Week 8: Multiple Regression

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[1]Many of these slides adapted from Matt Blackwell, Adam Glynn, Justin Grimmer, Jens Hainmueller and Erin Hartman.

Where We've Been and Where We're Going...

- Last Week
  - statistical theory for regression
  - regression with two variables
- This Week
  - matrix form of linear regression
  - inference and hypothesis tests
- Next Week
  - diagnostics
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

The Linear Model with New Notation

- Remember that we wrote the linear model as the following for all $i \in [1, \ldots, n]$:

$$y_i = \beta_0 + x_i\beta_1 + z_i\beta_2 + u_i$$

- Imagine we had an n of 4. We could write out each formula:

$$\begin{aligned}
y_1 &= \beta_0 + x_1\beta_1 + z_1\beta_2 + u_1 \quad \text{(unit 1)}\\
y_2 &= \beta_0 + x_2\beta_1 + z_2\beta_2 + u_2 \quad \text{(unit 2)}\\
y_3 &= \beta_0 + x_3\beta_1 + z_3\beta_2 + u_3 \quad \text{(unit 3)}\\
y_4 &= \beta_0 + x_4\beta_1 + z_4\beta_2 + u_4 \quad \text{(unit 4)}
\end{aligned}$$

The Linear Model with New Notation

$$\begin{aligned}
y_1 &= \beta_0 + x_1\beta_1 + z_1\beta_2 + u_1 \quad \text{(unit 1)} \\
y_2 &= \beta_0 + x_2\beta_1 + z_2\beta_2 + u_2 \quad \text{(unit 2)} \\
y_3 &= \beta_0 + x_3\beta_1 + z_3\beta_2 + u_3 \quad \text{(unit 3)} \\
y_4 &= \beta_0 + x_4\beta_1 + z_4\beta_2 + u_4 \quad \text{(unit 4)}
\end{aligned}$$

- We can write this as:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \beta_0 + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \beta_1 + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \beta_2 + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

- Outcome is a linear combination of the the $\mathbf{x}$, $\mathbf{z}$, and $\mathbf{u}$ vectors

Grouping Things into Matrices

$$\underset{(4\times 3)}{\mathbf{X}} = \begin{bmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{bmatrix} \quad \underset{(3\times 1)}{\boldsymbol{\beta}} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

Back to Regression

- $\mathbf{X}$ is the $n \times (k+1)$ design matrix of independent variables
- $\boldsymbol{\beta}$ be the $(k+1) \times 1$ column vector of coefficients.
- $\mathbf{X}\boldsymbol{\beta}$ will be $n \times 1$:

$$\mathbf{X}\boldsymbol{\beta} = \beta_0 + \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \cdots + \beta_k \mathbf{x}_k$$

- We can compactly write a linear regression as:

$$\underset{(n \times 1)}{\mathbf{y}} = \underset{(n \times 1)}{\mathbf{X}\boldsymbol{\beta}} + \underset{(n \times 1)}{\mathbf{u}}$$

- We can also write this at the individual level, where $\mathbf{x}_i'$ is the ith row of $\mathbf{X}$:

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i$$

Multiple Linear Regression in Matrix Form

- Let $\widehat{\boldsymbol{\beta}}$ be the matrix of estimated regression coefficients and $\widehat{\mathbf{y}}$ be the vector of fitted values:

$$\widehat{\boldsymbol{\beta}} = \begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_1 \\ \vdots \\ \widehat{\beta}_k \end{bmatrix} \qquad \widehat{\mathbf{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$$

- It might be helpful to see this again more written out:

$$\widehat{\mathbf{y}} = \begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \vdots \\ \widehat{y}_n \end{bmatrix} = \mathbf{X}\widehat{\boldsymbol{\beta}} = \begin{bmatrix} 1\widehat{\beta}_0 + x_{11}\widehat{\beta}_1 + x_{12}\widehat{\beta}_2 + \cdots + x_{1K}\widehat{\beta}_k \\ 1\widehat{\beta}_0 + x_{21}\widehat{\beta}_1 + x_{22}\widehat{\beta}_2 + \cdots + x_{2K}\widehat{\beta}_k \\ \vdots \\ 1\widehat{\beta}_0 + x_{n1}\widehat{\beta}_1 + x_{n2}\widehat{\beta}_2 + \cdots + x_{nK}\widehat{\beta}_k \end{bmatrix}$$

Residuals

- We can easily write the residuals in matrix form:

$$\widehat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\widehat{\beta}$$

- Our goal as usual is to minimize the sum of the squared residuals, which we saw earlier we can write:

$$\widehat{\mathbf{u}}'\widehat{\mathbf{u}} = (\mathbf{y} - \mathbf{X}\widehat{\beta})'(\mathbf{y} - \mathbf{X}\widehat{\beta})$$

OLS Estimator in Matrix Form

- Goal: minimize the sum of the squared residuals.
- Take (matrix) derivatives, set equal to 0.
- Resulting first order conditions:

$$\mathbf{X}'(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}) = 0$$

- Rearranging:

$$\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$$

- In order to isolate $\widehat{\boldsymbol{\beta}}$, we need to move the $\mathbf{X}'\mathbf{X}$ term to the other side of the equals sign.
- Here we need the equivalent of a matrix 'division' operator.

Matrix Inverses

Definition (Matrix Inverse)

If it exists, the **inverse** of square matrix $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, is the matrix such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

- We can use the inverse to solve (systems of) equations:

$$\begin{aligned}
\mathbf{A}\mathbf{u} &= \mathbf{b} \\
\mathbf{A}^{-1}\mathbf{A}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{I}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{u} &= \mathbf{A}^{-1}\mathbf{b}
\end{aligned}$$

- If the inverse exists, we say that $\mathbf{A}$ is invertible or nonsingular.

Back to OLS

- Recall:

$$\mathbf{X}'\mathbf{X}\widehat{\beta} = \mathbf{X}'\mathbf{y}$$

- Let's assume, for now, that the inverse of $\mathbf{X}'\mathbf{X}$ exists
- Then we can write the OLS estimator as the following:

$$\widehat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

- See Aronow and Miller Theorem 4.1.4 for proof.
- "ex prime ex inverse ex prime y" sear it into your soul.

Intuition for the OLS in Matrix Form

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

What's the intuition here?

- "Numerator" $\mathbf{X}'\mathbf{y}$: is approximately composed of the covariances between the columns of $\mathbf{X}$ and $\mathbf{y}$
- "Denominator" $\mathbf{X}'\mathbf{X}$ is approximately composed of the sample variances and covariances of variables within $\mathbf{X}$
- Thus, we have something like:

$$\widehat{\boldsymbol{\beta}} \approx (\text{variance of } \mathbf{X})^{-1}(\text{covariance of } \mathbf{X} \text{ \& } \mathbf{y})$$

  i.e. analogous to the simple linear regression case!

Disclaimer: the final equation is exactly true for all non-intercept coefficients if you remove the intercept from $\mathbf{X}$ such that $\hat{\boldsymbol{\beta}}_{-0} = V(\mathbf{X}_{-0})^{-1}\text{Cov}(\mathbf{X}_{-0}, \mathbf{y})$. The numerator and denominator are the variances and covariances if $\mathbf{X}$ and $\mathbf{y}$ are demeaned and normalized by the sample size minus 1.

Classical OLS Assumptions in Matrix Form

1. Linearity: $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$
2. Random/iid sample: $(y_i, \mathbf{x}_i')$ are a iid sample from the population.
3. No perfect collinearity: $\mathbf{X}$ is an $n \times (k+1)$ matrix with rank $k+1$
4. Zero conditional mean: $E[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
5. Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
6. Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I}_n)$

Assumption 3: No Perfect Collinearity

Definition (Rank)

The rank of a matrix is the maximum number of linearly independent columns.

- If $\mathbf{X}$ has rank $k+1$, then all of its columns are linearly independent
- ...If all of the columns are linearly independent, then the assumption of no perfect collinearity hold.
- If $\mathbf{X}$ has rank $k+1$, then $(\mathbf{X}'\mathbf{X})$ is invertible
  (see essentially any linear algebra book for proof)
- Just like variation in X led us to be able to divide by the variance in simple OLS

Expected Values of Vectors

- The expected value of the vector is just the expected value of its entries.
- Using the zero mean conditional error assumptions:

$$E[\mathbf{u}|\mathbf{X}] = \begin{bmatrix} E[u_1|\mathbf{X}] \\ E[u_2|\mathbf{X}] \\ \vdots \\ E[u_n|\mathbf{X}] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

Unbiasedness of $\hat{\beta}$

Is $\hat{\beta}$ still unbiased under assumptions 1-4? Does $E[\hat{\beta}] = \beta$?

$$\hat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} \text{ (linearity and no collinearity)}$$
$$\hat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'(\mathbf{X}\beta + \mathbf{u})$$
$$\hat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{X}\beta + \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{u}$$
$$\hat{\beta} = \mathbf{I}\beta + \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{u}$$
$$\hat{\beta} = \beta + \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{u}$$
$$E[\hat{\beta}|\mathbf{X}] = E[\beta|\mathbf{X}] + E[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}]$$
$$E[\hat{\beta}|\mathbf{X}] = \beta + \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'E[\mathbf{u}|\mathbf{X}]$$
$$E[\hat{\beta}|\mathbf{X}] = \beta \text{ (zero conditional mean)}$$
$$E[E[\hat{\beta}|\mathbf{X}]] = \beta \text{ (law of iterated expectations)}$$

So, yes!

Assumption 5: Homoskedasticity

- The stated homoskedasticity assumption is: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
- To really understand this we need to know what $\text{var}(\mathbf{u}|\mathbf{X})$ is in full generality.
- The variance of a vector is actually a matrix:

$$\text{var}[\mathbf{u}] = \Sigma_u = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_n) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_n) \\ \vdots & & \ddots & \\ \text{cov}(u_n, u_1) & \text{cov}(u_n, u_2) & \dots & \text{var}(u_n) \end{bmatrix}$$

- This matrix is always symmetric since $\text{cov}(u_i, u_j) = \text{cov}(u_j, u_i)$ by definition.

Assumption 5: The Meaning of Homoskedasticity

- What does $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$ mean?
- $\mathbf{I}_n$ is the $n \times n$ identity matrix, σ_u^2 is a scalar.
- Visually:

$$\text{var}[\mathbf{u}] = \sigma_u^2 \mathbf{I}_n = \begin{bmatrix} \sigma_u^2 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_u^2 & 0 & \ldots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \ldots & \sigma_u^2 \end{bmatrix}$$

- In less matrix notation:
  - $\text{var}(u_i) = \sigma_u^2$ for all i (constant variance)
  - $\text{cov}(u_i, u_j) = 0$ for all $i \neq j$ (implied by iid)

Rule: Variance of Linear Function of Random Vector

Recall that for a linear transformation of a random variable X we have $V[aX + b] = a^2 V[X]$ with constants a and b.

We will need an analogous rule for linear functions of random vectors.

Definition (Variance of Linear Transformation of Random Vector)

Let $f(\mathbf{u}) = \mathbf{Au} + \mathbf{B}$ be a linear transformation of a random vector $\mathbf{u}$ with non-random vectors or matrices $\mathbf{A}$ and $\mathbf{B}$. Then the variance of the transformation is given by:

$$V[f(\mathbf{u})] = V[\mathbf{Au} + \mathbf{B}] = \mathbf{A} V[\mathbf{u}] \mathbf{A}'$$

Conditional Variance of $\hat{\boldsymbol{\beta}}$

$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \boldsymbol{\beta} + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \boldsymbol{\beta}$ so the OLS estimator is a linear function of the errors. Thus:

$$
\begin{aligned}
V[\hat{\boldsymbol{\beta}}|\mathbf{X}] &= V[\boldsymbol{\beta}|\mathbf{X}] + V[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] \\
&= V[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' V[\mathbf{u}|\mathbf{X}]((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')' \quad (\mathbf{X} \text{ is nonrandom given } \mathbf{X}) \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' V[\mathbf{u}|\mathbf{X}]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \quad \text{(by homoskedasticity)} \\
&= \sigma^2\mathbf{I}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}
$$

This gives the $(k+1) \times (k+1)$ variance-covariance matrix of $\hat{\boldsymbol{\beta}}$.

To estimate $V[\hat{\boldsymbol{\beta}}|\mathbf{X}]$, we replace σ^2 with its unbiased estimator $\hat{\sigma}^2$, which is now written using matrix notation as:

$$\hat{\sigma}^2 = \frac{\sum_i \hat{u}_i^2}{n-(k+1)} = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$$

Sampling Variance for $\hat{\beta}$

Under assumptions 1-5, the variance-covariance matrix of the OLS estimators is given by:

$$V[\hat{\beta}|\mathbf{X}] = \sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1} =$$

$$\begin{array}{c|ccccc}
 & \widehat{\beta}_0 & \widehat{\beta}_1 & \widehat{\beta}_2 & \cdots & \widehat{\beta}_k \\
\hline
\widehat{\beta}_0 & V[\widehat{\beta}_0] & \mathrm{Cov}[\widehat{\beta}_0, \widehat{\beta}_1] & \mathrm{Cov}[\widehat{\beta}_0, \widehat{\beta}_2] & \cdots & \mathrm{Cov}[\widehat{\beta}_0, \widehat{\beta}_k] \\
\widehat{\beta}_1 & \mathrm{Cov}[\widehat{\beta}_0, \widehat{\beta}_1] & V[\widehat{\beta}_1] & \mathrm{Cov}[\widehat{\beta}_1, \widehat{\beta}_2] & \cdots & \mathrm{Cov}[\widehat{\beta}_1, \widehat{\beta}_k] \\
\widehat{\beta}_2 & \mathrm{Cov}[\widehat{\beta}_0, \widehat{\beta}_2] & \mathrm{Cov}[\widehat{\beta}_1, \widehat{\beta}_2] & V[\widehat{\beta}_2] & \cdots & \mathrm{Cov}[\widehat{\beta}_2, \widehat{\beta}_k] \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\widehat{\beta}_k & \mathrm{Cov}[\widehat{\beta}_0, \widehat{\beta}_k] & \mathrm{Cov}[\widehat{\beta}_k, \widehat{\beta}_1] & \mathrm{Cov}[\widehat{\beta}_k, \widehat{\beta}_2] & \cdots & V[\widehat{\beta}_k]
\end{array}$$

Recall that standard errors are the square root of the diagonals of this matrix.

Overview of Inference in the General Setting

- Under assumption 1-5 in large samples:

$$\frac{\widehat{\beta}_j - \beta_j}{\widehat{SE}[\widehat{\beta}_j]} \sim N(0,1)$$

- In small samples, under assumptions 1-6,

$$\frac{\widehat{\beta}_j - \beta_j}{\widehat{SE}[\widehat{\beta}_j]} \sim t_{n-(k+1)}$$

- Estimated standard errors are:

$$\begin{aligned}
\widehat{SE}[\hat{\beta}_j] &= \sqrt{\widehat{\text{var}}[\widehat{\boldsymbol{\beta}}]_{jj}} \\
\widehat{\text{var}}[\widehat{\boldsymbol{\beta}}] &= \widehat{\sigma}_u^2(\mathbf{X}'\mathbf{X})^{-1} \\
\widehat{\sigma}_u^2 &= \frac{\widehat{\mathbf{u}}'\widehat{\mathbf{u}}}{n-(k+1)}
\end{aligned}$$

- Thus, confidence intervals and hypothesis tests proceed in essentially the same way.

Properties of the OLS Estimator: Summary

Theorem

Under Assumptions 1–6, the $(k+1) \times 1$ vector of OLS estimators $\hat{\boldsymbol{\beta}}$, conditional on $\mathbf{X}$, follows a multivariate normal distribution *with mean $\boldsymbol{\beta}$ and variance-covariance matrix $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$:*

$$\hat{\boldsymbol{\beta}}|\mathbf{X} \sim \mathcal{N}\left(\boldsymbol{\beta}, \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}\right)$$

- *Each element of $\hat{\boldsymbol{\beta}}$ (i.e. $\hat{\beta}_0, ..., \hat{\beta}_{k+1}$) is normally distributed, and $\hat{\boldsymbol{\beta}}$ is an unbiased estimator of $\boldsymbol{\beta}$ as $E[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$*
- *Variances and covariances are given by $V[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$*
- *An unbiased estimator for the error variance σ^2 is given by*

$$\widehat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$$

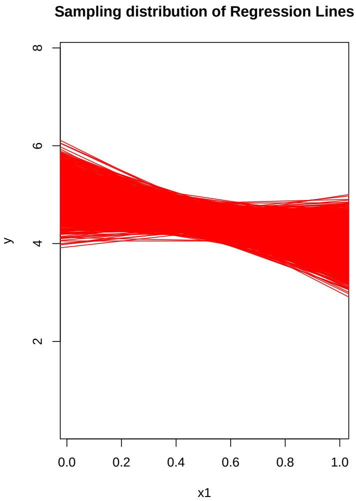
- *With a large sample, $\hat{\boldsymbol{\beta}}$ approximately follows the same distribution under Assumptions 1–5 only, i.e., without assuming the normality of $\mathbf{u}$.*

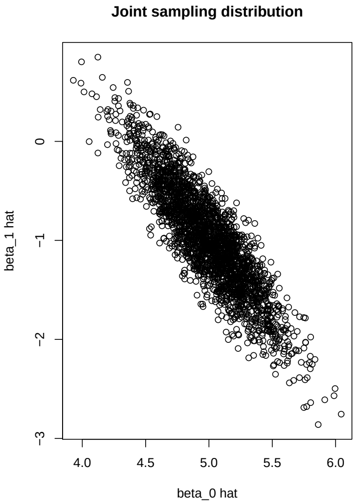
Implications of the Variance-Covariance Matrix

- Note that the sampling distribution is a joint distribution because it involves multiple random variables.
- This is because the sampling distribution of the terms in $\hat{\boldsymbol{\beta}}$ are correlated.
- In a practical sense, this means that our uncertainty about coefficients is correlated across variables.

Multivariate Normal: Simulation

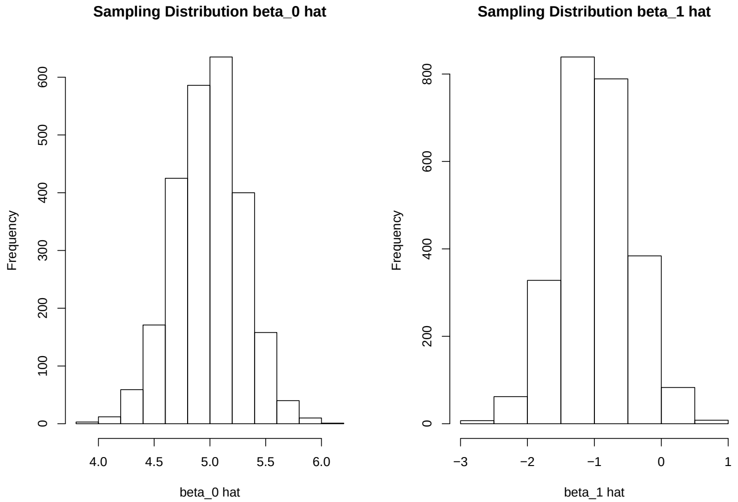
$Y = \beta_0 + \beta_1 X_1 + u$ with $u \sim N(0, \sigma_u^2 = 4)$ and $\beta_0 = 5$, $\beta_1 = -1$, and $n = 100$:





Marginals of Multivariate Normal RVs are Normal

$Y = \beta_0 + \beta_1 X_1 + u$ with $u \sim N(0, \sigma_u^2 = 4)$ and $\beta_0 = 5$, $\beta_1 = -1$, and $n = 100$:



Asymptotics of the OLS Estimator with IID Sampling

- Finite-sample guarantees came at the price of strong assumption (CEF is linear for unbiasedness, homoskedastic-normal errors for standard errors)
- Asymptotic results will hold more generally under essentially only IID sampling, but require more data.
- Define the Best Linear Projection (BLP) model as

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$$
$$E[\mathbf{X}'\mathbf{u}] = 0$$
$$E[\mathbf{X}'\mathbf{X}] > 0 \text{ (invertability)}$$

NB: ideally we would rewrite these all in terms of scalar random variable Y and random vectors for X. I've opted to keep them in matrices so the notation is parallel (you can imagine the vectors for Y are length 1 and matrices for X have one row), but the key idea remains that the expectations are over random draws.

Estimator

- We know the population value of β is:

$$\beta = E\left[\mathbf{X}'\mathbf{X}\right]^{-1} E\left[\mathbf{X}'\mathbf{y}\right]$$

- How do we get an estimator of this?
- Plug-in principle $\rightsquigarrow$ replace population expectation with sample versions:

$$\hat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$$

Asymptotic OLS inference

- With this representation, we can write the OLS estimator as follows:

$$\hat{\beta} = \beta + \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{u}$$

- Core idea: $\mathbf{X}'\mathbf{u}$ is the sum of random variables so the CLT applies.
- That, plus some asymptotic theory allows us to say:

$$\sqrt{N}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \Omega)$$

- The covariance matrix, Ω is given as:

$$\Omega = E[\mathbf{X}'\mathbf{X}]^{-1}E[\mathbf{X}'\text{Diag}(\mathbf{u^2})\mathbf{X}]E[\mathbf{X}'\mathbf{X}]^{-1}$$

- We will again be able to replace $\mathbf{u}$ with its empirical counterpart (the residuals) $\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\hat{\beta}$, and $\mathbf{X}$ with its sample counterpart.
- Note that we have only used IID sampling and no perfect collinearity so far.

Theorem (Asymptotic Normality of OLS Under Random Sampling)

Under the Best Linear Projection model,

$$\sqrt{N}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \mathbf{\Omega})$$

where

$$\mathbf{\Omega} = E[\mathbf{X}'\mathbf{X}]^{-1} E[\mathbf{X}' Diag(\mathbf{u^2})\mathbf{X}] E[\mathbf{X}'\mathbf{X}]^{-1}$$

and

- $\hat{\beta}$ *is approximately normal asymptotically with mean* β *and variance* $\mathbf{\Omega}/n$.
- $\mathbf{\Omega}/n$ *is the asymptotic covariance matrix of* $\hat{\beta}$ *and the square root of the diagonal are the standard errors.*
- *Asymptotically this provides us with all the necessary infrastructure for confidence intervals and tests.*

Comparing to the Classical Approach (Homoskedasticity)

- Remember what we did before:

$$\widehat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$$

- Let $\text{Var}[\mathbf{u}|\mathbf{X}] = \mathbf{\Sigma}$
- Recall before we used Assumptions 1-4 to show:

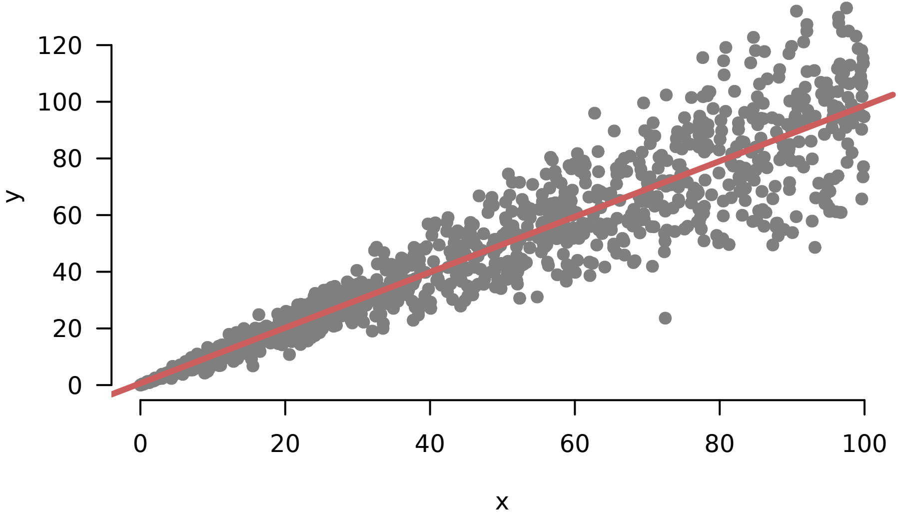
$$\text{Var}[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- With homoskedasticity, $\mathbf{\Sigma} = \sigma^2\mathbf{I}$, we simplified

$$\begin{aligned}
\text{Var}[\hat{\beta}|\mathbf{X}] &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} \\
&= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} \text{ (by homoskedasticity)} \\
&= \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} \\
&= \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}
\end{aligned}$$

- Replace σ^2 with estimate $\widehat{\sigma}^2$ will give us our estimate of the covariance matrix

What Does This Rule Out?



Non-constant Error Variance

- Homoskedastic:

$$V[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I} = \begin{bmatrix} \sigma^2 & 0 & 0 & \ldots & 0 \\ 0 & \sigma^2 & 0 & \ldots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \ldots & \sigma^2 \end{bmatrix}$$

- Heteroskedastic:

$$V[\mathbf{u}|\mathbf{X}] \;=\; \begin{bmatrix} \sigma_1^2 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2^2 & 0 & \ldots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \ldots & \sigma_n^2 \end{bmatrix}$$

- Independent, not identical
- $\text{Cov}(u_i, u_j|\mathbf{X}) = 0$
- $\text{Var}(u_i|\mathbf{X}) = \sigma_i^2$

Consequences of Heteroskedasticity Under Classical Model

- Standard error estimates incorrect:

$$\widehat{SE}[\widehat{\beta}_1] = \sqrt{\frac{\widehat{\sigma}^2}{\sum_i (X_i - \overline{X})^2}}$$

- α-level tests, the probability of Type I error $\neq \alpha$
- Coverage of $1 - \alpha$ CIs $\neq 1 - \alpha$
- OLS is not BLUE
- However:
  - Under Gauss-Markov Assumptions $\widehat{\boldsymbol{\beta}}$ still unbiased and consistent for $\boldsymbol{\beta}$
  - degree of the problem depends on how serious the heteroskedasticity is

Heteroskedasticity Consistent Estimator

- Under non-constant error variance:

$$\text{Var}[\mathbf{u}] = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

- When $\mathbf{\Sigma} \neq \sigma^2 \mathbf{I}$, we are stuck with this expression:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\Sigma\mathbf{X} \left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- Idea: If we can consistently estimate the components of $\mathbf{\Sigma}$, we could directly use this expression by replacing $\mathbf{\Sigma}$ with its estimate, $\hat{\mathbf{\Sigma}}$.

White's Heteroskedasticity Consistent Estimator

Suppose we have heteroskedasticity of unknown form (but zero covariance):

$$V[\mathbf{u}] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

then $V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$ and White (1980) shows that

$$\widehat{V[\hat{\beta}|\mathbf{X}]} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{u}_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \hat{u}_n^2 \end{bmatrix} \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

is a consistent estimator of $V[\hat{\beta}|\mathbf{X}]$ under any form of heteroskedasticity consistent with $V[\mathbf{u}]$ above.

The estimate based on the above is called the heteroskedasticity consistent (HC) or robust standard errors. This also coincides with the agnostic standard errors!

Intuition for Robust Standard Errors

Core intuition: while $\widehat{\boldsymbol{\Sigma}}$ is an $n \times n$ matrix, $\mathbf{X}'\widehat{\boldsymbol{\Sigma}}\mathbf{X}$ is a $(k+1) \times (k+1)$ matrix. So there is hope of estimating it consistently as sample size grows *even when every true error variance is unique*.

$$\widehat{\boldsymbol{\Sigma}} = \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{u}_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \hat{u}_n^2 \end{bmatrix}$$

$$\mathbf{X}'\boldsymbol{\Sigma}\mathbf{X} = \begin{bmatrix} \sum_i x_{i,1}x_{i,1}\hat{u}_i^2 & \sum_i x_{i,1}x_{i,2}\hat{u}_i^2 & \dots & \sum_i x_{i,1}x_{i,k+1}\hat{u}_i^2 \\ \sum_i x_{i,2}x_{i,1}\hat{u}_i^2 & \sum_i x_{i,2}x_{i,2}\hat{u}_i^2 & \dots & \sum_i x_{i,2}x_{i,k+1}\hat{u}_i^2 \\ & & \vdots & \\ \sum_i x_{i,k+1}x_{i,1}\hat{u}_i^2 & \sum_i x_{i,k+1}x_{i,2}\hat{u}_i^2 & \dots & \sum_i x_{i,k+1}x_{i,k+1}\hat{u}_i^2 \end{bmatrix}$$

White's Heteroskedasticity Consistent Estimator

Robust standard errors are easily computed with the "sandwich" formula:

1. Fit the regression and obtain the residuals $\hat{\mathbf{u}}$
2. Construct the "meat" matrix $\widehat{\mathbf{\Sigma}}$ with squared residuals in diagonal:

$$\widehat{\mathbf{\Sigma}} = \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{u}_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \hat{u}_n^2 \end{bmatrix}$$

3. Plug $\widehat{\mathbf{\Sigma}}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix

$$V[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\widehat{\mathbf{\Sigma}}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- There are various small sample corrections to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|\mathbf{X}] \;=\; \frac{n}{n-k-1}\cdot\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\widehat{\mathbf{\Sigma}}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

Notes on White's Robust Standard Errors

- Doesn't change estimate $\widehat{\boldsymbol{\beta}}$.
- Provides a plug-and-play estimate of $V[\widehat{\beta}]$ which can be used with SEs, confidence intervals etc.—does not provide $V[u]$.
- Consistent for $V[\widehat{\beta}]$ under any form of heteroskedasticity (i.e. where the covariances are 0).
- This is a large sample result, best with large n
- For small n, performance might be poor and the estimates are downward biased (correction factors exist but are often insufficient)
- As we saw, we can arrive at White's heteroskedasticity consistent standard errors using the plug-in principle and thus in some ways, these are the natural way of getting standard errors in the agnostic regression framework.
- Robust SEs converge to same point as the bootstrap.

Clustered Dependence: Intuition

- Think back to the Gerber, Green, and Larimer (2008) social pressure mailer example.
- Their design: randomly sample households and randomly assign them to different treatment conditions.
- But the measurement of turnout is at the individual level and we have one unit per person.
- Violation of iid/random sampling called clustering or clustered dependence.
- Loosely speaking, you are tricking the regression into thinking it has more information than it does.

Sampling Variance under Cluster Dependence

- Recall our general sandwich expression:

$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Sigma\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

- We need to adjust to address our new sampling strategy.
- We assume that respondents are independent across clusters, but dependent within clusters. Thus, we have

$$\text{Var}[\boldsymbol{u}_g|\mathbf{X}_g] = \boldsymbol{\Sigma}_g$$

- This leads to a block diagonal expression for $\boldsymbol{\Sigma}$ overall

$$V[\boldsymbol{u}] = \boldsymbol{\Sigma} = \left[\begin{array}{c|c|c|c} \boldsymbol{\Sigma}_1 & 0 & \dots & 0 \\ \hline \mathbf{0} & \boldsymbol{\Sigma}_2 & \dots & \mathbf{0} \\ \hline & & \ddots & \\ \hline \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\Sigma}_M \end{array}\right]$$

- Under this clustered dependence, we can write this as:

$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1}\left(\sum_{g=1}^{m}\mathbf{X}_g'\Sigma_g\mathbf{X}_g\right)(\mathbf{X}'\mathbf{X})^{-1}$$

Cluster-Robust Variance Estimation

- The independence is now across clusters, so we will need asymptotics in the number of clusters for CLT.
- We can use a plugin estimator for the sampling variance on the previous slide with a small sample correction (many variants available, here we give the CR1 variant Stata uses):

$$\hat{V}_{CR1}[\hat{\beta}|\mathbf{X}] = \frac{m}{(m-1)}\frac{n-1}{n-k}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\sum_{g=1}^{m}\mathbf{X}_g'\hat{\boldsymbol{u}}_g\hat{\boldsymbol{u}}_g{}'\mathbf{X}_g\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- Another excellent strategy is to use the block bootstrap (resample clusters instead of units).

Notes on Cluster-Robust Standard Errors

- CRSE do not change our estimates $\widehat{\boldsymbol{\beta}}$, cannot fix bias.
- Consistency of the CRSE are in the number of groups, not the number of individuals.
- Works well in the following settings:
  - sampling complete blocks of units where sampled clusters are a small fraction of the population.
  - sampling a small fraction of units from all or nearly all clusters in the population.
  - assignment of key variable is constant within the cluster.
- Just because something is a group, doesn't mean you need to cluster standard errors, key is to think about the sampling strategy.
- Abadie, Athey, Imbens and Wooldridge (2022) "When Should You Adjust Standard Errors for Clustering" *QJE* provides guidance and approaches for other cases.
- NB: this broad strategy of sandwich estimators works for other kinds of settings (time-series etc.). See `sandwich` package in `R`.

Running Example: Chilean Referendum on Pinochet

- The 1988 Chilean national plebiscite was a national referendum held to determine whether or not dictator Augusto Pinochet would extend his rule for another eight-year term in office.
- Data: national survey conducted in April and May of 1988 by FLACSO in Chile.
- Outcome: 1 if respondent intends to vote for Pinochet, 0 otherwise. We can interpret the β slopes as marginal "effects" on the probability that respondent votes for Pinochet.
- Plebiscite was held on October 5, 1988. The No side won with 56% of the vote, with 44% voting Yes.
- We model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

Hypothesis Testing in R

Model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,       Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

The t-Value for Multiple Linear Regression

- Consider testing a hypothesis about a single regression coefficient β_j:

$$H_0 : \beta_j = c$$

- In the simple linear regression we used the t-value to test this kind of hypothesis.
- We can consider the same t-value about β_j for the multiple regression:

$$T = \frac{\hat{\beta}_j - c}{\hat{SE}(\hat{\beta}_j)}$$

- How do we compute $\hat{SE}(\hat{\beta}_j)$?

$$\hat{SE}(\hat{\beta}_j) = \sqrt{\widehat{V}(\hat{\beta}_j)} = \sqrt{\widehat{V}(\hat{\boldsymbol{\beta}})_{(j,j)}} = \sqrt{\hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}_{(j,j)}}$$

where $\mathbf{A}_{(j,j)}$ is the (j, j) element of matrix $\mathbf{A}$.
That is, take the variance-covariance matrix of $\hat{\boldsymbol{\beta}}$ and square root the diagonal element corresponding to j.

Getting the Standard Errors

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
```

We can pull classical variance-covariance matrix in R from the lm() object:

R Code

```
> V <- vcov(fit)
> V
              (Intercept)           fem          educ           age
(Intercept)  2.642311e-03 -3.455498e-04 -5.270913e-04 -3.357119e-05
fem         -3.455498e-04  5.623170e-04  2.249973e-05  8.285291e-07
educ        -5.270913e-04  2.249973e-05  1.922354e-04  3.411049e-06
age         -3.357119e-05  8.285291e-07  3.411049e-06  6.914098e-07

> sqrt(diag(V))
 (Intercept)          fem         educ          age
0.0514034097 0.0237132251 0.0138648980 0.0008315105
```

Using the t-Value as a Test Statistic

The procedure for testing this null hypothesis ($\beta_j = c$) is identical to the simple regression case, except that our reference distribution is t_{n-k-1} instead of t_{n-2}.

1. Compute the t-value as $T = (\hat{\beta}_j - c)/\hat{SE}[\hat{\beta}_j]$
2. Compare the value to the critical value $t_{\alpha/2}$ for the α level test, which under the null hypothesis satisfies

$$P\left(-t_{\alpha/2} \leq T \leq t_{\alpha/2}\right) = 1 - \alpha$$

3. Decide whether the realized value of T in our data is unusual given the distribution of the test statistic under the null hypothesis.
4. Finally, either declare that we reject H_0 or not, or report the p-value.

Confidence Intervals

To construct confidence intervals, there is again no difference compared to the case of $k = 1$, except that we need to use t_{n-k-1} instead of t_{n-2}

Since we know the sampling distribution for our t-value:

$$T \;=\; \frac{\hat{\beta}_j - c}{\hat{SE}[\hat{\beta}_j]} \;\sim t_{n-k-1}$$

So we also know the probability that the value of our test statistics falls into a given interval:

$$P\left(-t_{\alpha/2} \leq \frac{\widehat{\beta}_j - \beta_j}{\hat{SE}[\hat{\beta}_j]} \leq t_{\alpha/2}\right) = 1 - \alpha$$

We rearrange:

$$\left[\widehat{\beta}_j - t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\,,\; \widehat{\beta}_j + t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\right]$$

and thus can construct the confidence intervals as usual using:

$$\hat{\beta}_j \pm t_{\alpha/2} \cdot \hat{SE}[\hat{\beta}_j]$$

Confidence Intervals in R

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
```

R Code

```
> confint(fit)
                   2.5 %      97.5 %
(Intercept)  0.303407780  0.50504909
fem          0.089493169  0.18251357
educ        -0.087954435 -0.03356629
age          0.002147755  0.00540954
```

Testing Hypothesis About a Linear Combination of β_j

R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> summary(fit)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       4452.7      783.4   5.684 2.07e-07 ***
RegionAfrica     -2552.8     1204.5  -2.119   0.0372 *
RegionAsia         148.9     1149.8   0.129   0.8973
RegionLatAmerica  -271.3     1007.0  -0.269   0.7883
RegionOecd        9671.3     1007.0   9.604 5.74e-15 ***
```

- $\hat{\beta}_{Asia}$ and $\hat{\beta}_{LAm}$ are close. So we may want to test the null hypothesis:

$$H_0 : \ \beta_{LAm} = \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} = 0$$

against the alternative of

$$H_1 : \ \beta_{LAm} \neq \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} \neq 0$$

- What would be an appropriate test statistic for this hypothesis?

Testing Hypothesis About a Linear Combination of β_j

R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> summary(fit)

Coefficients:
                  Estimate Std. Error t value Pr(>|t|)
(Intercept)         4452.7      783.4   5.684 2.07e-07 ***
RegionAfrica       -2552.8     1204.5  -2.119   0.0372 *
RegionAsia           148.9     1149.8   0.129   0.8973
RegionLatAmerica    -271.3     1007.0  -0.269   0.7883
RegionOecd          9671.3     1007.0   9.604 5.74e-15 ***
```

- Let's consider a t-value:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})}$$

  We will reject H_0 if T is sufficiently different from zero.

- Note that unlike the test of a single hypothesis, both $\hat{\beta}_{LAm}$ and $\hat{\beta}_{Asia}$ are random variables, hence the denominator.

Testing Hypothesis About a Linear Combination of β_j

- Our test statistic:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \sim t_{n-k-1}$$

- How do you find $\hat{SE}(\hat{\beta}_{LAm} - \hat{\beta}_{Asia})$?
- Is it $\hat{SE}(\hat{\beta}_{LAm}) - \hat{SE}(\hat{\beta}_{Asia})$? No!
- Is it $\hat{SE}(\hat{\beta}_{LAm}) + \hat{SE}(\hat{\beta}_{Asia})$? No!
- Recall the following property of the variance:

$$V(X \pm Y) = V(X) + V(Y) \pm 2\text{Cov}(X, Y)$$

  Therefore, the standard error for a linear combination of coefficients is:

$$\hat{SE}(\widehat{\beta}_1 \pm \widehat{\beta}_2) = \sqrt{\widehat{V}(\widehat{\beta}_1) + \widehat{V}(\widehat{\beta}_2) \pm 2\widehat{\text{Cov}}[\widehat{\beta}_1, \widehat{\beta}_2]}$$

  which we can calculate from the estimated covariance matrix of $\hat{\boldsymbol{\beta}}$.

- Since the estimates of the coefficients are correlated, we need the covariance term.

Example: GDP per capita on Regions

R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> V <- vcov(fit)
> V
                 (Intercept) RegionAfrica RegionAsia RegionLatAmerica
(Intercept)         613769.9    -613769.9  -613769.9        -613769.9
RegionAfrica       -613769.9    1450728.8   613769.9         613769.9
RegionAsia         -613769.9     613769.9  1321965.9         613769.9
RegionLatAmerica   -613769.9     613769.9   613769.9        1014054.6
RegionOecd         -613769.9     613769.9   613769.9         613769.9
                 RegionOecd
(Intercept)       -613769.9
RegionAfrica       613769.9
RegionAsia         613769.9
RegionLatAmerica   613769.9
RegionOecd        1014054.6
```

Example: GDP per capita on Regions

We can then compute the test statistic for the hypothesis of interest:

R Code

```
> se <- sqrt(V[4,4] + V[3,3] - 2*V[3,4])
> se
[1] 1052.844
>
> tstat <- (coef(fit)[4] - coef(fit)[3])/se
> tstat
RegionLatAmerica
      -0.3990977
```

$$t = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \quad \text{where}$$

$$\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}) = \sqrt{\widehat{V}(\widehat{\beta}_{LAm}) + \widehat{V}(\widehat{\beta}_{Asia}) - 2\widehat{Cov}[\widehat{\beta}_{LAm}, \widehat{\beta}_{Asia}]}$$

Plugging in we get $t \approx -0.40$. So what do we conclude?

We cannot reject the null.

NB: this same strategy can be used to get standard errors for interactions at any value of the other variables.

Adjusted R^2

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,	Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

Adjusted R^2

- R^2 often used to assess in-sample model fit. Recall

$$R^2 = 1 - \frac{\text{SS}_{\text{res}}}{\text{SS}_{\text{tot}}}$$

  where SS_{res} are the sum of squared residuals and the SS_{tot} are the sum of the squared deviations from the mean.

- Perhaps problematically, it can be shown that R^2 always stays constant or increases with more explanatory variables
- So, how do we penalize more complex models? Adjusted R^2
- This makes R^2 more 'comparable' across models with different numbers of variables, but the next section will show you an even better way to approach that problem in a testing framework.
- Still since people report it, the next slide derives adjusted R^2.

Adjusted R^2

- Key idea: rewrite R^2 in terms of variances

$$R^2 = 1 - \frac{\text{SS}_{\text{res}}/n}{\text{SS}_{\text{tot}}/n} = 1 - \frac{\tilde{V}(\text{SS}_{\text{res}})}{\tilde{V}(\text{SS}_{\text{tot}})}$$

where $\tilde{V}$ is a biased estimator of the population variance.

- What if we replace the biased estimator with the unbiased estimators

$$\hat{V}(\text{SS}_{\text{res}}) = \text{SS}_{\text{res}}/(n-k-1)$$
$$\hat{V}(\text{SS}_{\text{tot}}) = \text{SS}_{\text{tot}}/(n-1)$$

- Some algebra gets us to

$$R^2_{adj} = R^2 - \underbrace{(1-R^2)\frac{k-1}{n-k}}_{\text{model complexity penalty}}$$

- Adjusted R^2 will always be smaller than R^2 and can sometimes be negative!

Limits of R^2

- In-sample model fit is not a particularly good indicator of model fit on a new sample.
- Adjusted R^2 is solving a problem about increasingly complex models, but by the time you reach this problem, you should be using held-out data.
- Stay tuned for more on that next week.
- Regardless the intuition here is useful: as we add more variables R^2 must stay constant or go up mechanically and that creates complications.

F Test for Joint Significance of Coefficients

- In research we often want to test a joint hypothesis which involves multiple linear restrictions (e.g. $\beta_1 = \beta_2 = \beta_3 = 0$)
- Suppose our regression model is:

$$Voted = \beta_0 + \gamma_1 FEMALE + \beta_1 EDUCATION +$$

$$\gamma_2(FEMALE \cdot EDUCATION) + \beta_2 AGE + \gamma_3(FEMALE \cdot AGE) + u$$

  and we want to test

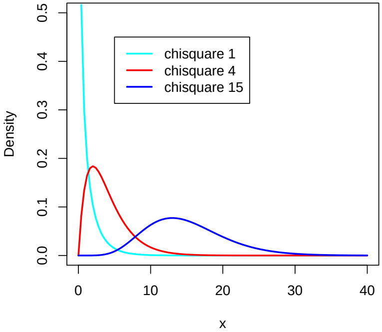
$$H_0: \ \gamma_1 = \gamma_2 = \gamma_3 = 0.$$

- Substantively, what question are we asking?

  $\rightarrow$ Do females and males vote systematically differently from each other? (Under the null, there is no difference in either the intercept or slopes between females and males).
- This is an example of a joint hypothesis test involving three restrictions: $\gamma_1 = 0$, $\gamma_2 = 0$, and $\gamma_3 = 0$.
- If all the interaction terms and the group lower order term are close to zero, then we fail to reject the null hypothesis of no gender difference.
- F tests allows us to to test joint hypothesis

The χ^2 Distribution

- To test more than one hypothesis jointly we need to introduce some new probability distributions.
- Suppose $Z_1, ..., Z_n$ are n i.i.d. random variables following $\mathcal{N}(0,1)$.
- Then, the sum of their squares, $X = \sum_{i=1}^{n} Z_i^2$, is distributed according to the χ^2 distribution with n degrees of freedom, $X \sim \chi^2_n$.



Properties: $X > 0$, $E[X] = n$ and $V[X] = 2n$. In R: `dchisq()`, `pchisq()`, `rchisq()`

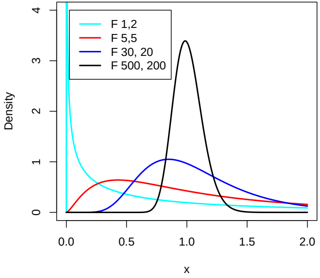
The F distribution

The F distribution arises as a ratio of two independent chi-squared distributed random variables:

$$F = \frac{X_1/df_1}{X_2/df_2} \sim \mathcal{F}_{df_1,df_2}$$

where $X_1 \sim \chi^2_{df_1}$, $X_2 \sim \chi^2_{df_2}$, and $X_1 \perp\!\!\!\perp X_2$.

df_1 and df_2 are called the numerator degrees of freedom and the denominator degrees of freedom.



In R: `df()`, `pf()`, `rf()`

F Test against $H_0: \gamma_1 = \gamma_2 = \gamma_3 = 0.$

The F statistic can be calculated by the following procedure:

1. Fit the Unrestricted Model (UR) which *does not* impose H_0:

   $$Vote = \beta_0 + \gamma_1 FEM + \beta_1 EDUC + \gamma_2 (FEM * EDUC) + \beta_2 AGE + \gamma_3 (FEM * AGE) + u$$

2. Fit the Restricted Model (R) which *does* impose H_0:

   $$Vote = \beta_0 + \beta_1 EDUC + \beta_2 AGE + u$$

3. From the two results, compute the F Statistic:

   $$F_0 = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}$$

   where SSR=sum of squared residuals, q=number of restrictions, k=number of predictors in the unrestricted model, and n= # of observations.

   Intuition:

   $$\frac{\text{increase in prediction error}}{\text{original prediction error}}$$

The F statistics have the following sampling distributions:

- Under Assumptions 1–6, $F_0 \sim \mathcal{F}_{q,n-k-1}$ regardless of the sample size.
- Asymptotically under IID sampling, $F_0 \overset{d}{\sim} \mathcal{F}_{q,n-k-1}$

Unrestricted Model (UR)

R Code

```
> fit.UR <- lm(vote1 ~ fem + educ + age + fem:age + fem:educ, data = Chile)
> summary(fit.UR)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
educ        -0.038571   0.019578  -1.970 0.048988 *
age          0.005482   0.001114   4.921 9.44e-07 ***
fem:age     -0.003779   0.001673  -2.259 0.024010 *
fem:educ    -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,	Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Restricted Model (R)

R Code

```
> fit.R <- lm(vote1 ~ educ + age, data = Chile)
> summary(fit.R)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4878039  0.0497550   9.804  < 2e-16 ***
educ        -0.0662022  0.0139615  -4.742 2.30e-06 ***
age          0.0035783  0.0008385   4.267 2.09e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4921 on 1700 degrees of freedom
Multiple R-squared:  0.03275,	Adjusted R-squared:  0.03161
F-statistic: 28.78 on 2 and 1700 DF,  p-value: 5.097e-13
```

F Test in R

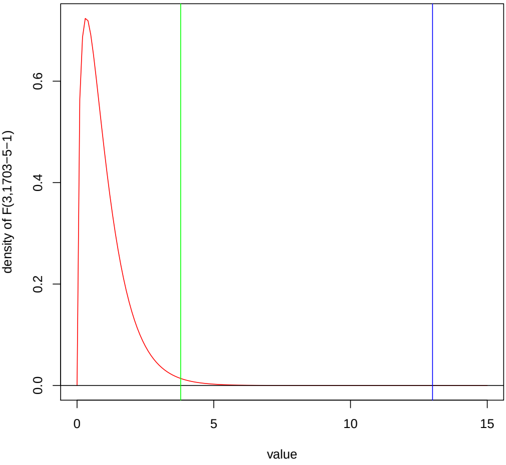
R Code
```
> SSR.UR <- sum(resid(fit.UR)^2)  # = 402
> SSR.R <- sum(resid(fit.R)^2)    # = 411

> DFdenom <- df.residual(fit.UR)  # = 1703
> DFnum <- 3

> F <- ((SSR.R - SSR.UR)/DFnum) / (SSR.UR/DFdenom)
> F
[1] 13.01581

> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

$F_0 = 13$ is greater than the critical value for a .01 level test. So we *reject* the null hypothesis.

Null Distribution, Critical Value, and Test Statistic

Note that the F statistic is always positive, so we only look at the right tail of the reference F (or χ^2 in a large sample) distribution.



F Test Examples I

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + ... + \beta_k X_k + u$$

We may want to test:

$$H_0 : \beta_1 = \beta_2 = ... = \beta_k = 0$$

- Have any of you used an F-test like this in your research?
- This is called the omnibus test and is routinely reported by statistical software.

Omnibus Test in R

R Code

```
> summary(fit.UR)
~~~~~~

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
educ        -0.038571   0.019578  -1.970 0.048988 *
age          0.005482   0.001114   4.921 9.44e-07 ***
fem:age     -0.003779   0.001673  -2.259 0.024010 *
fem:educ    -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,       Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Omnibus Test in R with Random Noise

R Code

```
> set.seed(08540)
> p <- 10; x <- matrix(rnorm(p*1000), nrow=1000)
> y <- rnorm(1000); summary(lm(y~x))
~~~~~

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0115475  0.0320874  -0.360   0.7190
x1          -0.0019803  0.0333524  -0.059   0.9527
x2           0.0666275  0.0314087   2.121   0.0341 *
x3          -0.0008594  0.0321270  -0.027   0.9787
x4           0.0051185  0.0333678   0.153   0.8781
x5           0.0136656  0.0322592   0.424   0.6719
x6           0.0102115  0.0332045   0.308   0.7585
x7          -0.0103903  0.0307639  -0.338   0.7356
x8          -0.0401722  0.0318317  -1.262   0.2072
x9           0.0553019  0.0315548   1.753   0.0800 .
x10          0.0410906  0.0319742   1.285   0.1991
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.011 on 989 degrees of freedom
Multiple R-squared:  0.01129,       Adjusted R-squared:  0.001294
F-statistic: 1.129 on 10 and 989 DF,  p-value: 0.3364
```

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + ... + \beta_k X_k + u$$

Next, let's consider:

$$H_0 : \beta_1 = \beta_2 = \beta_3$$

- What question are we asking?

  $\rightarrow$ Are the coefficients X_1, X_2 and X_3 different from each other?

- How many restrictions?

  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)

- How do we fit the restricted model?

  $\rightarrow$ The null hypothesis implies that the model can be written as:

  $$Y = \beta_0 + \beta_1 (X_1 + X_2 + X_3) + ... + \beta_k X_k + u$$

  So we create a new variable $X^* = X_1 + X_2 + X_3$ and fit:

  $$Y = \beta_0 + \beta_1 X^* + ... + \beta_k X_k + u$$

Testing Equality of Coefficients in R

R Code

```
> fit.UR2 <- lm(REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> summary(fit.UR2)
~~~~~
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1899.9      914.9   2.077   0.0410 *
Asia          2701.7     1243.0   2.173   0.0327 *
LatAmerica    2281.5     1112.3   2.051   0.0435 *
Transit       2552.8     1204.5   2.119   0.0372 *
Oecd         12224.2     1112.3  10.990   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 3034 on 80 degrees of freedom
Multiple R-squared: 0.7096,       Adjusted R-squared: 0.6951
F-statistic: 48.88 on 4 and 80 DF,  p-value: < 2.2e-16
```

Are the coefficients on *Asia*, *LatAmerica* and *Transit* statistically significantly different?

Testing Equality of Coefficients in R

R Code

```
> D$Xstar <- D$Asia + D$LatAmerica + D$Transit
> fit.R2 <- lm(REALGDPCAP ~ Xstar + Oecd, data = D)

> SSR.UR2 <- sum(resid(fit.UR2)^2)
> SSR.R2 <- sum(resid(fit.R2)^2)

> DFdenom <- df.residual(fit.UR2)

> F <- ((SSR.R2 - SSR.UR2)/2) / (SSR.UR2/DFdenom)
> F
[1] 0.08786129

> pf(F, 2, DFdenom, lower.tail = F)
[1] 0.9159762
```

So, what do we conclude?

The three coefficients are statistically indistinguishable from each other, with the p-value of 0.916.

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2)/\hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the t test.
- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR})/(SSR_R/(n-k-1))$ and do the F test.

It turns out these two tests give identical results. This is because

$$X \ \sim \ t_{n-k-1} \quad \Longleftrightarrow \quad X^2 \ \sim \ \mathcal{F}_{1,n-k-1}$$

- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.
- Usually, the t test is used for single hypotheses and the F test is used for joint hypotheses.

Some More Notes on F Tests

- The F-value can also be calculated from R^2:

$$F = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n-k-1)}$$

- F tests only work for testing nested models, i.e. the restricted model must be a special case of the unrestricted model.

  For example F tests cannot be used to test

$$Y = \beta_0 + \beta_1 X_1 \qquad\qquad + \beta_3 X_3 + u$$

against

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \qquad\qquad + u$$

Some More Notes on F Tests

Joint significance does not necessarily imply the significance of individual coefficients, or vice versa:

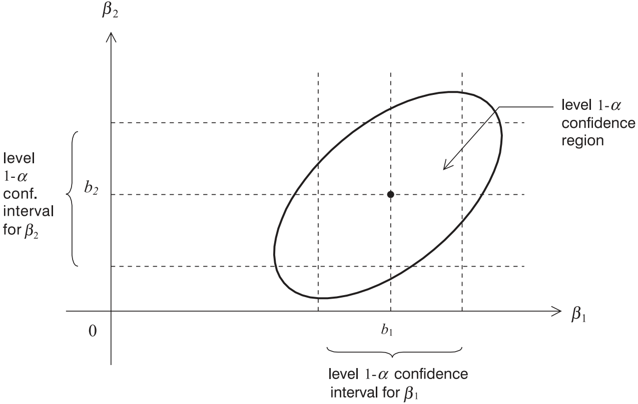


Figure 1.5: t- versus F-Tests

Image Credit: Hayashi (2011) *Econometrics*

Goal Check: Understand `lm()` Output

```
Call:
lm(formula = sr ~ pop15, data = LifeCycleSavings)

Residuals:
   Min     1Q Median     3Q    Max
-8.637 -2.374  0.349  2.022 11.155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.49660    2.27972   7.675 6.85e-10 ***
pop15       -0.22302    0.06291  -3.545 0.000887 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.03 on 48 degrees of freedom
Multiple R-squared:  0.2075,Adjusted R-squared:  0.191
F-statistic: 12.57 on 1 and 48 DF,  p-value: 0.0008866
```

This Week in Review

You now have seen the full linear regression model!

- Multiple regression is much like the regression formulations we have already seen.
- We showed how to estimate the coefficients and get the variance-covariance matrix.
- Under strong classical linear model assumptions we have lots of nice finite sample properties.
- Under the BLP model, we can be more agnostic and assume only IID and no-perfect collinearity. This leads to different standard errors and properties only hold asymptotically (and converge to the linear projection rather than the true CEF).
- Much of the hypothesis test/CI infrastructure ports over nicely, plus there are new joint tests we can use.

Next week: Diagnostics!