sum(resid(fit.UR2)^2)  
> SSR.R2 <- sum(resid(fit.R2)^2)  
  
> DFdenom <- df.residual(fit.UR2)  
  
> F <- ((SSR.R2 - SSR.UR2)/2) / (SSR.UR2/DFdenom)  
> F  
[1] 0.08786129  
  
> pf(F, 2, DFdenom, lower.tail = F)  
[1] 0.9159762
```

So, what do we conclude?

Testing Equality of Coefficients in R

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_____ R Code _____  
> D$Xstar <- D$Asia + D$LatAmerica + D$Transit  
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> SSR.UR2 <- sum(resid(fit.UR2)^2)  
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> F  
[1] 0.08786129  
  
> pf(F, 2, DFdenom, lower.tail = F)  
[1] 0.9159762
```

So, what do we conclude?

The three coefficients are statistically indistinguishable from each other, with the p-value of 0.916.

t Test vs. F Test

Consider the hypothesis test of

$$H_0 : \beta_1 = \beta_2 \quad \text{vs.} \quad H_1 : \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

t Test vs. F Test

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$$H_0 : \beta_1 = \beta_2 \quad \text{vs.} \quad H_1 : \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2) / \hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the **t test**.

t Test vs. F Test

Consider the hypothesis test of

$$H_0 : \beta_1 = \beta_2 \quad \text{vs.} \quad H_1 : \beta_1 \neq \beta_2$$

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- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR}) / (SSR_R / (n - k - 1))$ and do the **F test**.

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It turns out these two tests give **identical** results. This is because

$$X \sim t_{n-k-1} \iff X^2 \sim \mathcal{F}_{1,n-k-1}$$

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- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.

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It turns out these two tests give **identical** results. This is because

$$X \sim t_{n-k-1} \iff X^2 \sim \mathcal{F}_{1,n-k-1}$$

- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.
- Usually, the t test is used for single hypotheses and the F test is used for joint hypotheses.

Some More Notes on F Tests

- The F-value can also be calculated from R^2 :

$$F = \frac{(R_{UR}^2 - R_R^2)/q}{(1 - R_{UR}^2)/(n - k - 1)}$$

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- The F-value can also be calculated from R^2 :

$$F = \frac{(R_{UR}^2 - R_R^2)/q}{(1 - R_{UR}^2)/(n - k - 1)}$$

- F tests only work for testing **nested** models, i.e. the restricted model must be a special case of the unrestricted model.

For example F tests cannot be used to test

$$Y = \beta_0 + \beta_1 X_1 \quad + \beta_3 X_3 + u$$

against

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \quad + u$$

Some More Notes on F Tests

Joint significance does not necessarily imply the significance of individual coefficients, or vice versa:

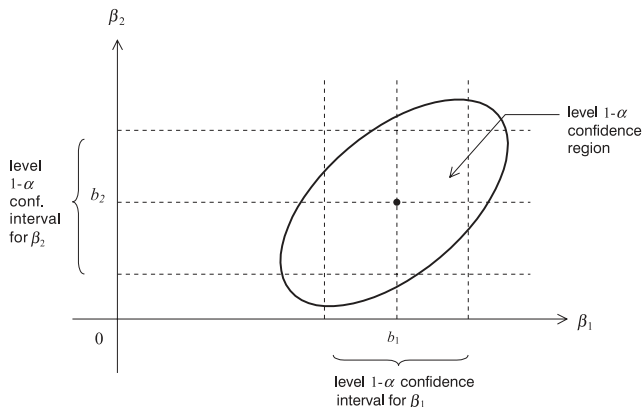


Figure 1.5: t - versus F -Tests

Image Credit: Hayashi (2011) *Econometrics*

Goal Check: Understand `lm()` Output

Call:

```
lm(formula = sr ~ pop15, data = LifeCycleSavings)
```

Residuals:

Min	1Q	Median	3Q	Max
-8.637	-2.374	0.349	2.022	11.155

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	17.49660	2.27972	7.675	6.85e-10 ***
pop15	-0.22302	0.06291	-3.545	0.000887 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.03 on 48 degrees of freedom

Multiple R-squared: 0.2075, Adjusted R-squared: 0.191

F-statistic: 12.57 on 1 and 48 DF, p-value: 0.0008866

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- Under the **BLP** model, we can be more agnostic and assume only IID and no-perfect collinearity. This leads to different **standard errors** and properties only hold **asymptotically** (and converge to the linear projection rather than the true CEF).

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- Much of the **hypothesis test/CI** infrastructure ports over nicely, plus there are new **joint tests** we can use.

Next week: Diagnostics!