Week 6: Estimating Conditional Expectation Functions

Brandon Stewart[1]

Princeton

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[1]Many of the slides are adapted from Matt Blackwell, Adam Glynn, Jens Hainmueller, and Erin Hartman

Where We've Been and Where We're Going...

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  - causal inference
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- Long Run
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2. Best Linear Predictor
3. Fun With Linearity
4. Estimation of the BLP
5. Additional Techniques
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   - LOESS
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  - Causal Inference: with more assumptions we can talk about how intervening to change the value of X will change Y.

The Conditional Expectation Function

Recall, a conditional expectation, for random variables X and Y:
Discrete with joint PMF f:

$$E[Y \mid X = x] = \sum_y y f_{Y|X}(y|x), \forall x \in \text{Supp}(X)$$

Continuous with joint PDF f:

$$E[Y \mid X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy, \forall x \in \text{Supp}(X)$$

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We will generally denote it $E[Y \mid X]$ or $\mu(\mathbf{x})$.

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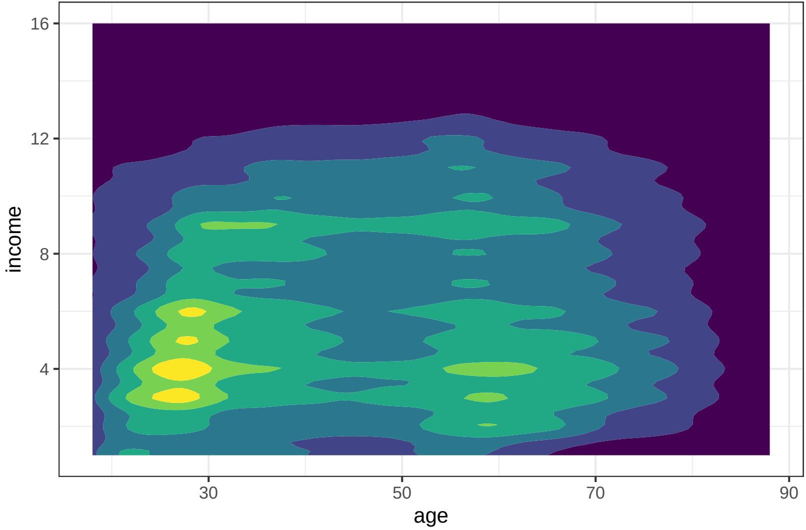

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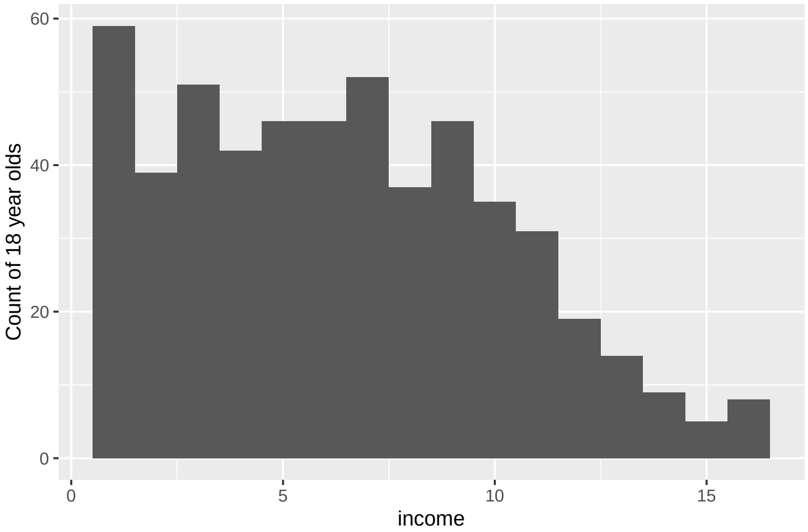

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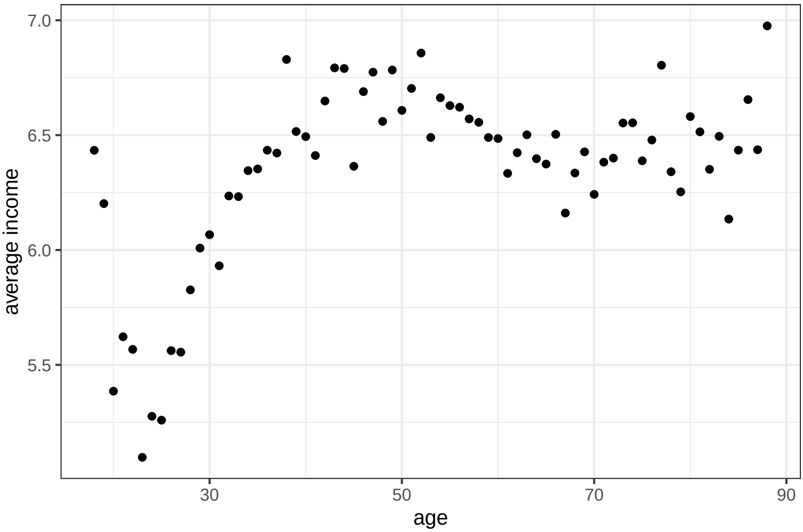

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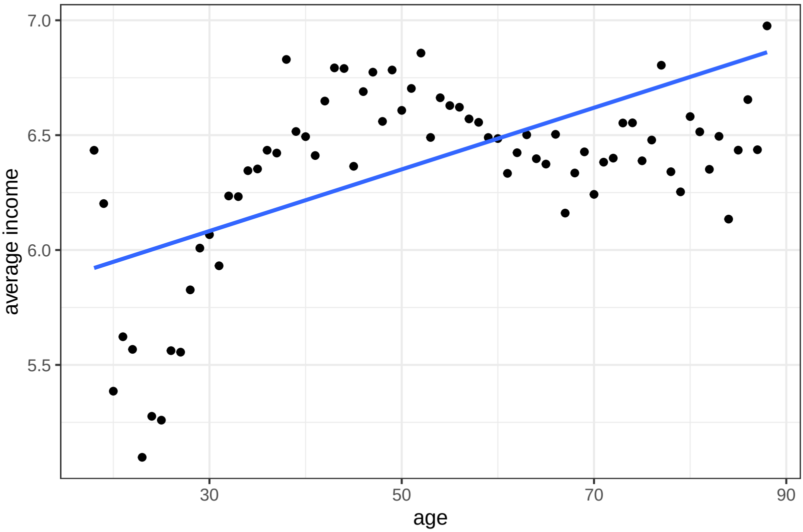

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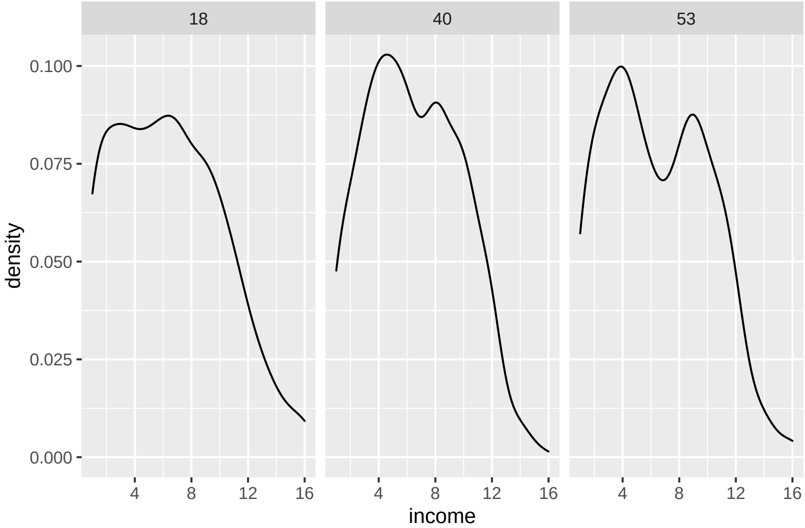

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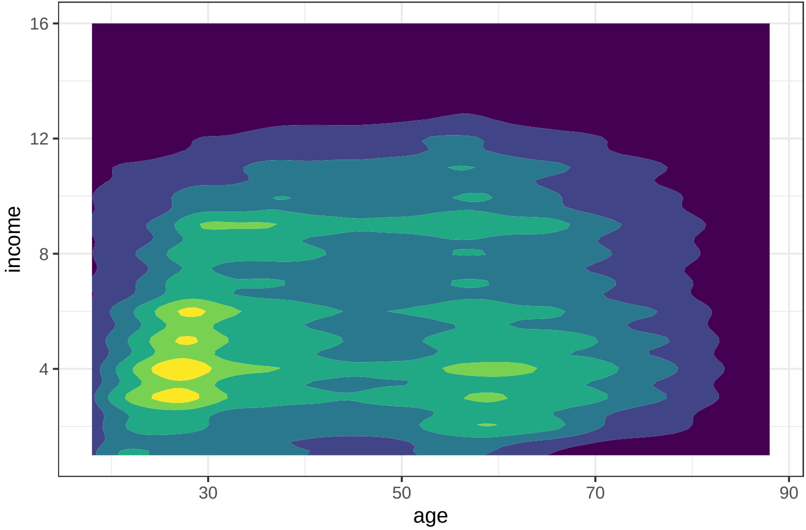

Properties of the CEF

The CEF is very powerful. Let X and Y be random variables and let $\epsilon = Y - E[Y \mid X]$, then:

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5. $V[\epsilon] = E\big[V[Y \mid X]\big]$
   - By the Law of Total Variance

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  - There is no better way (in terms of MSE) to approximate $Y \mid X$ than with the CEF
  - If we know the CEF, we know *a lot* about how X relates to Y
  - But, just because it is the best predictor doesn't mean that it is simple
    - ⋆ It could take on *any* shape!

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- Estimation just involves taking the means within the groups.

Non-binary CEFs

- If X is discrete with not too many categories, we can estimate the conditional expectation using the means within groups:
  - $\hat{\mu}(1) = \hat{E}[Y|X = 1] = \frac{1}{n_{x=1}} \sum_{i:X_i=1} Y_i$
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- When X can take on many possible values (think income) or we have few observation for a given value of X, we have to write out a more general function.
- These functional forms are unknown which makes life hard.

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- We use two variables:
  - Y: income
  - X: educational attainment
- Goal is to characterize the conditional expectation $E[Y|X = x]$, i.e. how average income varies with education level

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`educ`: Respondent's education:

- 1. 8 grades or less and no diploma or
- 2. 9-11 grades
- 3. High school diploma or equivalency test
- 4. More than 12 years of schooling, no higher degree
- 5. Junior or community college level degree (AA degrees)
- 6. BA level degrees; 17+ years, no postgraduate degree
- 7. Advanced degree

Nonparametric Regression with Discrete X

`income`: Respondent's family income:

- 1. None or less than \$2,999
- 2. \$3,000-\$4,999
- 3. \$5,000-\$6,999
- 4. \$7,000-\$8,999
- 5. \$9,000-\$9,999
- 6. \$10,000-\$10,999

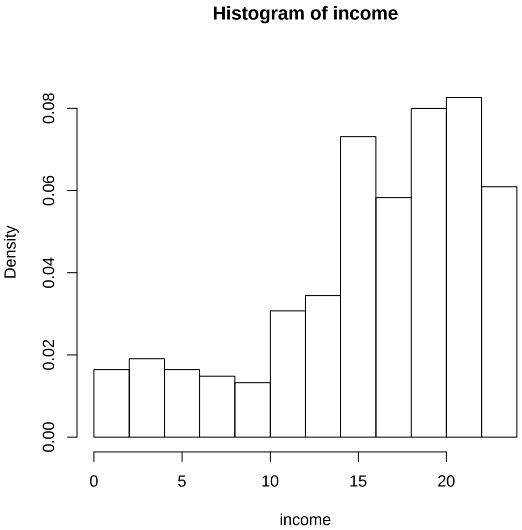
$\vdots$

- 17. \$35,000-\$39,999
- 18. \$40,000-\$44,999

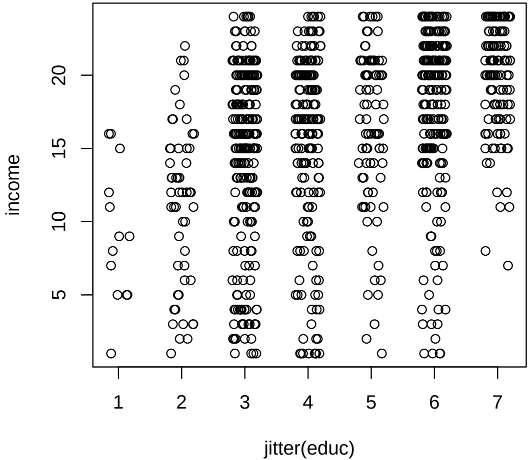
$\vdots$

- 23. \$90,000-\$104,999
- 24. \$105,000 and over

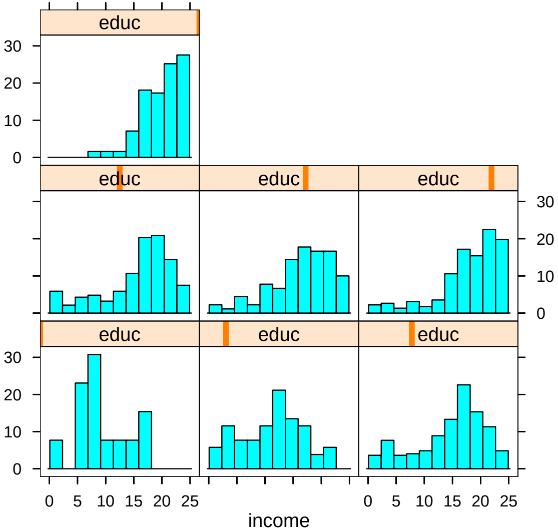
Marginal Distribution of Y (income)



Joint Distribution of X and Y (Income and Education)



Distribution of income given education $p(y|x)$



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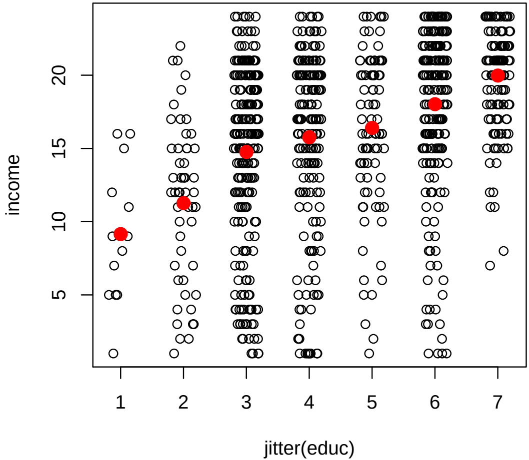

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- But what do we do when X is continuous and has many values?
- Let's talk through a few options.

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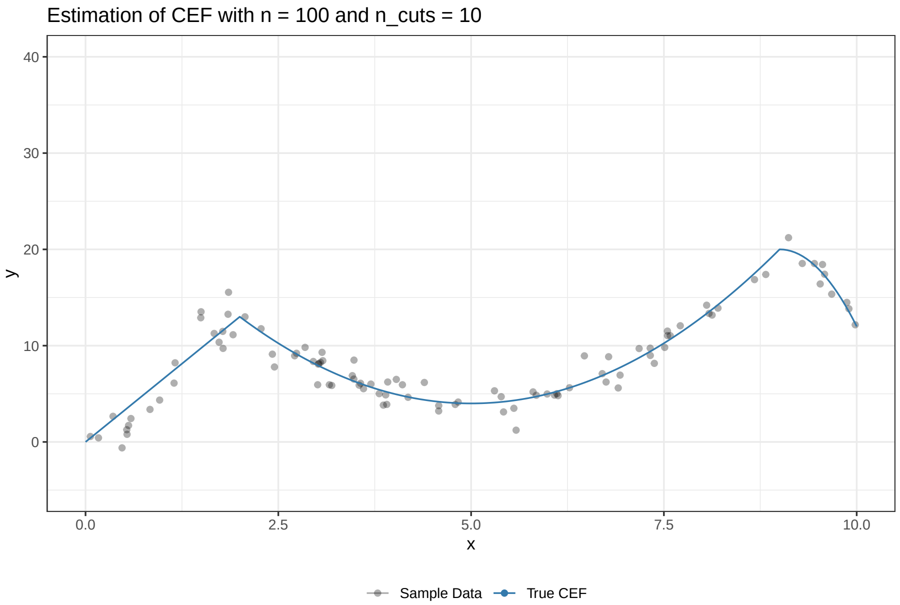

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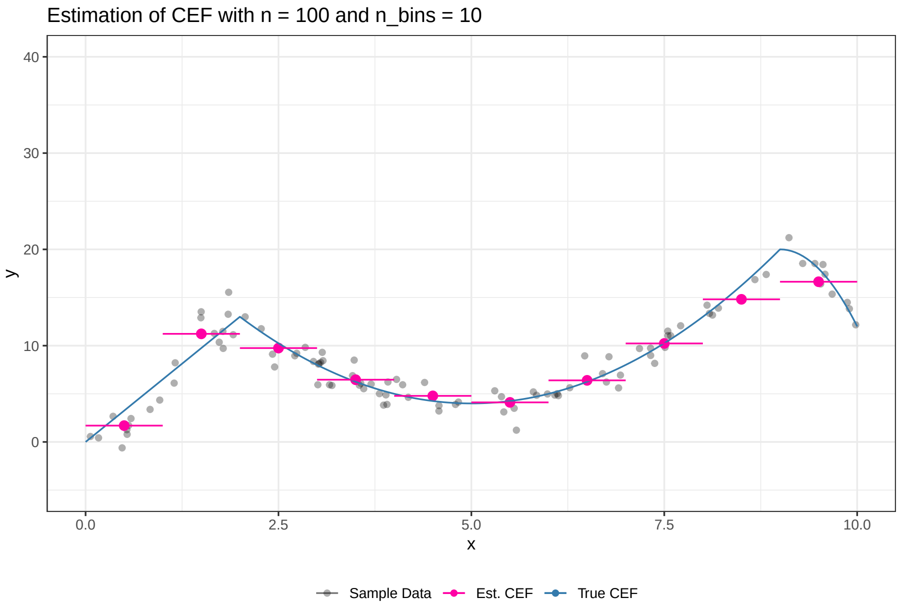

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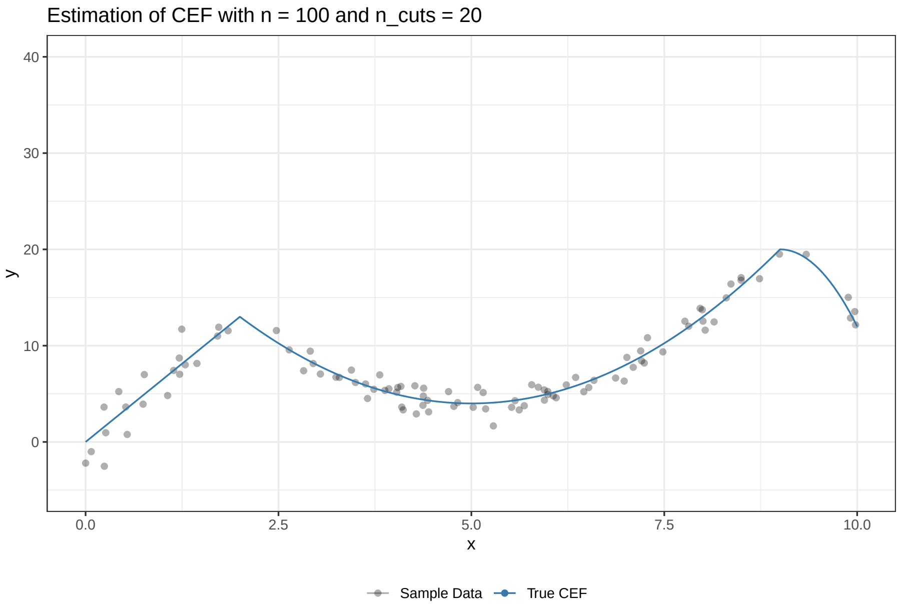

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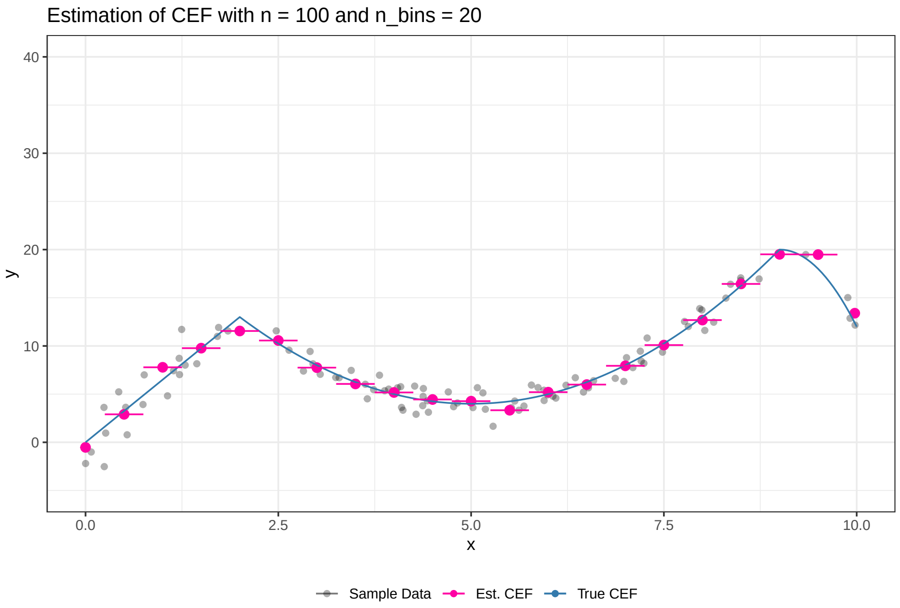

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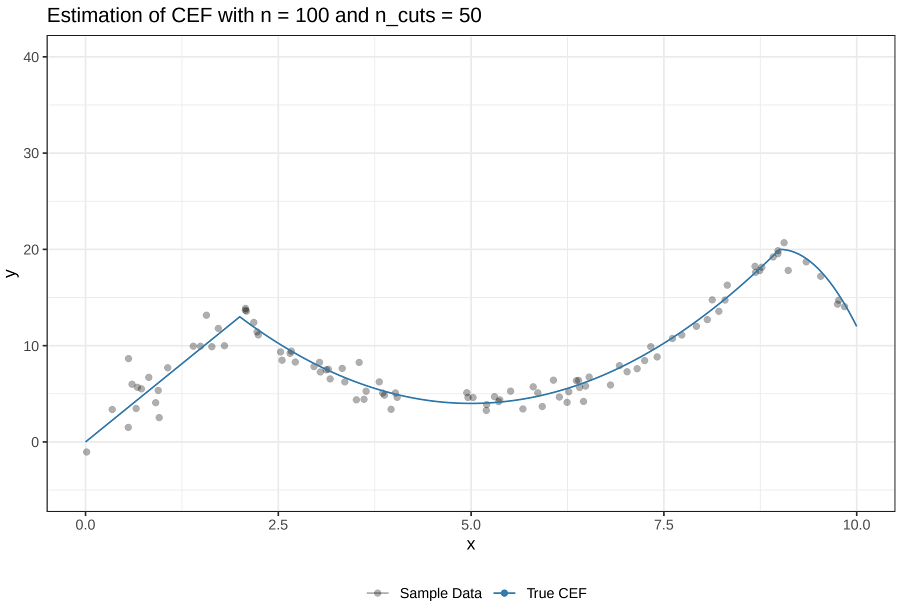

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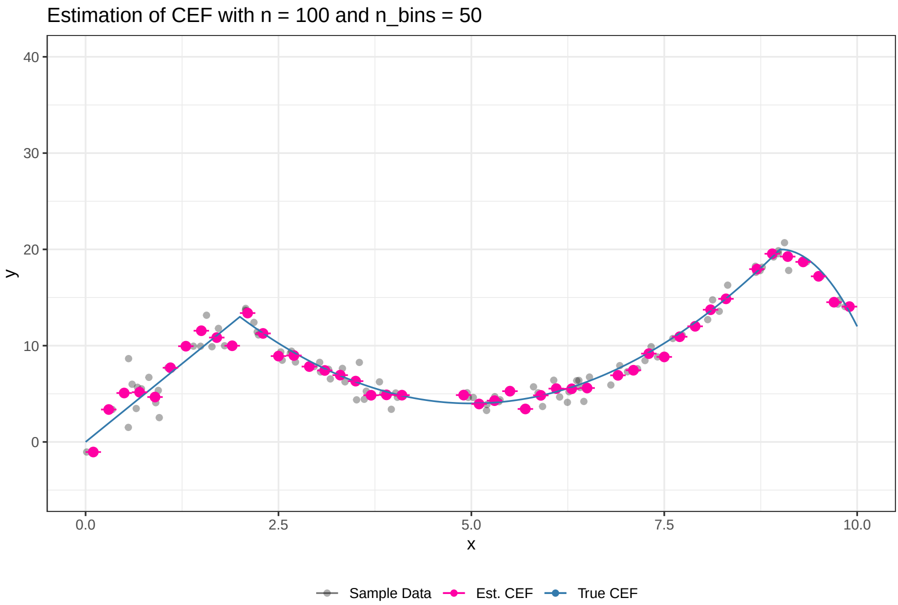

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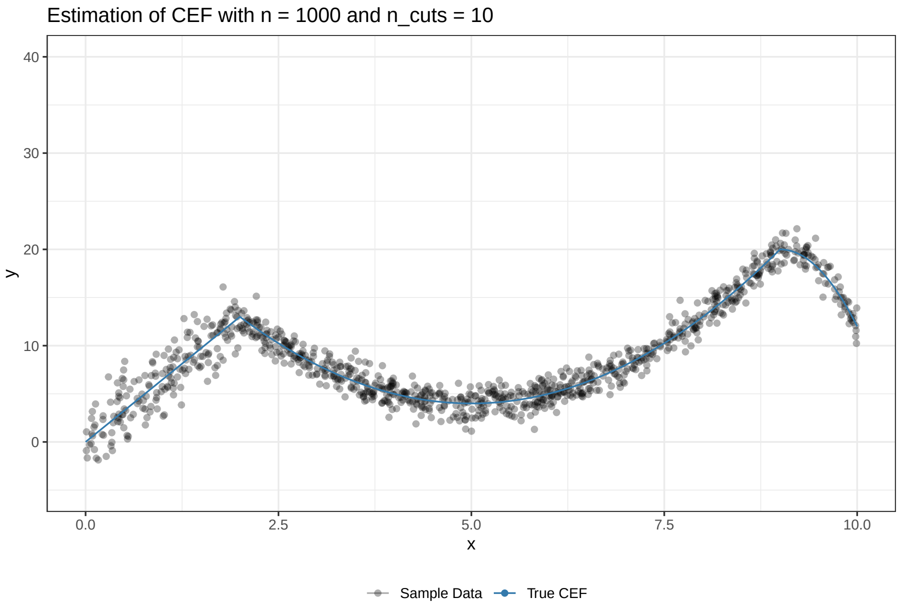

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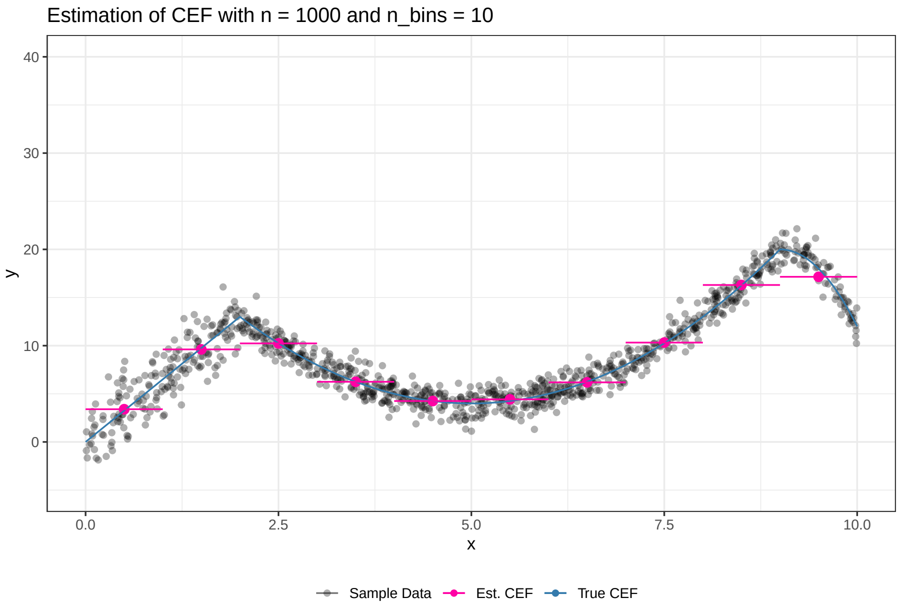

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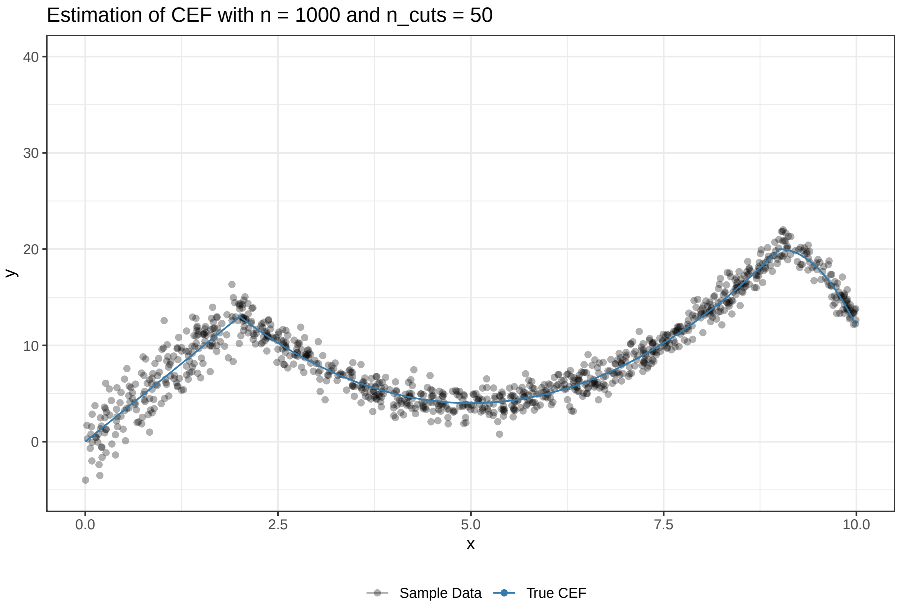

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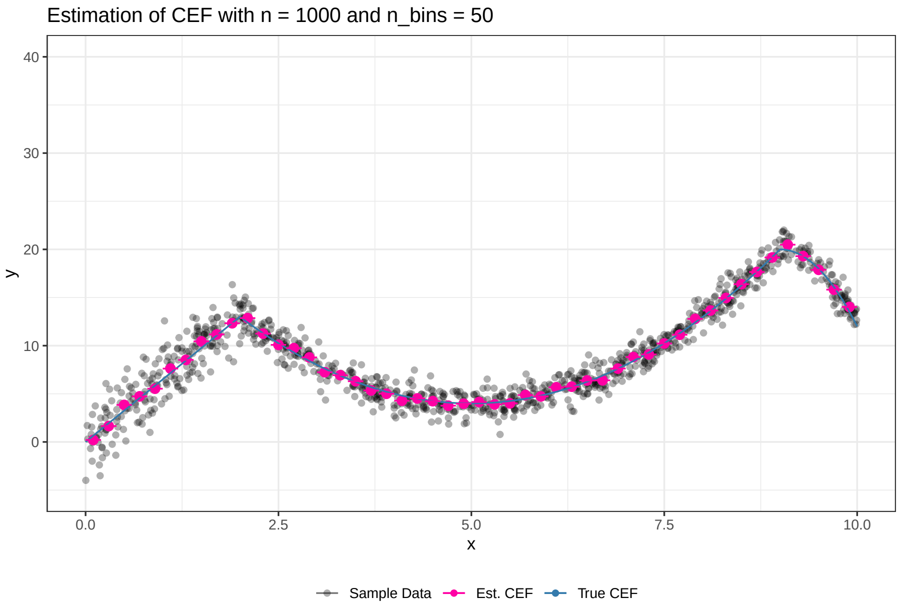

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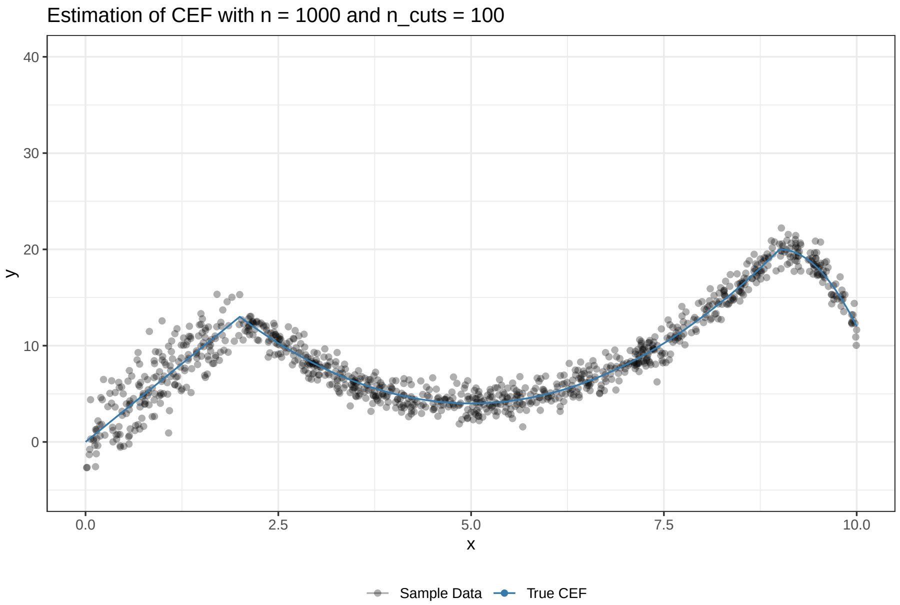

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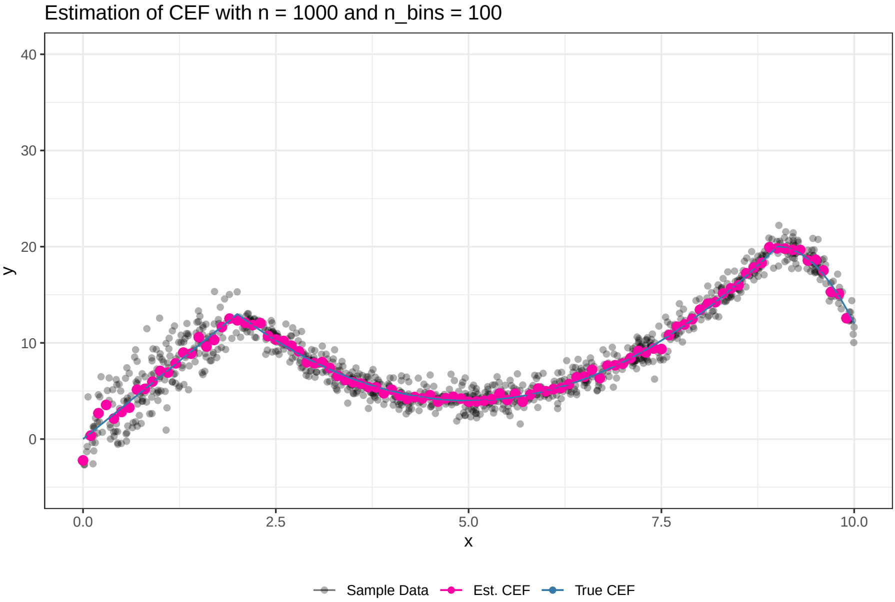

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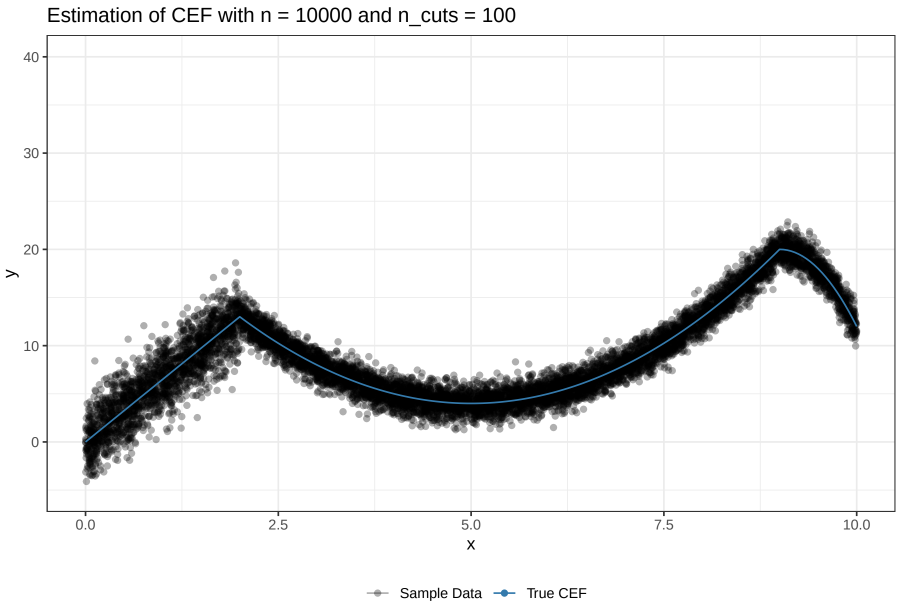

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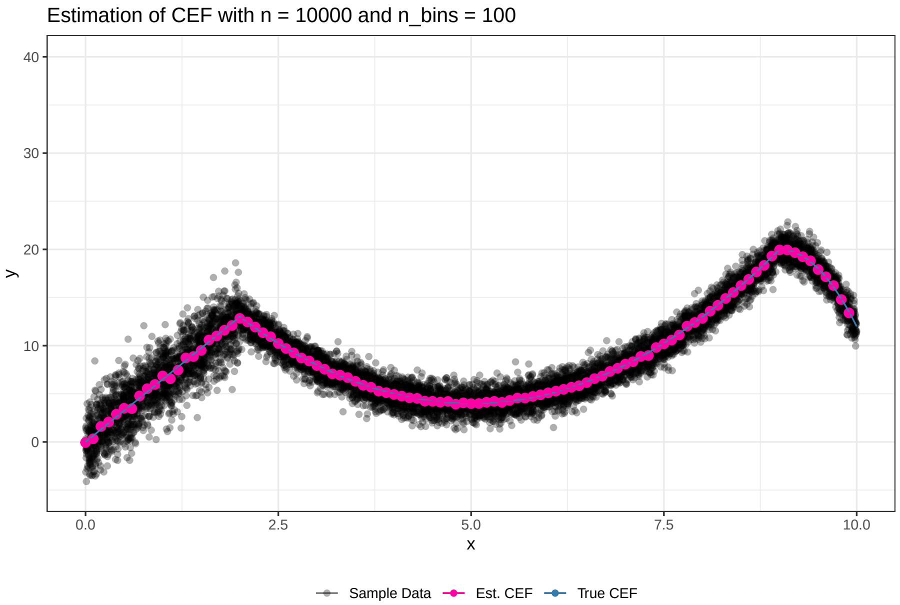

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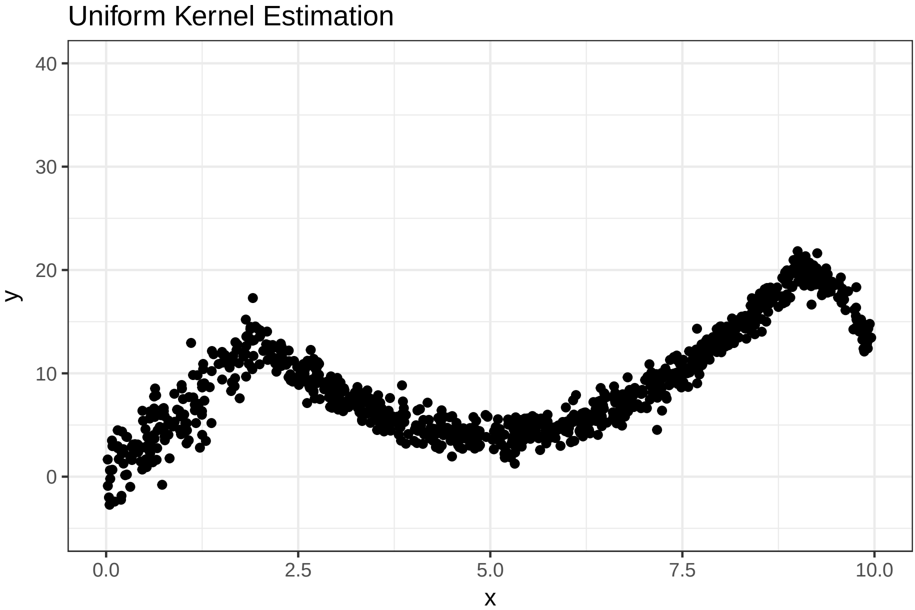
- We will call this approach uniform kernel regression for a reason that will become clear shortly.

Example of Kernel CEF Estimation



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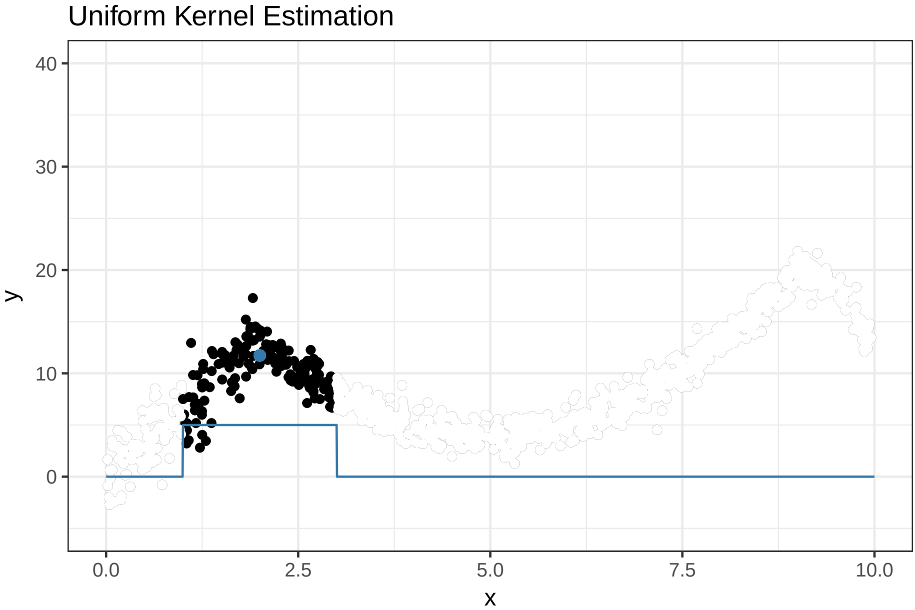

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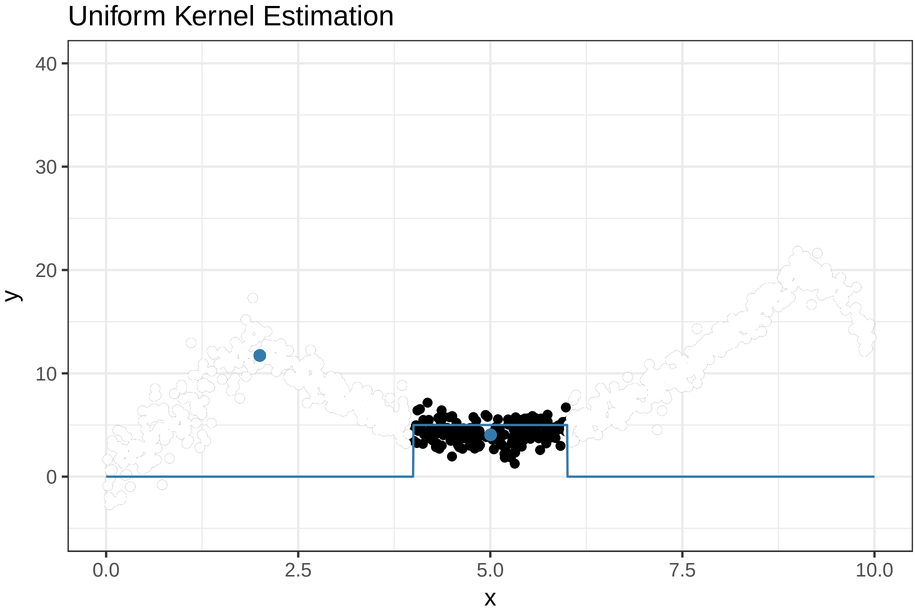

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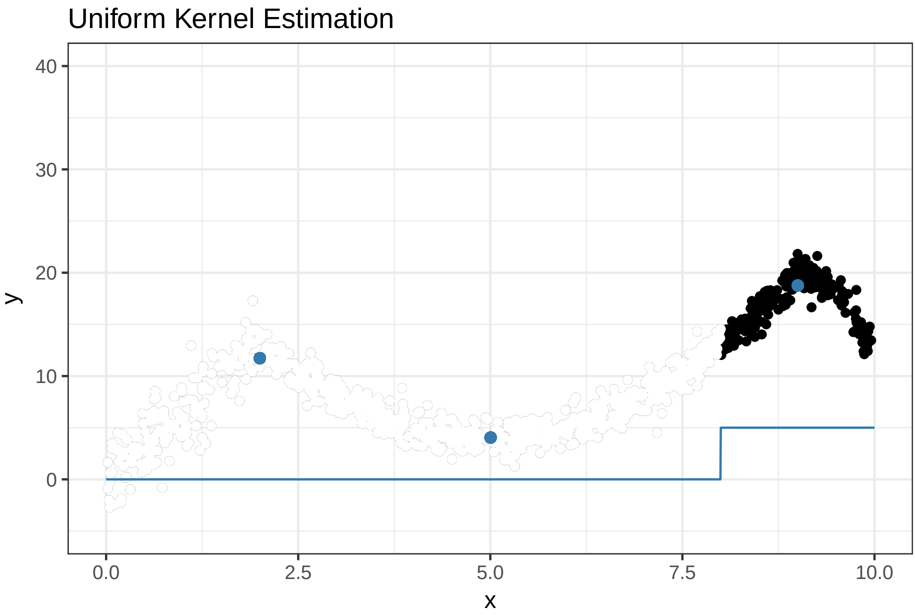

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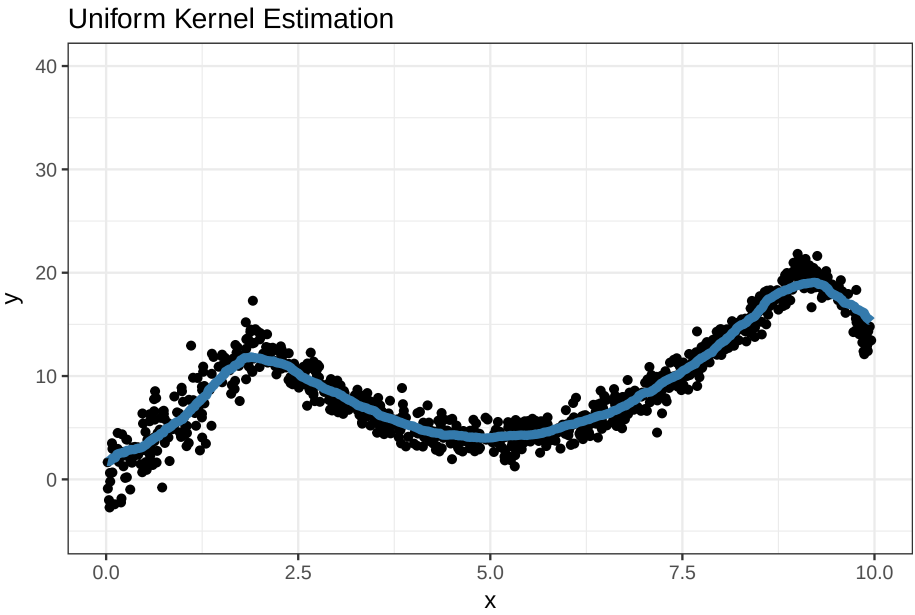

Uniform Kernel Regression: Simple Local Averages

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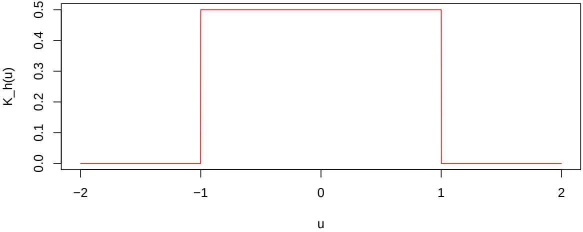

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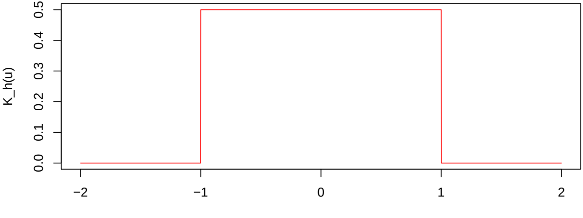
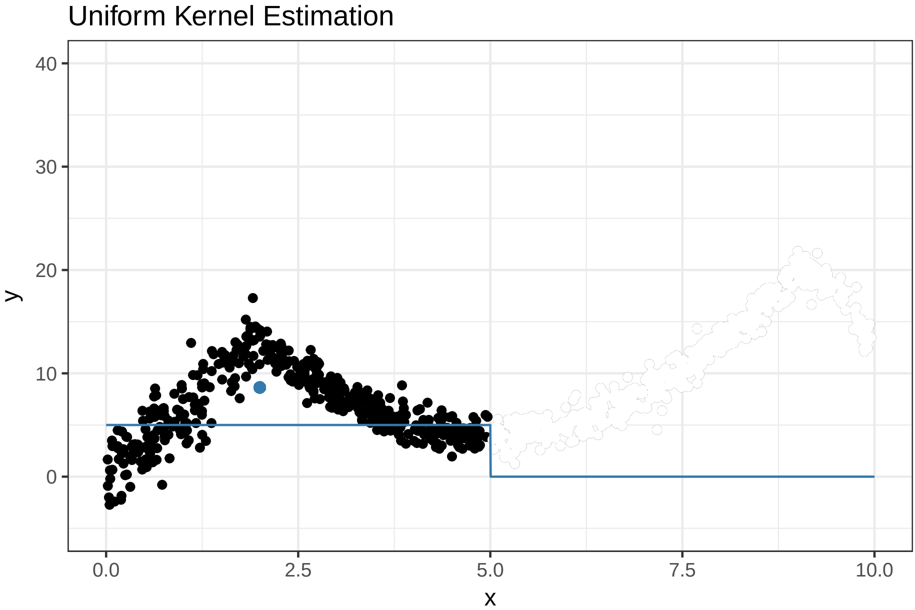


- This gives the uniform kernel regression:

$$\widehat{E}[Y|X = x_0] = \frac{\sum_{i=1}^{N} K_h((X_i - x_0)/h) Y_i}{\sum_{i=1}^{N} K_h((X_i - x_0)/h)} \text{ where } K_h(u) = \frac{1}{2}\,\mathbf{1}_{\{|u|\le 1\}}$$

Changing the Bandwidth



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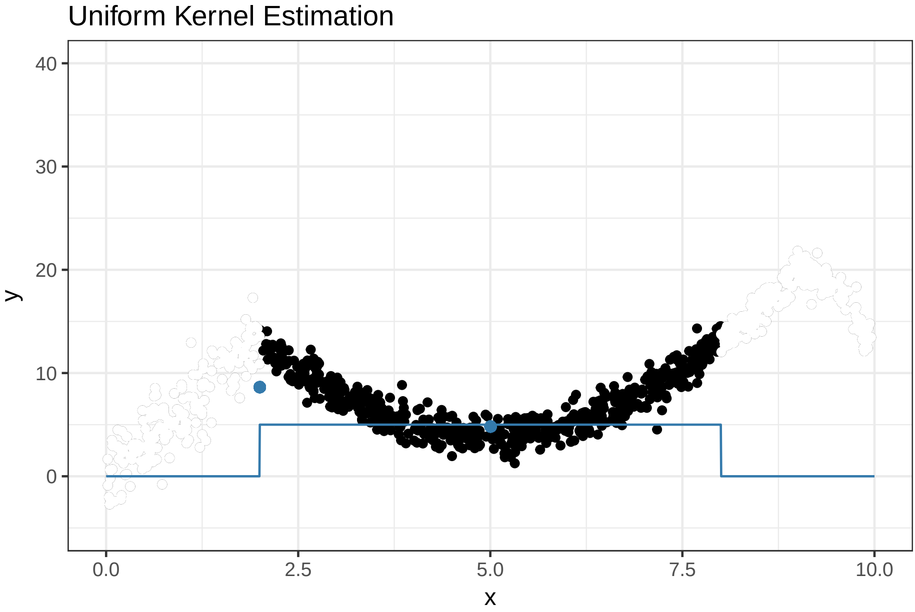

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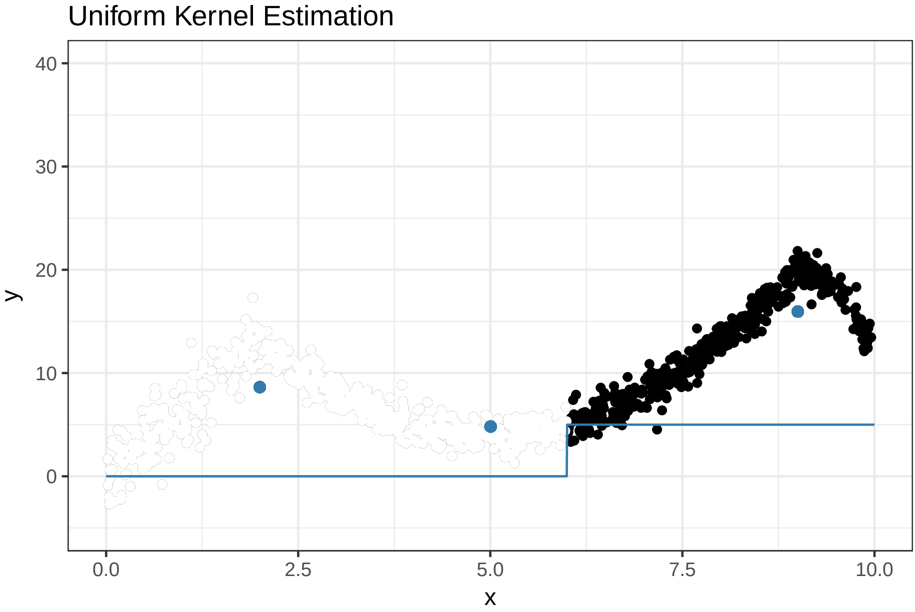

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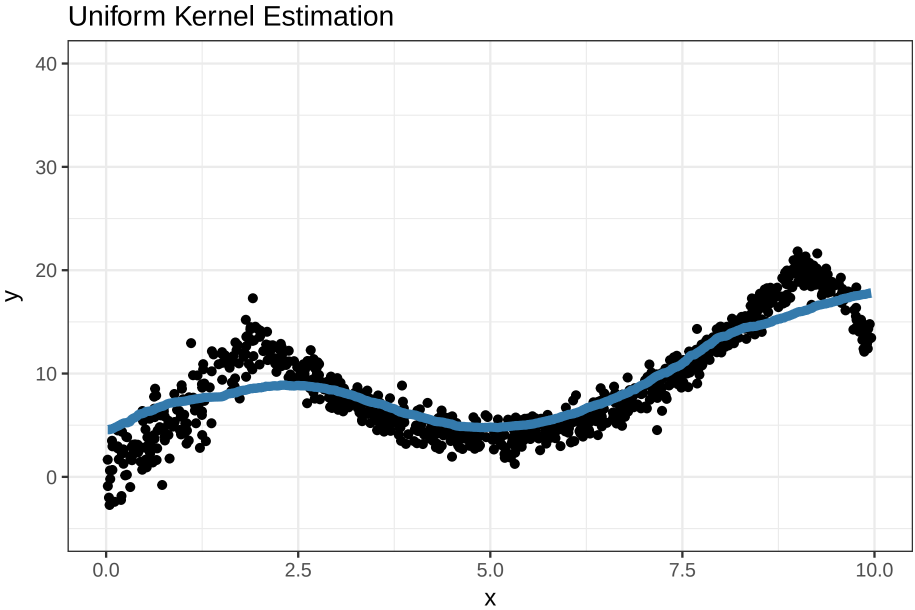

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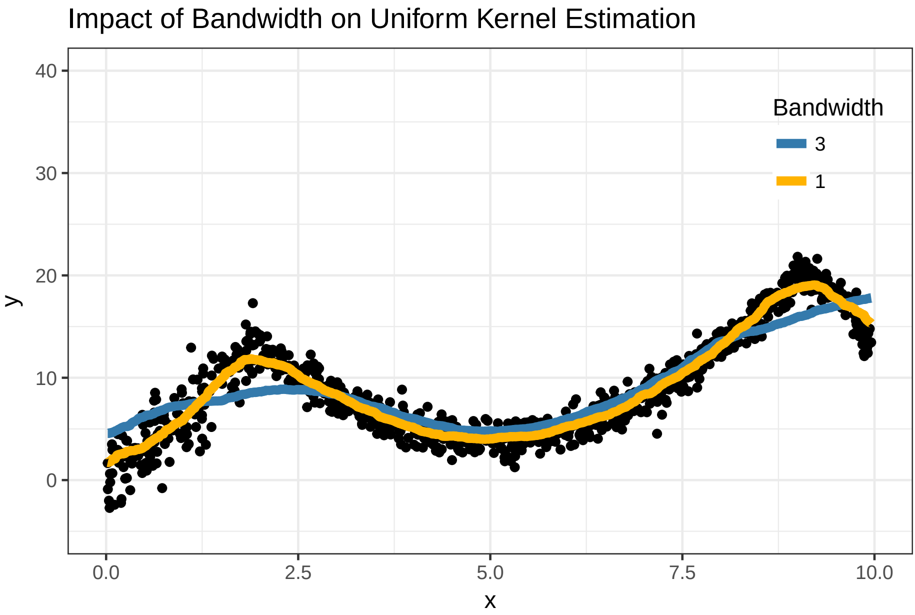

Uniform Kernel Regression: Properties

Theorem (Consistency of the Uniform Kernel Density Estimator)

For iid continuous random variables $X_1, X_2, \ldots X_n$, $\forall x \in \mathbb{R}$,

- *if the kernel is uniform, and*
- *if $h \to 0$ and*
- *$nh \to \infty$ as*
- *$n \to \infty$, then*

$$\hat{f}_K(x) \xrightarrow{p} f(x).$$

Aronow and Miller Theorem 3.3.8. Proof by weak law of large numbers and the plug-in principle.

The More General Form of the Estimator

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Let $X_1, X_2, \ldots X_n$ be iid continuous random variables with common PDF f.

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Let $K : \mathbb{R} \to \mathbb{R}$ be a function which is symmetric about the y-axis satisfying $\int_{-\infty}^{\infty} K(x)dx = 1$, and let $K_h(x) = \frac{1}{h}K(\frac{x}{h}) \forall x \in \mathbb{R}$ and $h > 0$.

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Then a kernel density estimator of $f(x)$ is

$$\hat{f}_K(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i), \forall x \in \mathbb{R}$$

The function K is called the kernel and the scaling parameter h is called the bandwidth.
(Aronow and Miller Definition 3.3.7)

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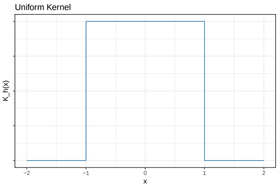
- Each observation within the interval is given equal weight, each observation outside the interval is given 0 weight



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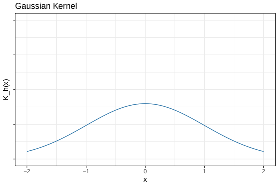
- Gaussian Kernel
- Distance weighted by how far from x_0 following the normal density

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$



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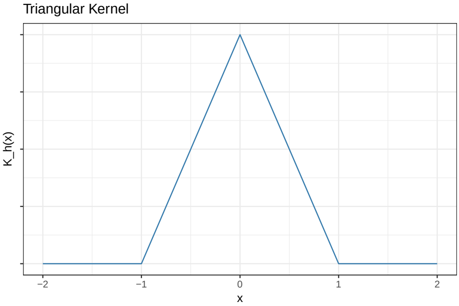
- Triangular
- Distance weighted by how far from x_0 using linear distance



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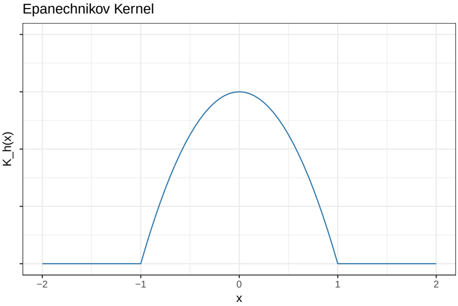
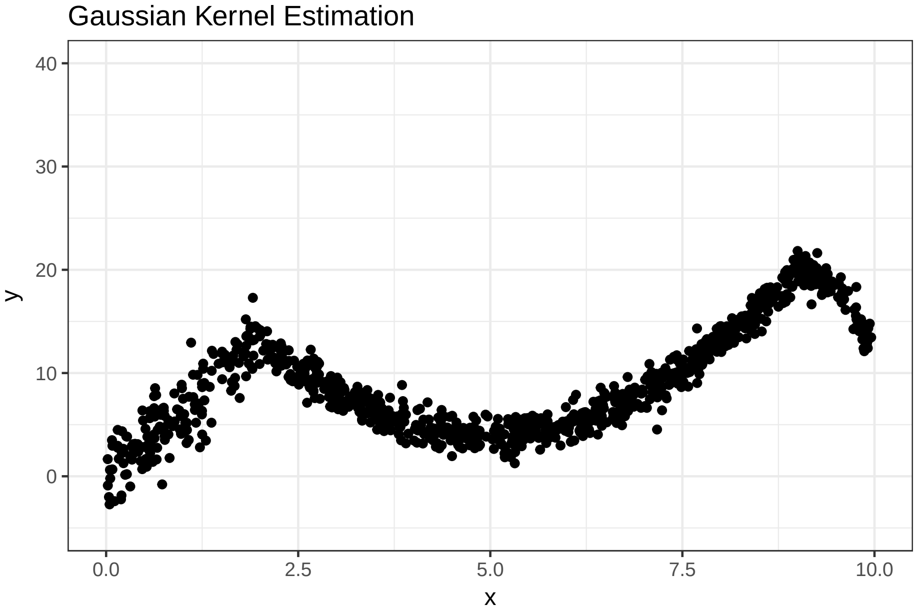
- Distance weighted by how far from x_0 using a parabolic function



Example of Gaussian Kernel



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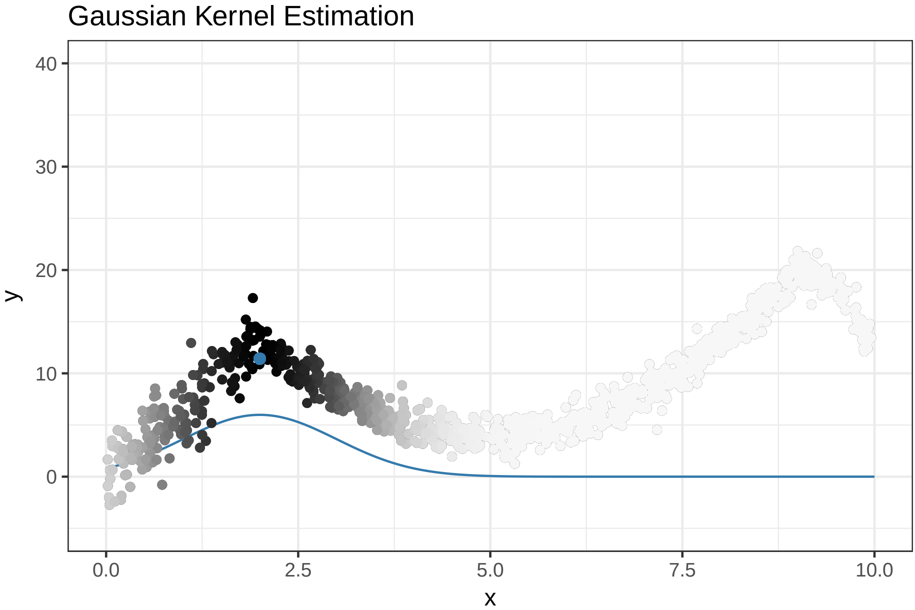

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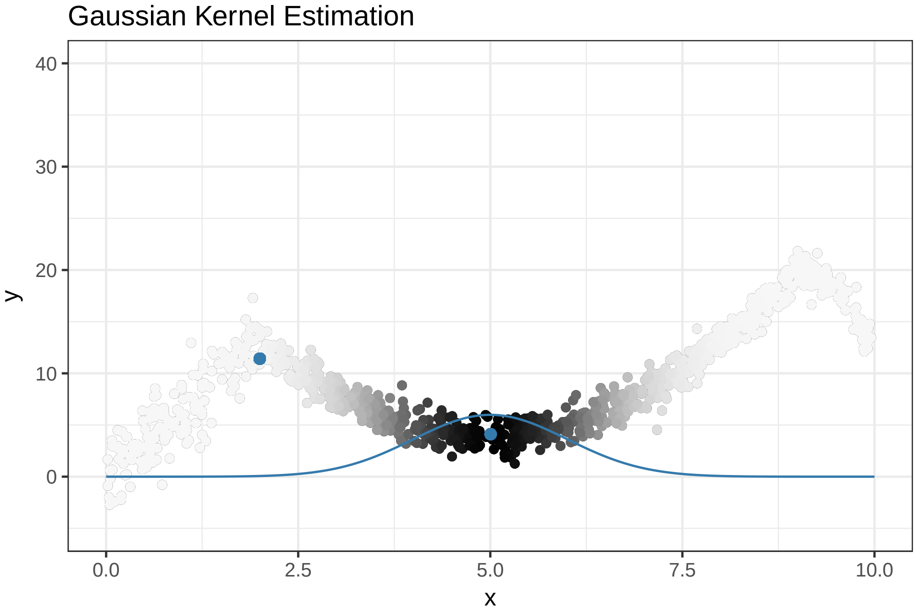

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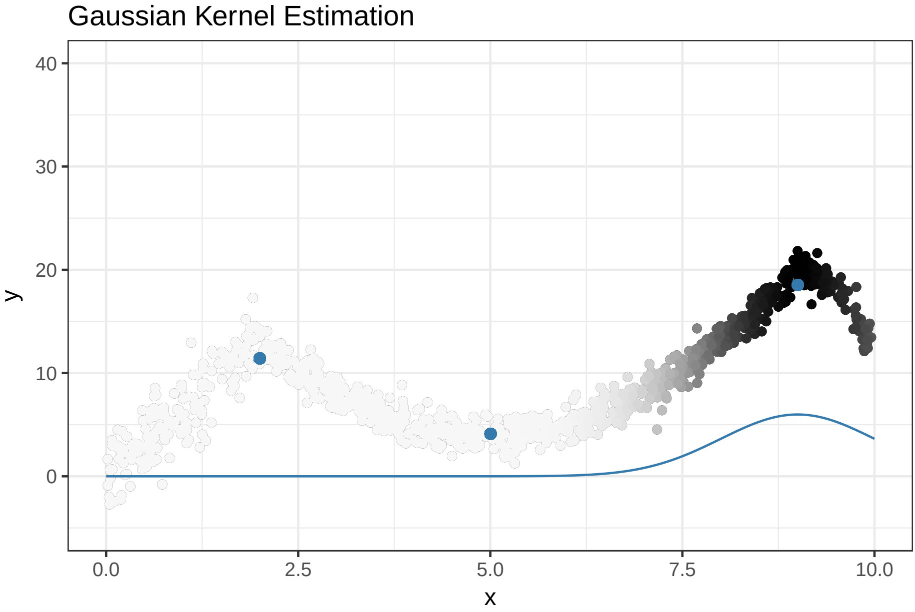

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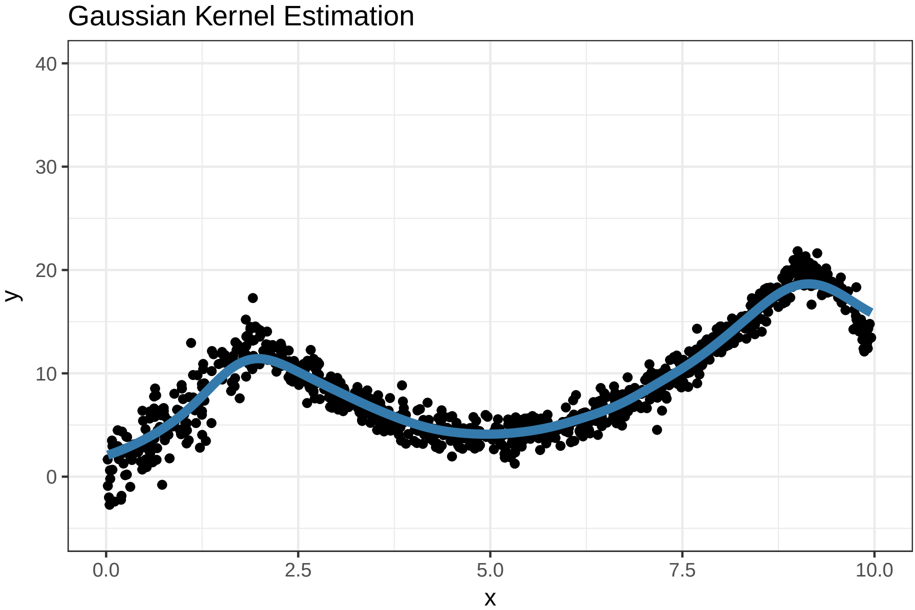

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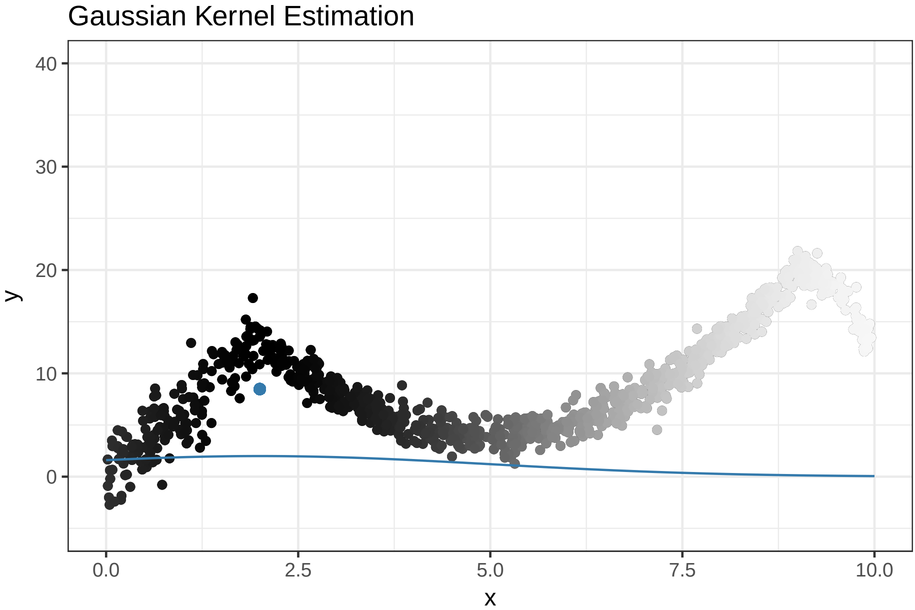

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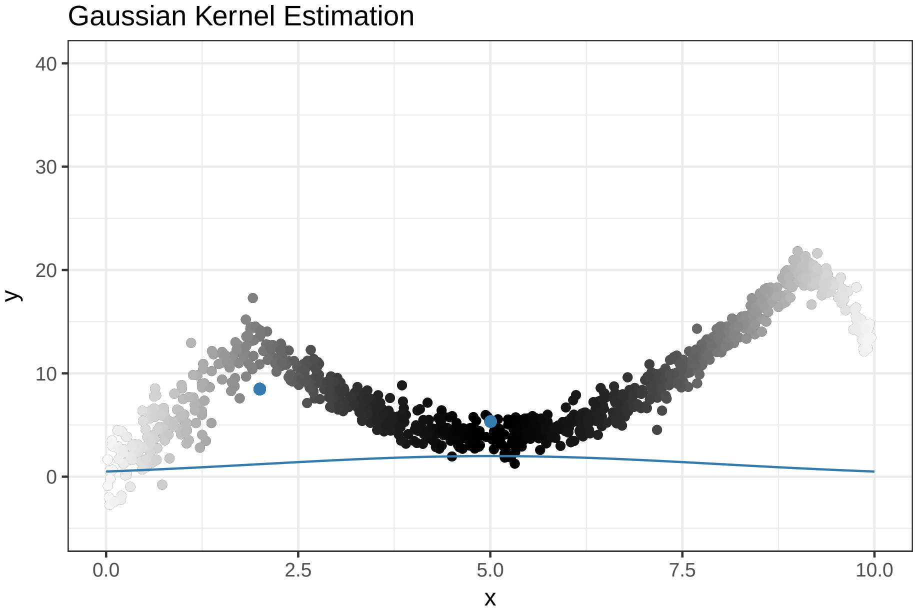

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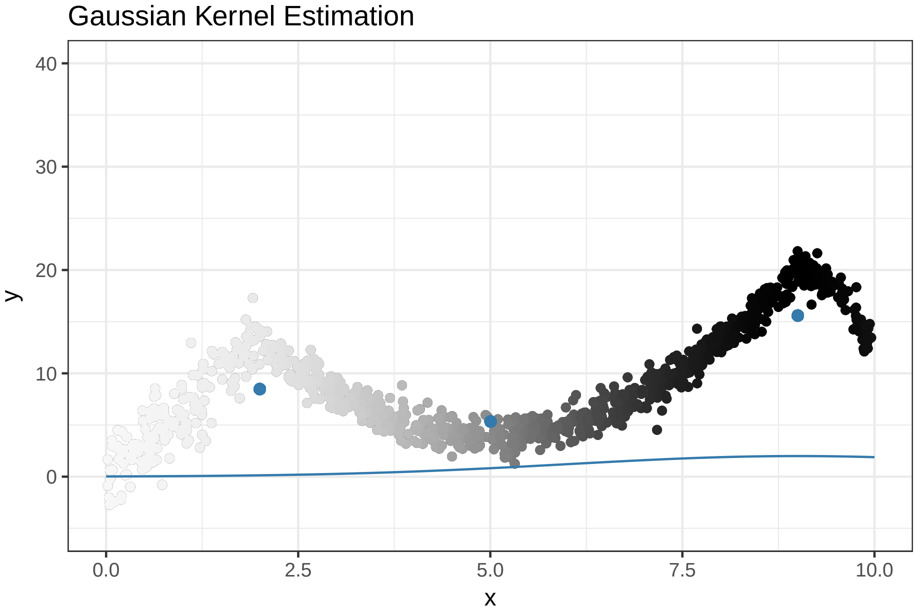

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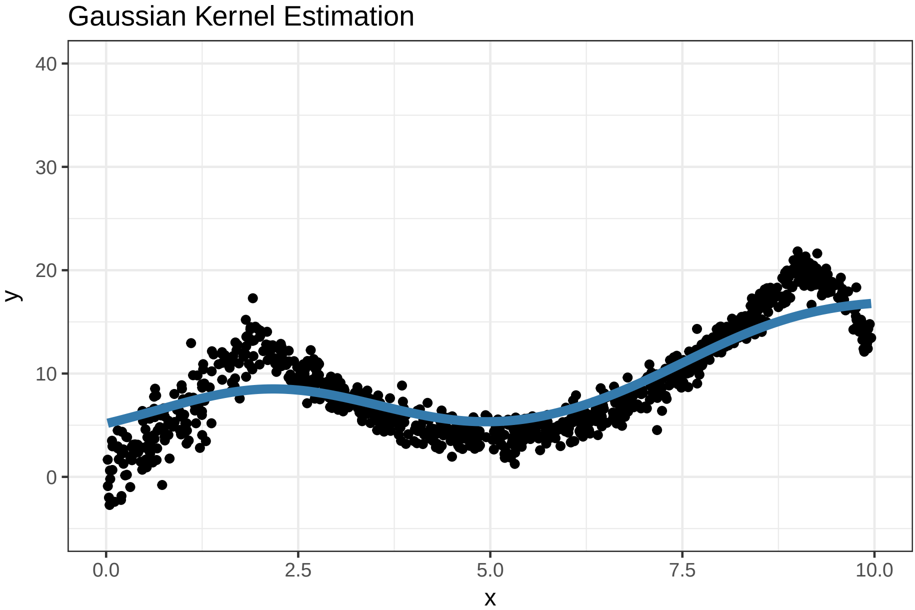

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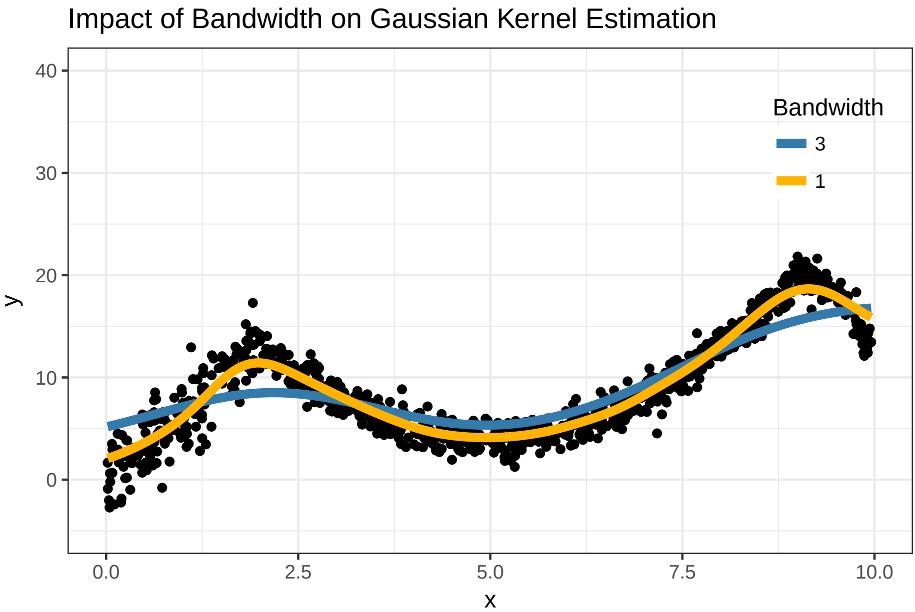

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  - A very flexible estimator allows the shape of the function to vary (e.g. a kernel regression with a small bandwidth)
  - A very inflexible estimator restricts the shape of the function to a particular form
    (e.g. a kernel regression with a very wide bandwidth)

Best Linear Predictor

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- What function minimizes MSE, among this class of functional forms?

Best Linear Predictor

Theorem

For a random variable X and Y, if $V[X] > 0$, then the best linear predictor (BLP) of Y given X is $g(X) = \alpha + \beta X$ where,

$$\alpha = E[Y] - \frac{Cov[X,Y]}{V[X]} E[X]$$

$$\beta = \frac{Cov(X,Y)}{V(X)}$$

Corollary

- *The BLP is the best linear predictor of the CEF. I.e. setting $a = \alpha$ and $b = \beta$ minimizes*

$$E\left[(E[Y \mid X] - (a + bX))^2\right]$$

- *If the CEF is linear, the CEF is the BLP*

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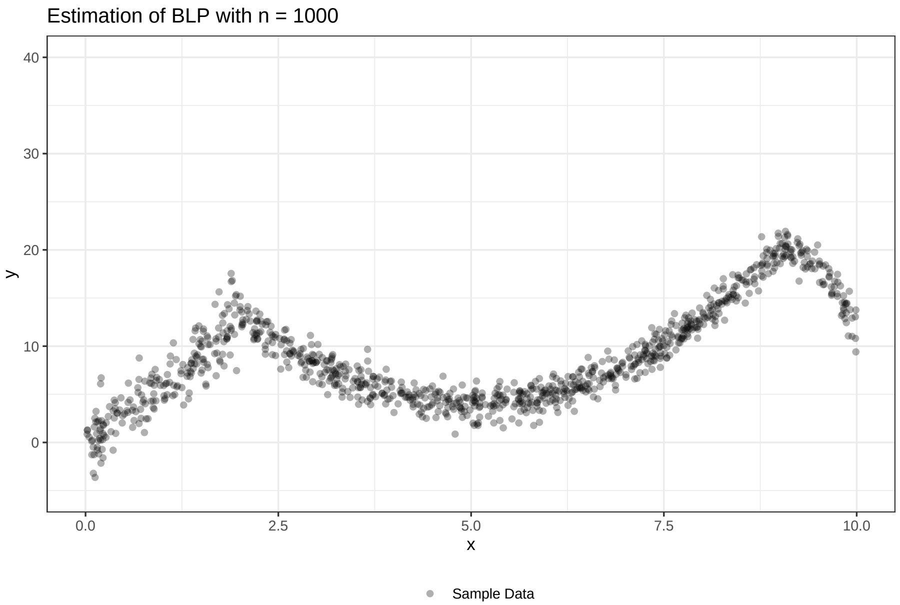

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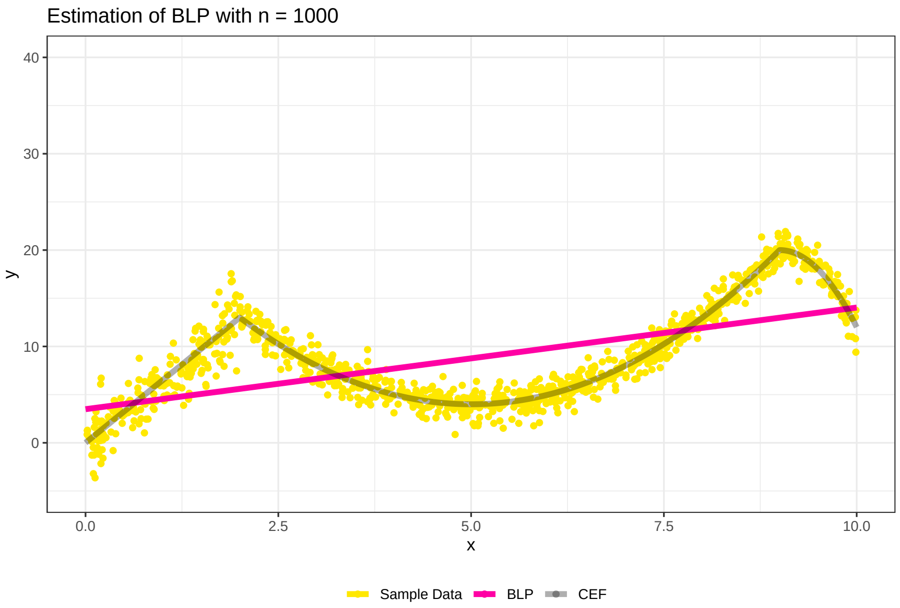

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  - Linear projection is the best predictor among linear functions.
- The nice thing about the linear projection is that it exists and is well-defined even if the CEF is non-linear.

BLP Approximations Depend on the Marginal Distribution of X

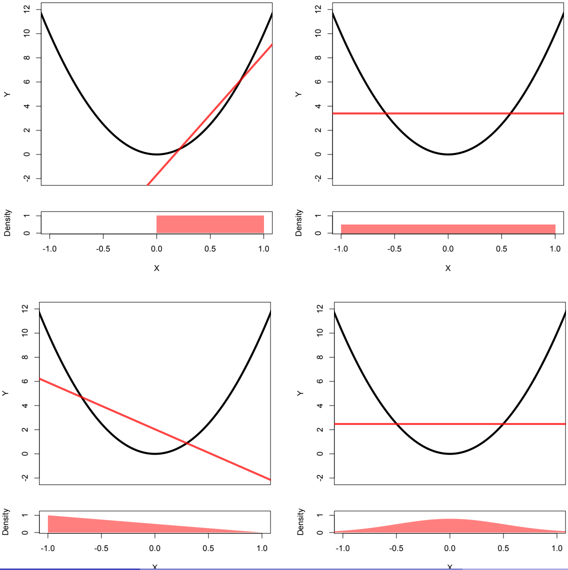

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Figure: 'If I fits, I sits'

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The BLP is always a line regardless of the data.

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- This is called the ordinary least squares (OLS) estimator.

Plug-in Estimation of the BLP

Noting we can rewrite $\frac{Cov[X,Y]}{V[X]} = \frac{E[XY]-E[X]E[Y]}{E[X^2]-E[X]^2}$, and using the plug-in principle, we can estimate the parameters of the BLP as:

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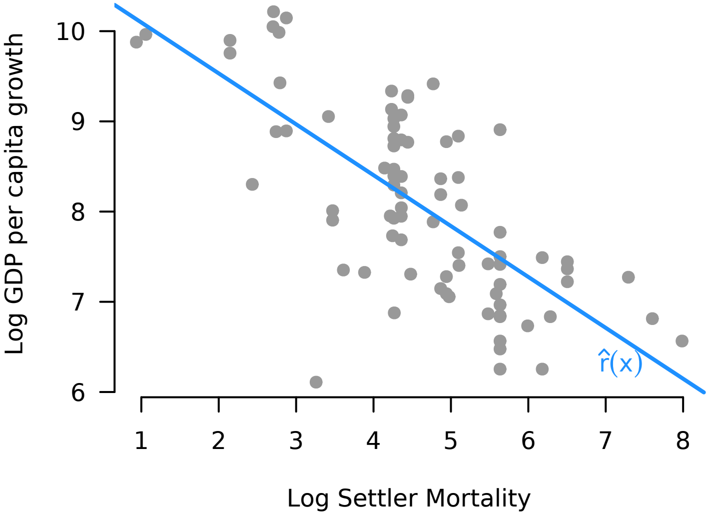
This corresponds to the linear projection which minimizes the sum of squared errors.

Fitted linear CEF/regression function



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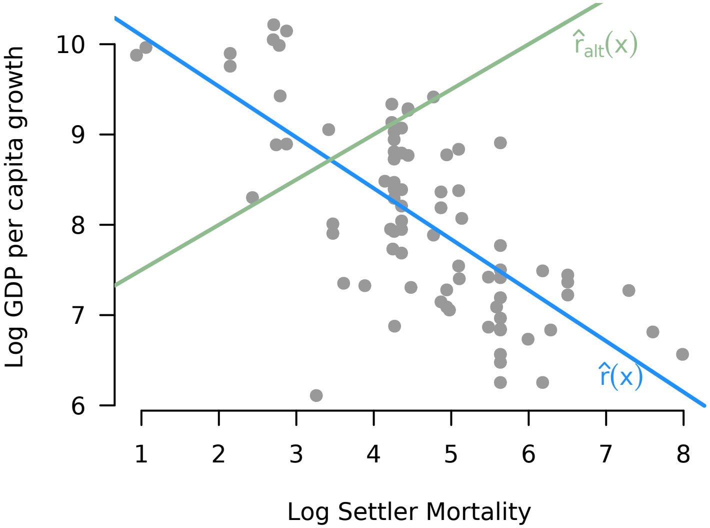

Fitted values and residuals

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A **fitted value** or **predicted value** is the estimated conditional mean of Y_i for a particular observation with independent variable X_i:

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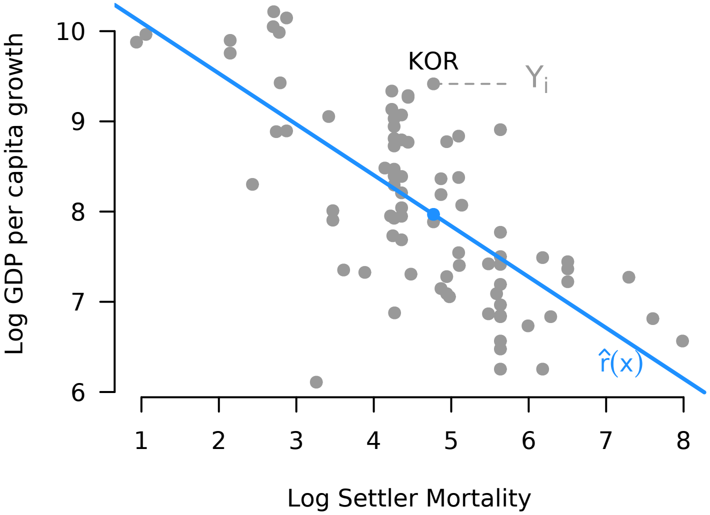
Definition (Residual)

The **residual** is the difference between the actual value of Y_i and the predicted value, $\widehat{Y}_i$:

$$\widehat{u}_i = Y_i - \widehat{Y}_i = Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 X_i$$

Fitted linear CEF/regression function



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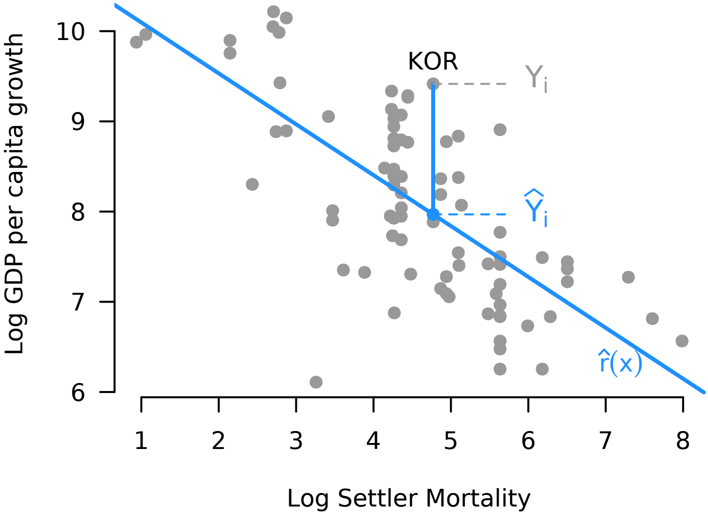

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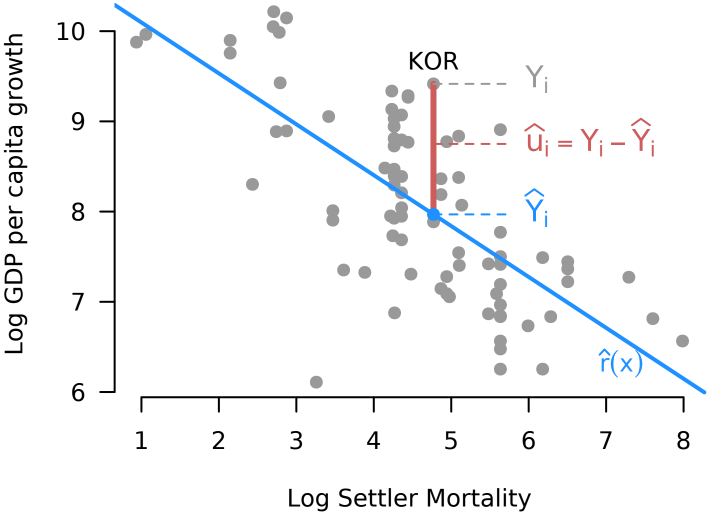
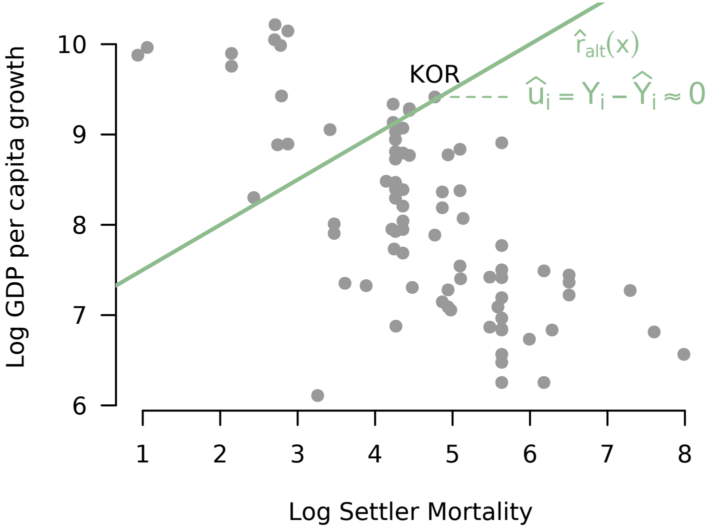


Why not this line?



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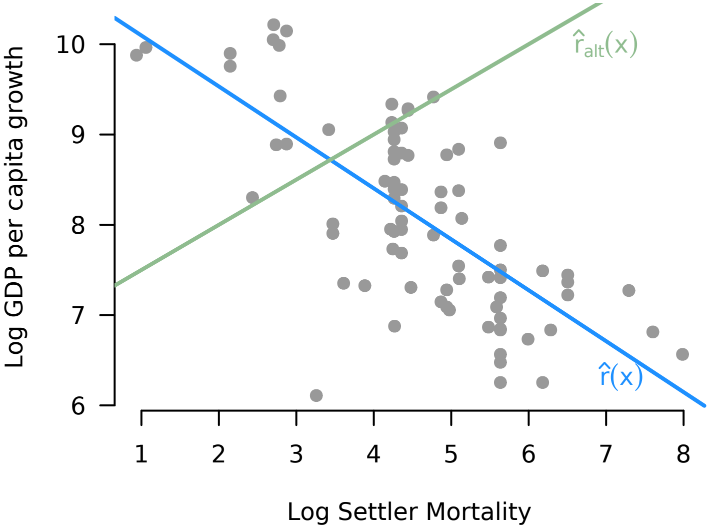
- Choose the line that minimizes the residuals

Which is better at minimizing residuals?



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- Perhaps the biggest reason is that it extends easily to the case where X is a vector of random variables.

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- This helps clear that it is an approximation to the CEF and that the units being described are different.

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- While the line is defined over all regions of the data we may be concerned about:
  - interpolation
  - extrapolation
  - predicting in ranges of X with sparse data

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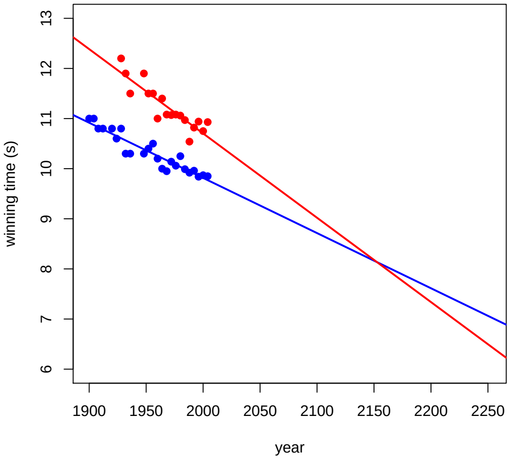
How do the authors make this conclusion?

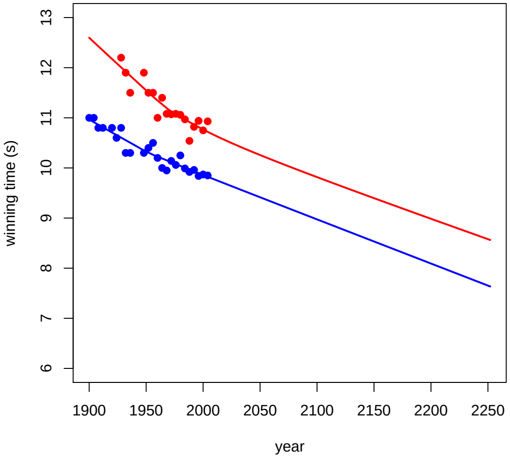
Using data from 1900 to 2004, they fit linear regression models of the winning 100 meter time on year for both men and women. They then use the estimates from these models to extrapolate 152 years into the future.

Tatem et al. Extrapolation



Tatem et al.'s predictions. Men's times are in blue, women's times are in red.

Alternate Models Fit Well, Yield Different Predictions



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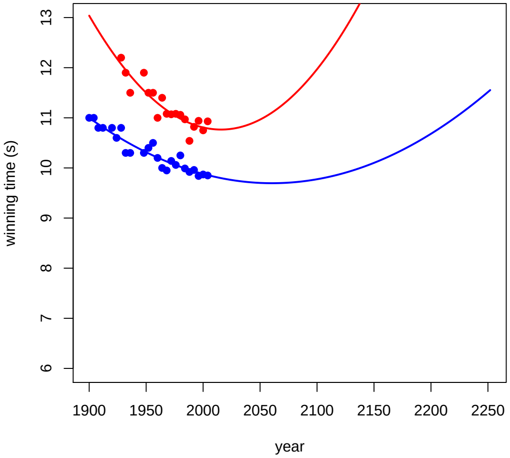

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- This problem gets much harder in high dimensions

A More Subtle Example

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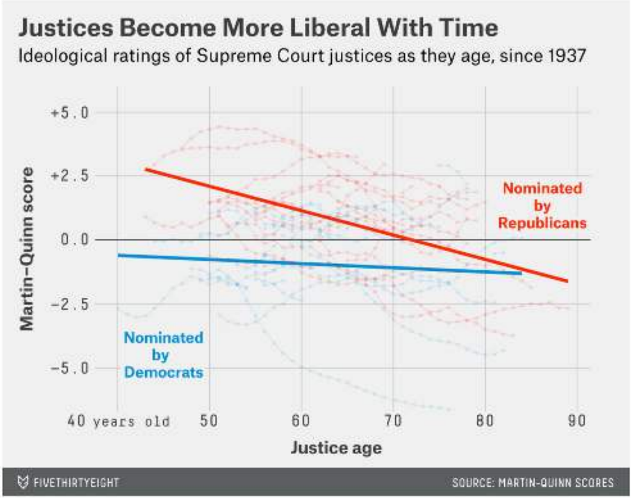


A More Subtle Example

Fun with Linearity

"The Siren's Song of Linearity"

Fun with Linearity

Psychonomic Bulletin & Review
2007, 14 (2), 288-294

Iterated learning: Intergenerational knowledge transmission reveals inductive biases

MICHAEL L. KALISH
University of Louisiana, Lafayette, Louisiana

THOMAS L. GRIFFITHS
University of California, Berkeley, California

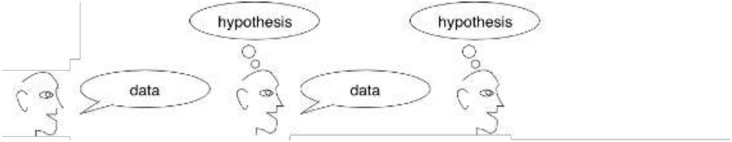
AND

STEPHAN LEWANDOWSKY
University of Western Australia, Perth, Australia

Cultural transmission of information plays a central role in shaping human knowledge. Some of the most complex knowledge that people acquire, such as languages or cultural norms, can only be learned from other people, who themselves learned from previous generations. The prevalence of this process of "iterated learning" as a mode of cultural transmission raises the question of how it affects the information being transmitted. Analyses of iterated learning utilizing the assumption that the learners are Bayesian agents predict that this process should converge to an equilibrium that reflects the inductive biases of the learners. An experiment in iterated function learning with human participants confirmed this prediction, providing insight into the consequences of intergenerational knowledge transmission and a method for discovering the inductive biases that guide human inferences.

Images on following slides courtesy of Tom Griffiths

Fun with Linearity



The Design

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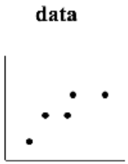

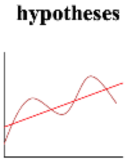

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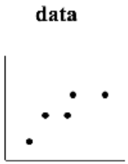

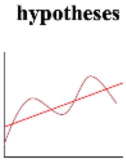

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The Design

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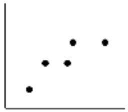
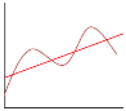
The Design

data



hypotheses



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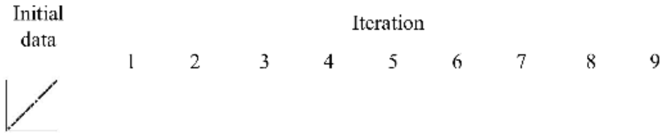
- Predictions are data for the next learner

Results



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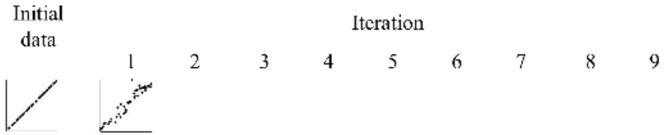

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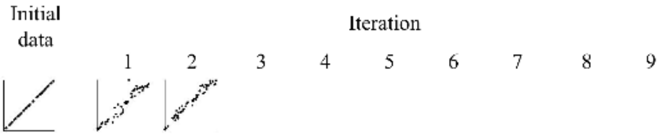

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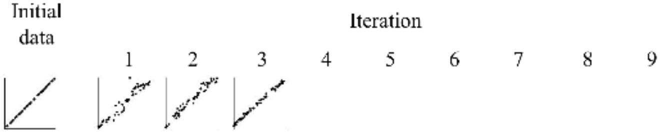

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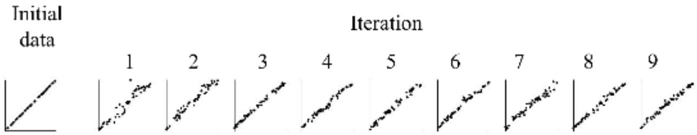

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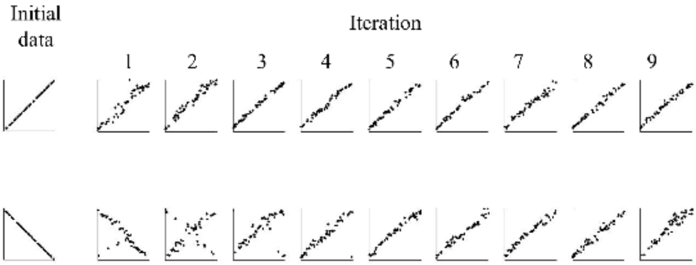

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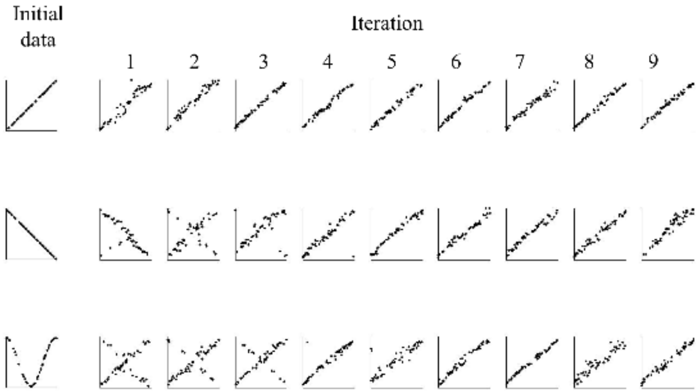

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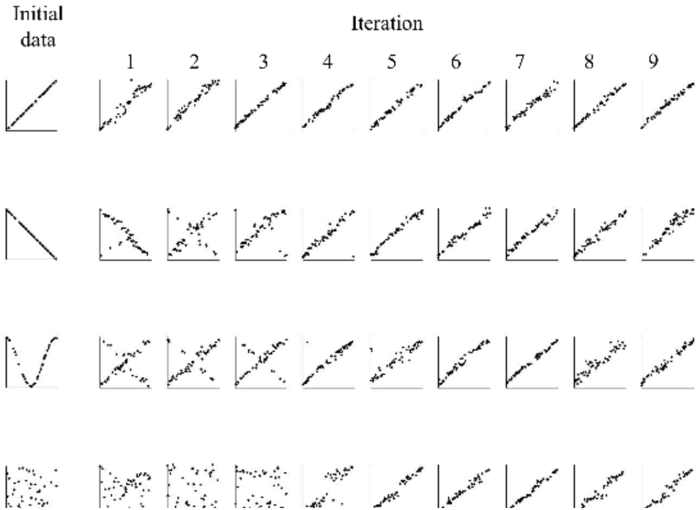

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- Derivations are on the next few slides if you are interested.

Step 1: Take Partial Derivatives

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Solving for the Intercept

$$\frac{\partial S(b_0, b_1)}{\partial b_0} = -2\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)$$

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b_0 &= \overline{Y} - b_1 \overline{X}
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A Helpful Lemma on Deviations from Means

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&= \left(\sum_{i=1}^{n} X_i\right) - \sum_{i=1}^{n} X_i \\
&= 0
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$$b_1 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n} (X_i - \overline{X})^2}$$

The OLS estimator

- Now we're done! Here are the **OLS estimators**:

$$\widehat{\beta}_0 = \overline{Y} - \widehat{\beta}_1 \overline{X}$$

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

Intuition of the OLS estimator

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- The slope for the regression line can be written as the following:

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{\text{Sample Covariance between } X \text{ and } Y}{\text{Sample Variance of } X}$$

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- The higher the covariance between X and Y, the higher the slope will be.
- Negative covariances $\rightarrow$ negative slopes;
  positive covariances $\rightarrow$ positive slopes
- If X_i doesn't vary, the denominator is undefined.
- If Y_i doesn't vary, you get a flat line.

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3. The residuals will be uncorrelated with the fitted values:

$$\sum_{i=1}^{n} \widehat{Y}_i\widehat{u}_i = 0 \implies \widehat{\text{Cov}}(\widehat{Y}_i, \widehat{u}_i) = 0$$

OLS slope as a weighted sum of the outcomes

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- This is important for two reasons. First, it'll make derivations later much easier. And second, it shows that is just the sum of a random variable. Therefore it is also a random variable.

Lemma 2: OLS as a Weighted Sum of Outcomes

Lemma (OLS as Weighted Sum of Outcomes)

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\widehat{\beta}_1 &= \frac{\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^n (X_i - \overline{X})^2} \\
&= \frac{\sum_{i=1}^n (X_i - \overline{X})Y_i}{\sum_{i=1}^n (X_i - \overline{X})^2} - \frac{\sum_{i=1}^n (X_i - \overline{X})\overline{Y}}{\sum_{i=1}^n (X_i - \overline{X})^2}
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Where the weights, W_i are:

$$
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OLS

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Sample 1: $\{(Y_1, X_1), \dots, (Y_n, X_n)\} \longrightarrow \boxed{\text{OLS}} \longrightarrow (\widehat{\beta}_0, \widehat{\beta}_1)_1$

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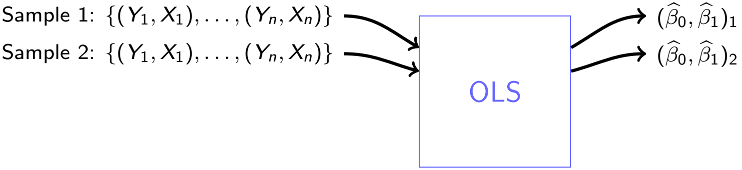

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Sample 1: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$ $\rightarrow$ OLS $\rightarrow$ $(\widehat{\beta}_0, \widehat{\beta}_1)_1$

Sample 2: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$ $\rightarrow$ OLS $\rightarrow$ $(\widehat{\beta}_0, \widehat{\beta}_1)_2$

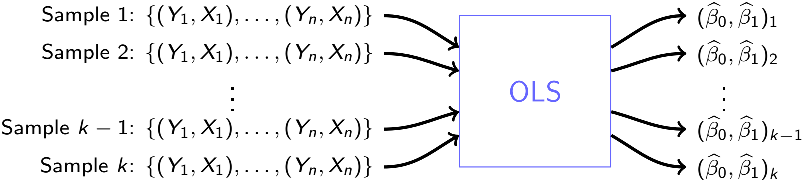
$\vdots$

Sample $k-1$: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$ $\rightarrow$ OLS $\rightarrow$ $(\widehat{\beta}_0, \widehat{\beta}_1)_{k-1}$

Sample k: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$ $\rightarrow$ OLS $\rightarrow$ $(\widehat{\beta}_0, \widehat{\beta}_1)_k$

Sampling distribution of the OLS estimator

- Remember: OLS is an estimator—it's a machine that we plug samples into and we get out estimates.



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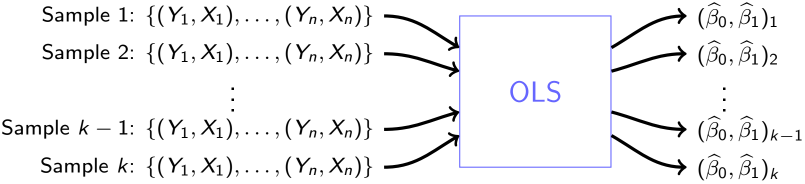

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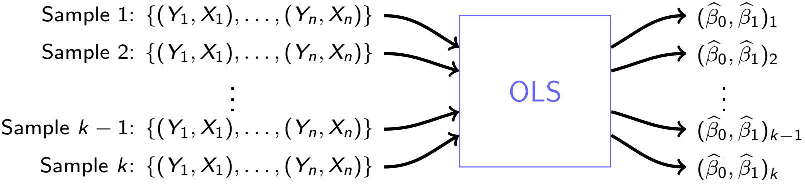

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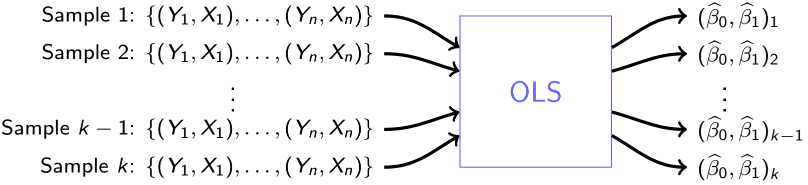

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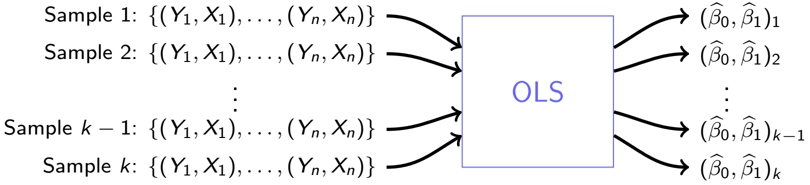

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    - See how the line varies from sample to sample

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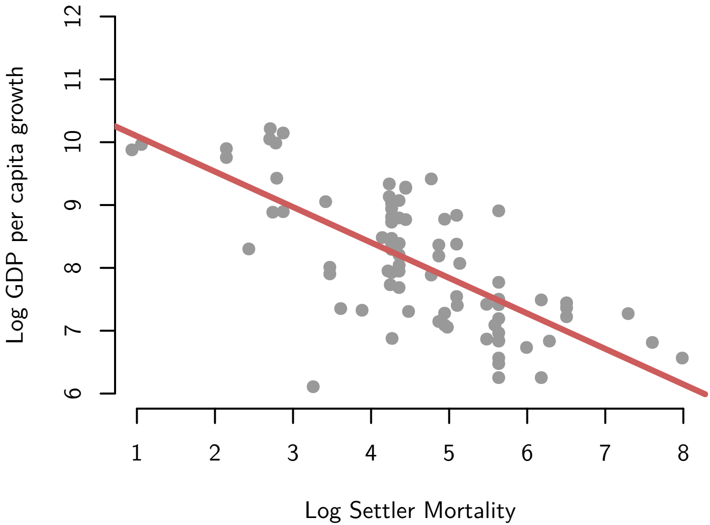
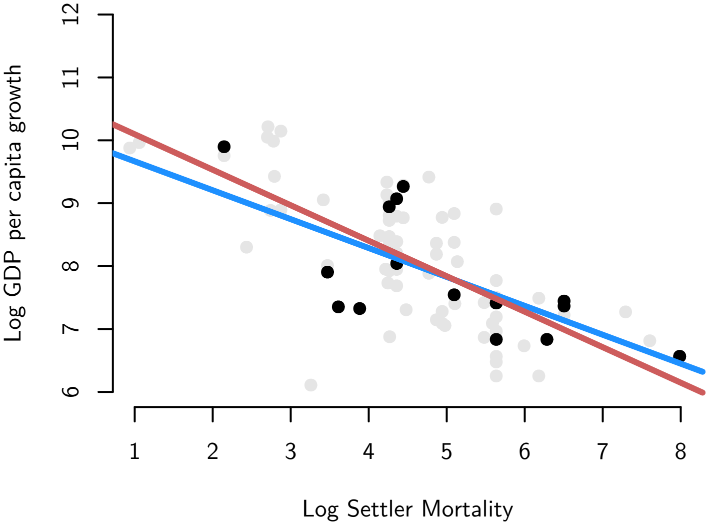
Population Regression



Randomly sample from AJR



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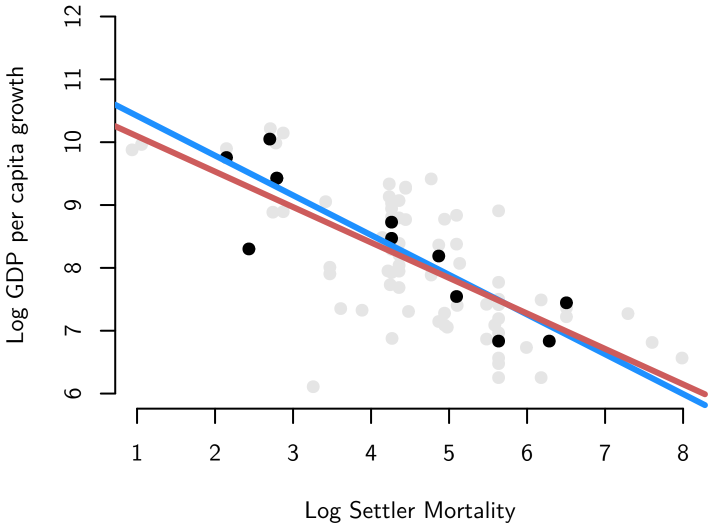

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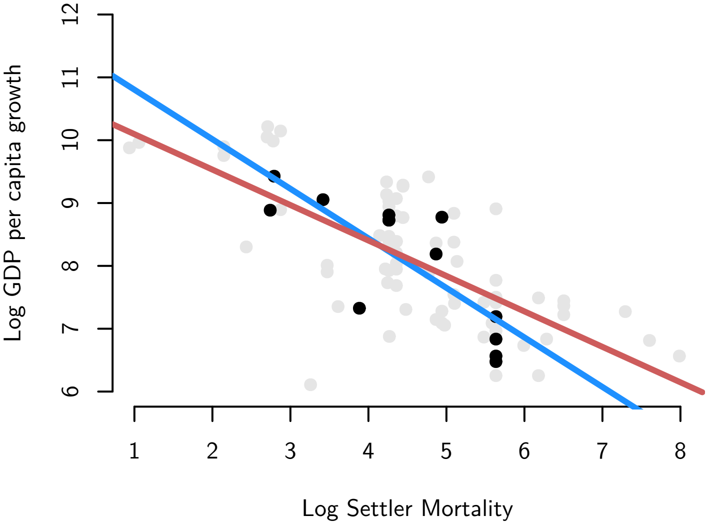

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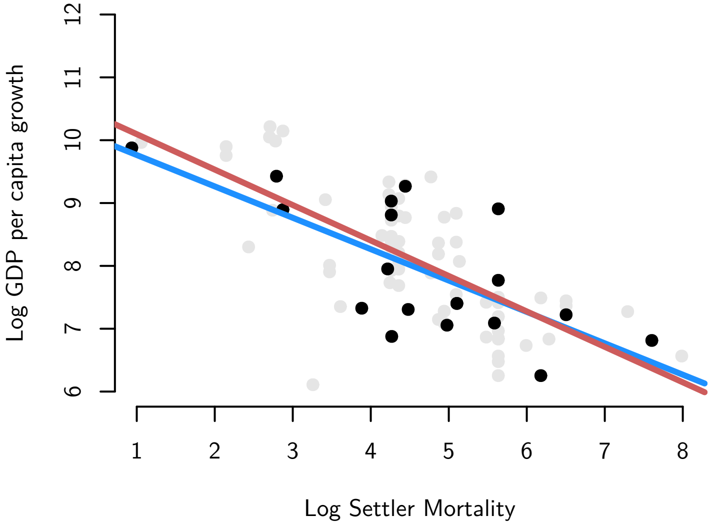

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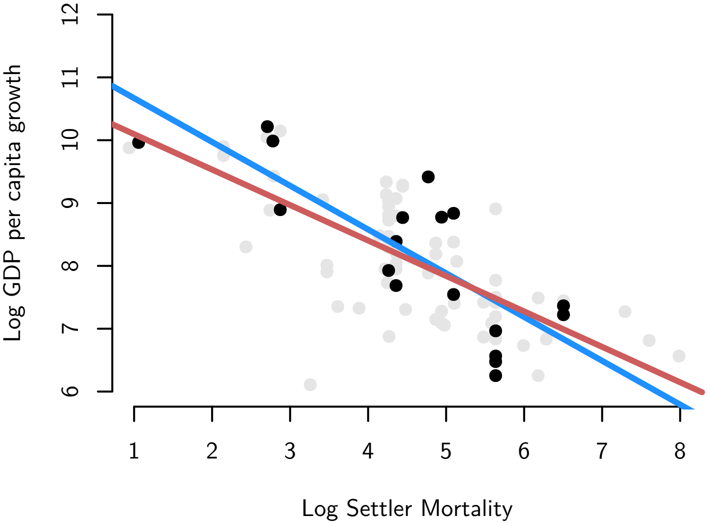

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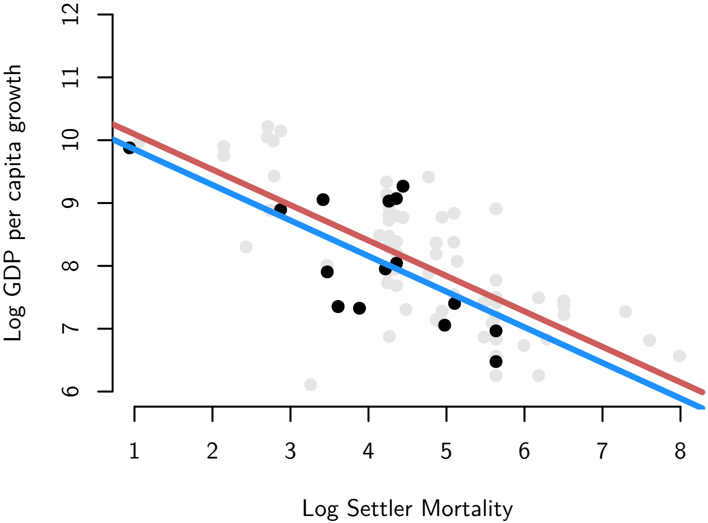

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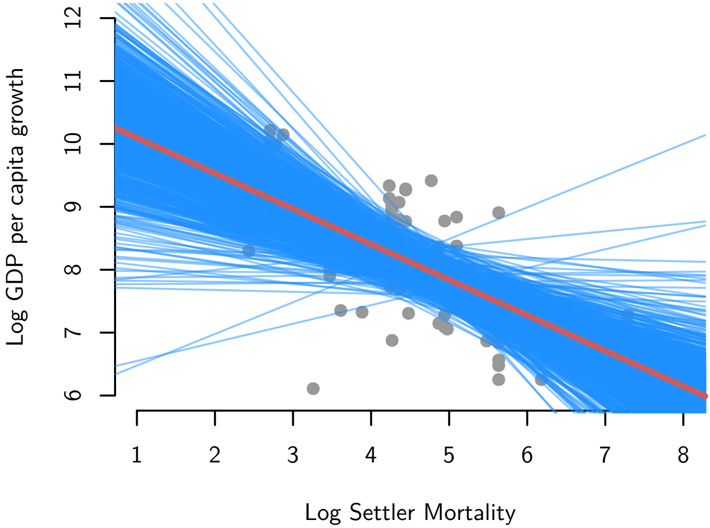

Sampling distribution of OLS

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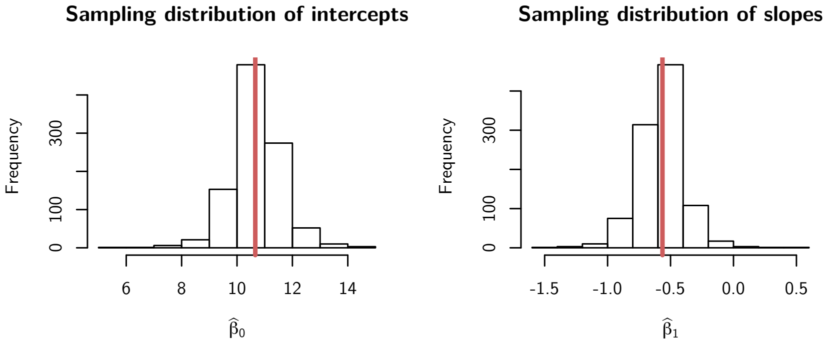
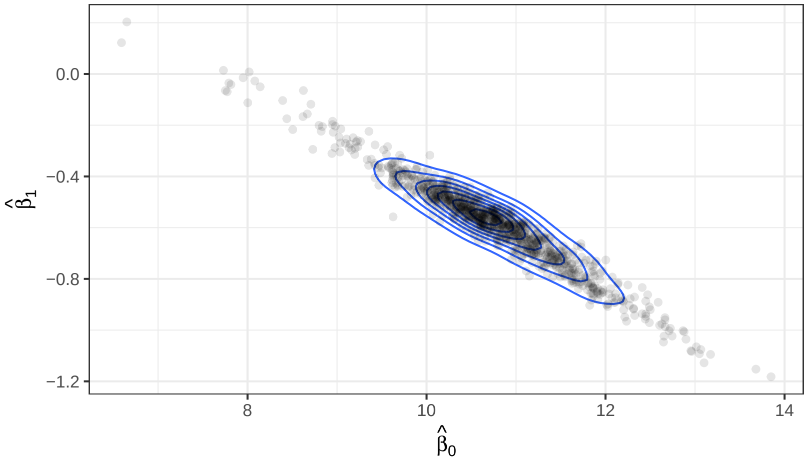
- You can see that the estimated slopes and intercepts vary from sample to sample, but that the "average" of the lines looks about right.



The Sampling Distribution is a Joint Distribution!

While both the intercept and the slope vary, they vary together.



1. What is Regression?
2. Best Linear Predictor
3. Fun With Linearity
4. Estimation of the BLP
5. Additional Techniques
   - R^2
   - LOESS
   - Log Transformations
   - Bootstrap

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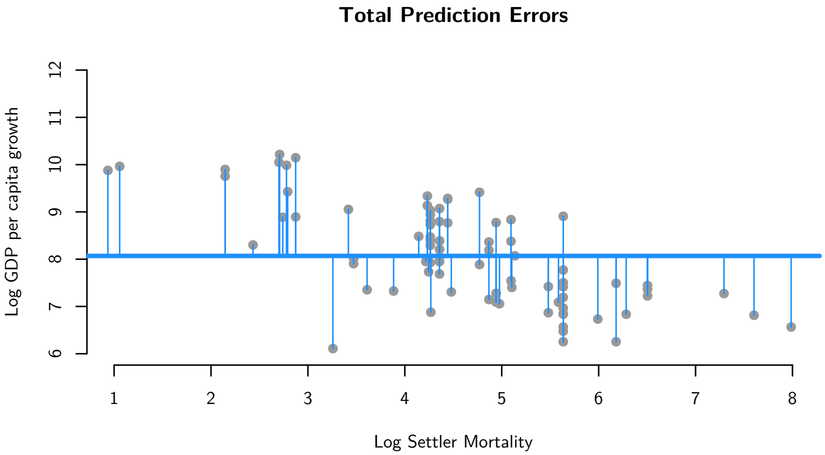
- Once we have estimated our model, we have new prediction errors, which are just the sum of the squared residuals or SS_{res}:

$$SS_{res} = \sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2$$

Sum of Squares



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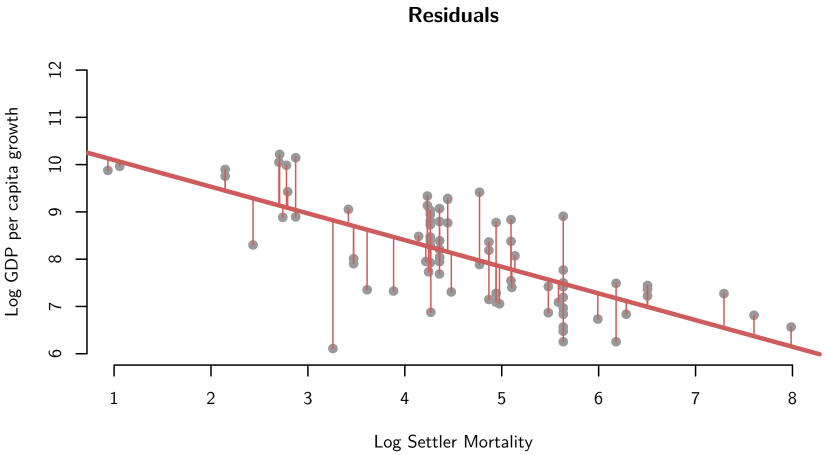

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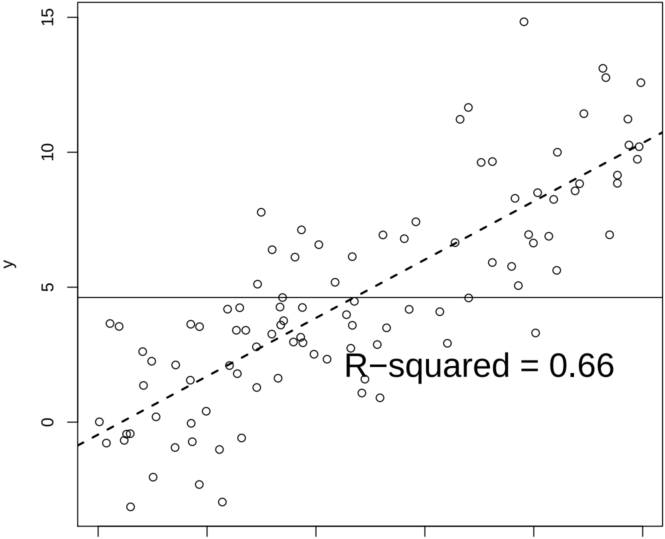
- $R^2 = 1$ implies perfect linear fit

Is R-squared useful?



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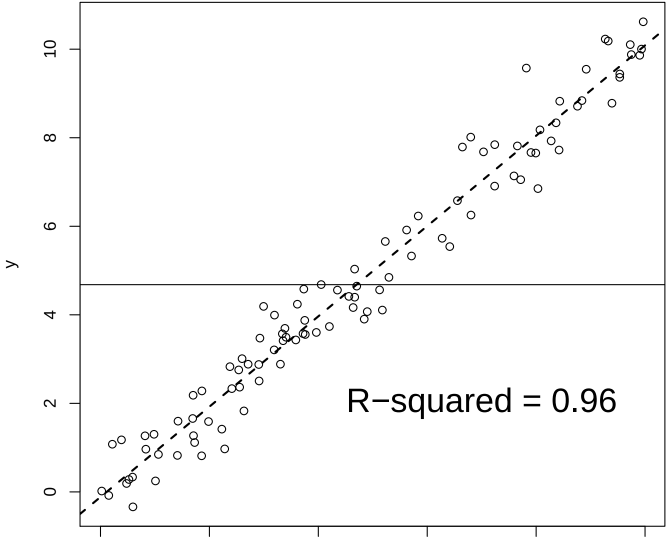

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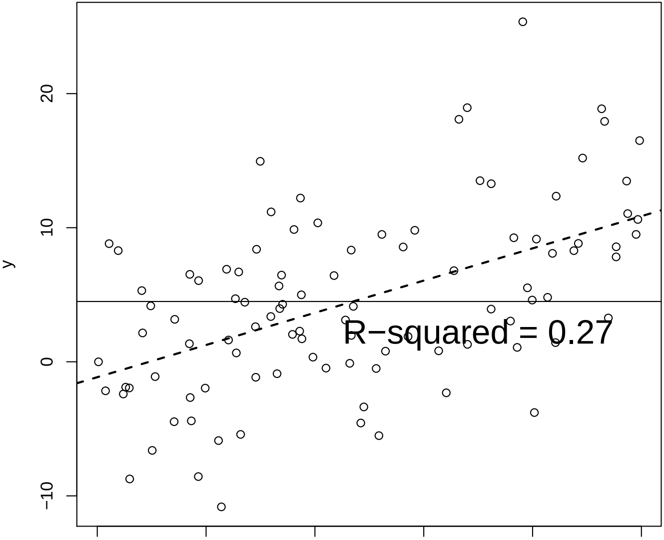

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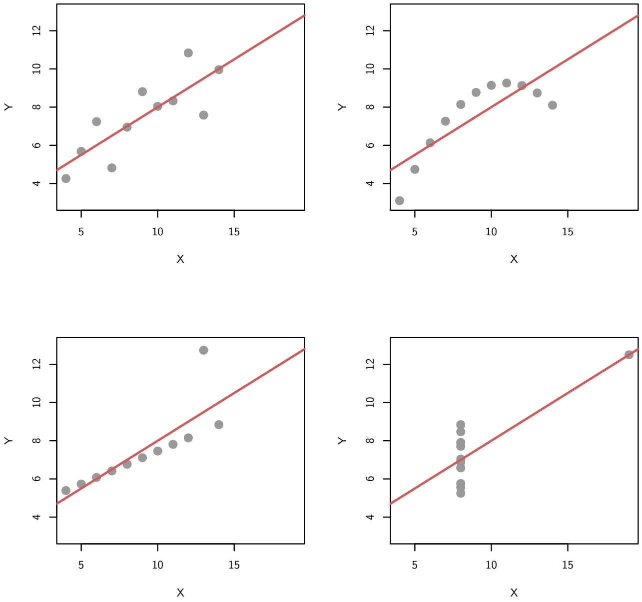
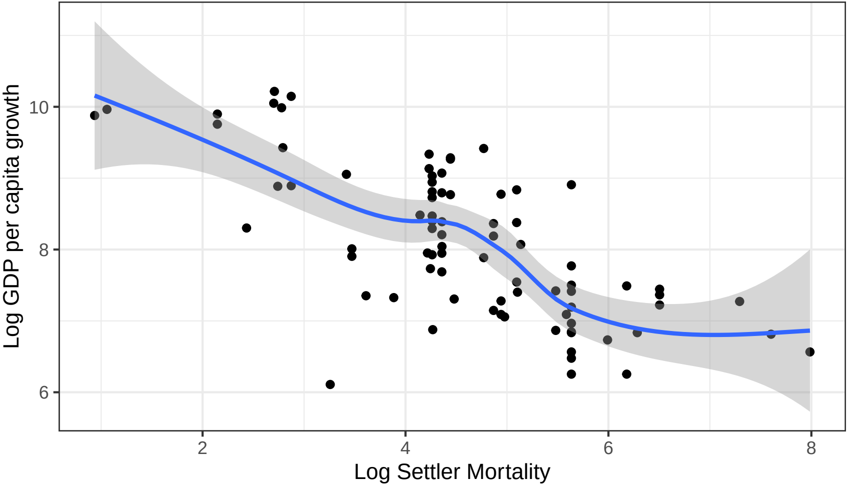


So what is ggplot2 doing?



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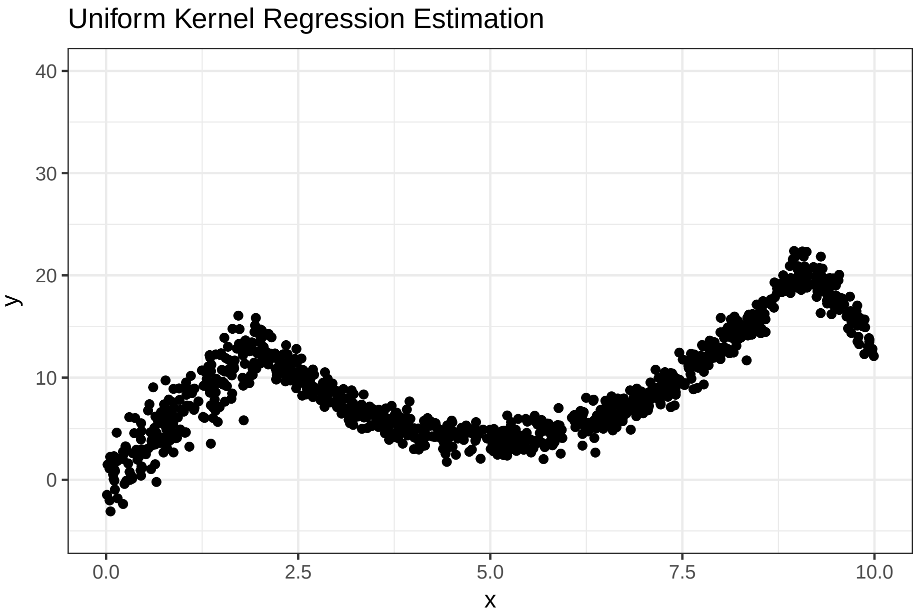
  3. Use the fitted regression line to predict the expected value of $E[Y|X = x_0]$

LOESS Example



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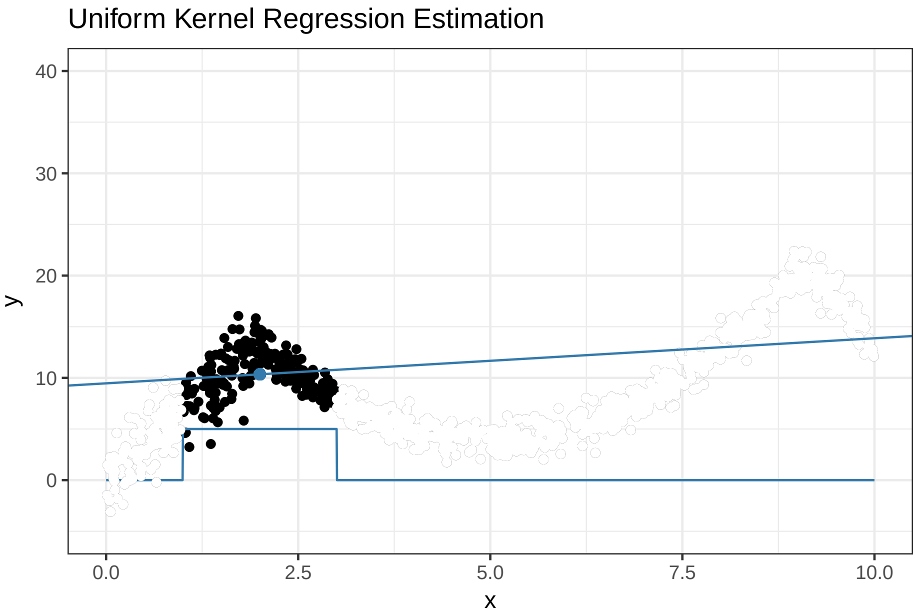

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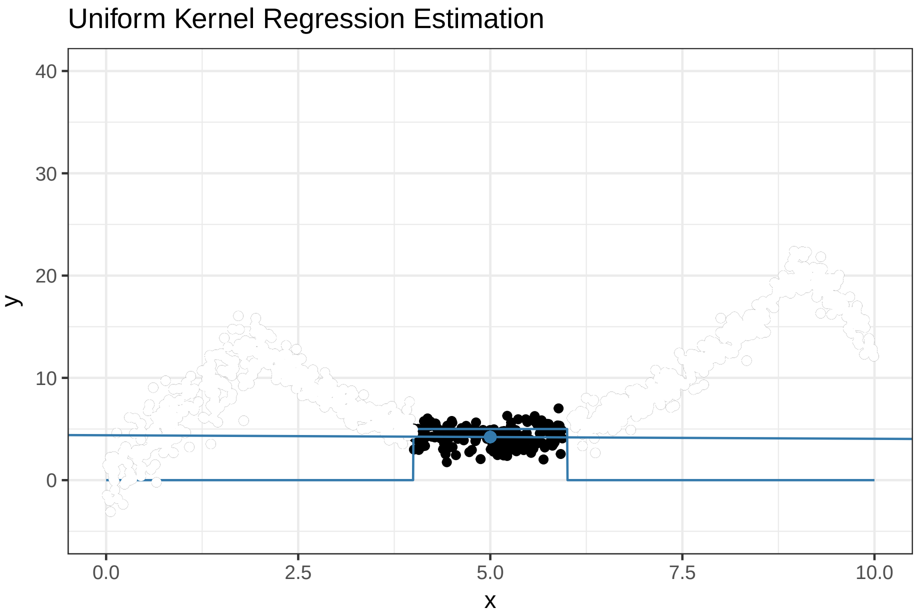

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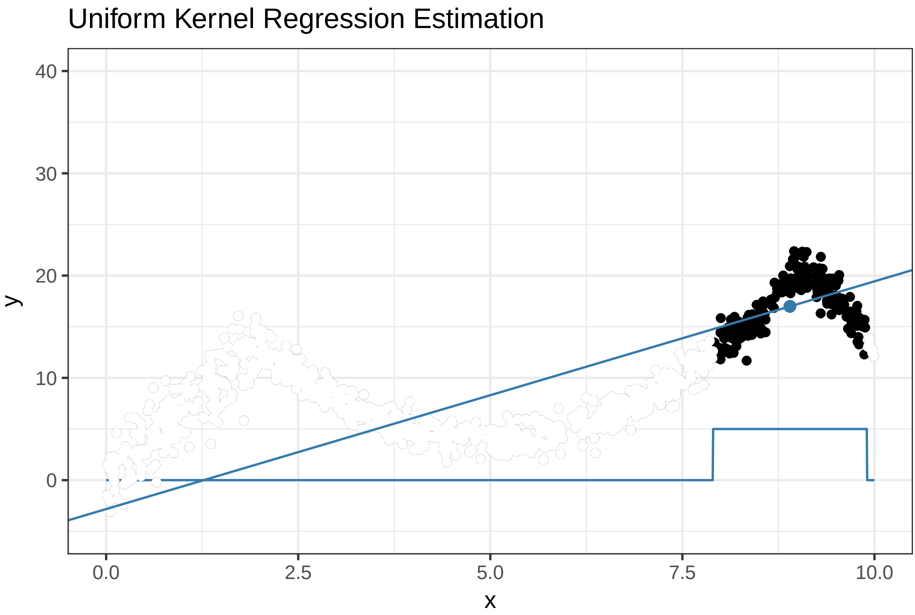

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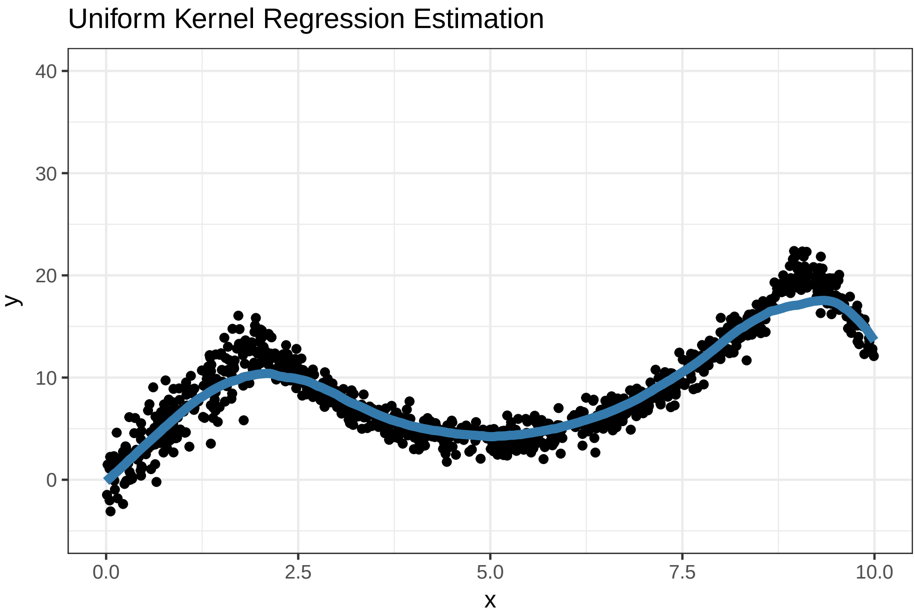

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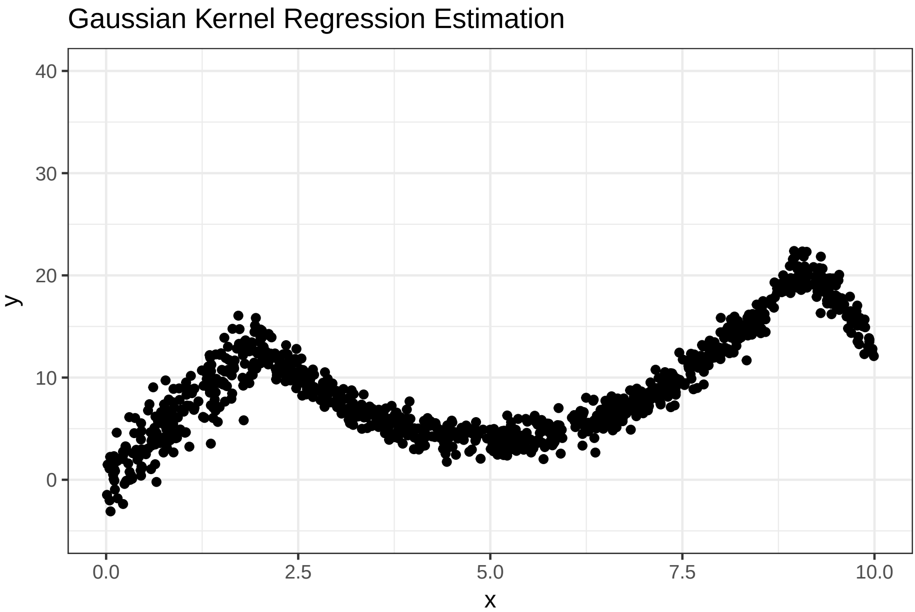

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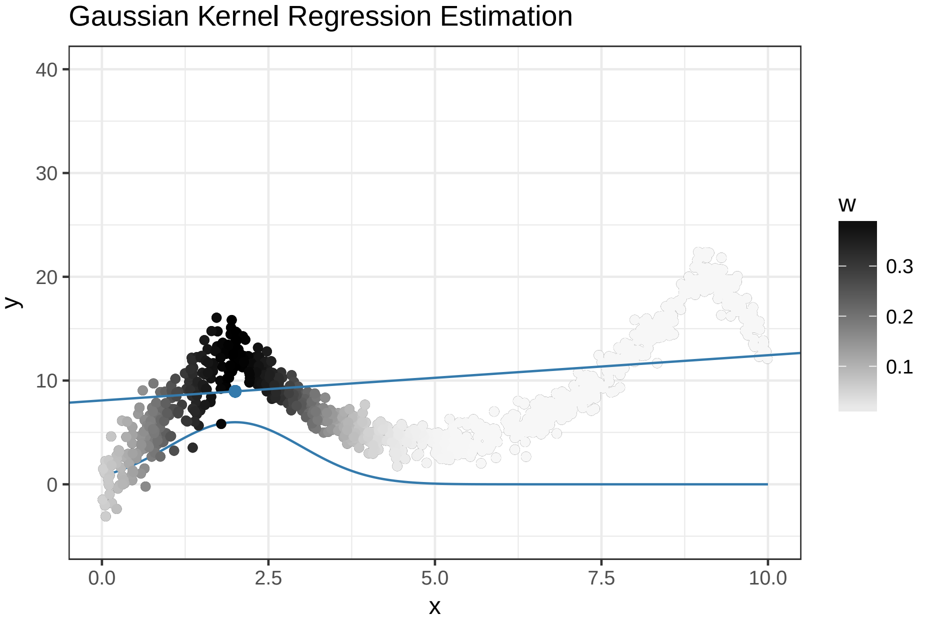

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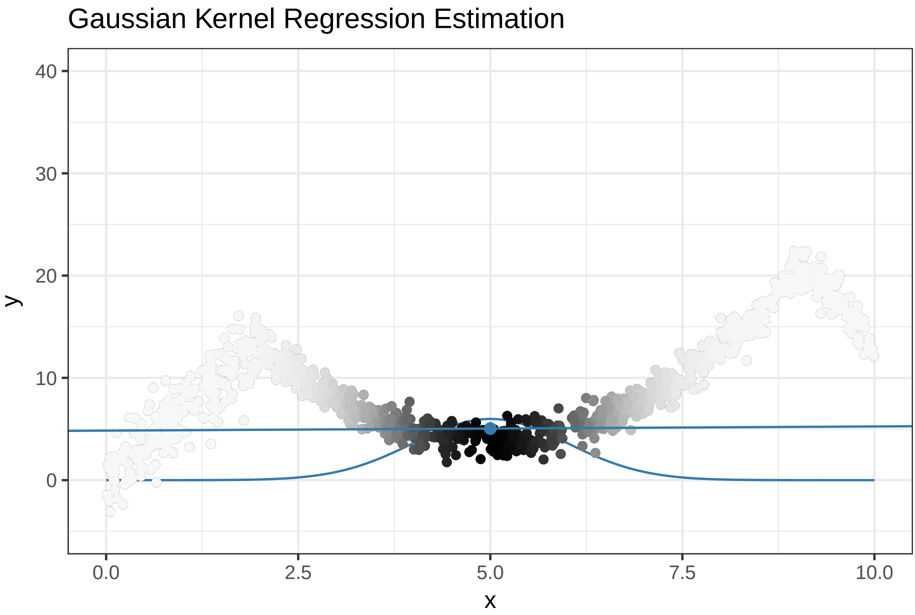

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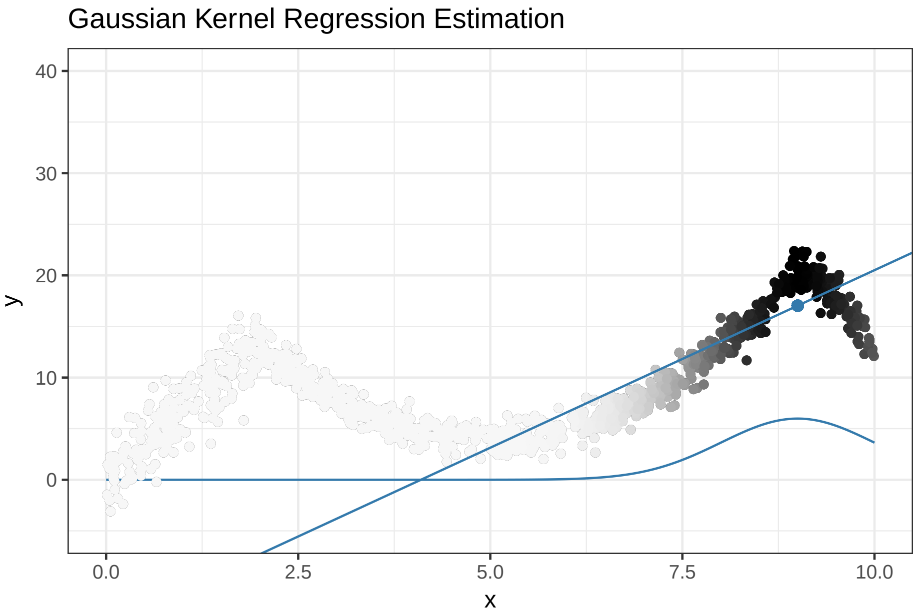

Non-linear CEFs

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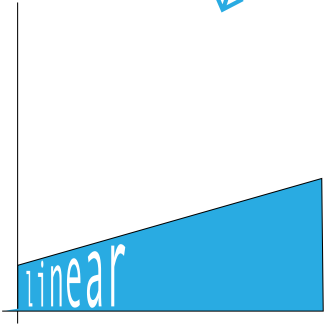
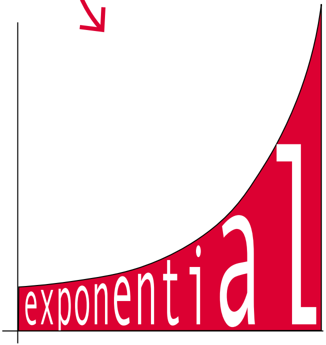
- The function $\log(\cdot)$ is one common transformation that has only one parameter.
- This is particularly useful for positive and right-skewed variables.

Why does everyone keep logging stuff??

Logs linearize exponential growth.



grows by a fixed amount.



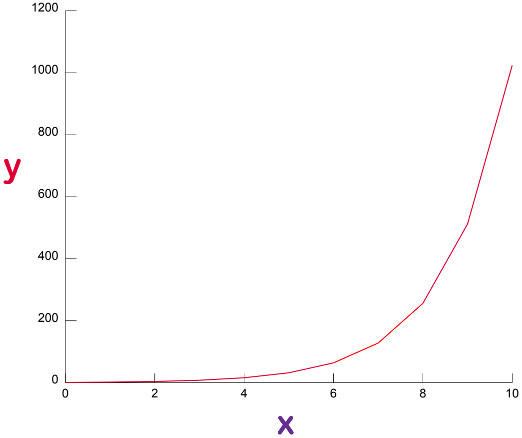
grows by a fixed percent.

How? Let's look.

First, here's a graph showing exponential growth.

We're going to use $y = 2^x$, but any other exponent will work

x	$y=(2^x)$
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024



What happens when we take the log of y?

$\log y = z \qquad e^{z} = y$

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0	1	
1	2	
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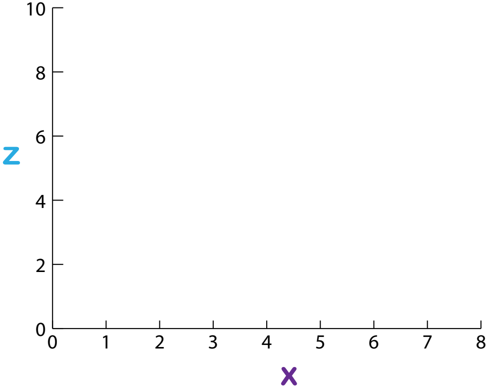

What happens when we take the log of y?

$$\log 1 = 0 \qquad e^0 = 1$$

We're going to use $y = 2^x$, but any other exponent will work

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0	1	0
1	2	
2	4	
3	8	
4	16	
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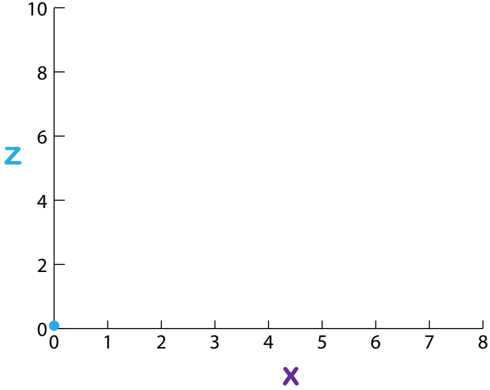
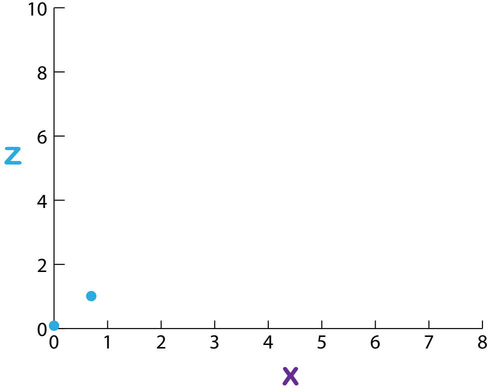


What happens when we take the log of y?

$$\log 2 = .69 \qquad e^{.69} = 2$$

We're going to use $y = 2^x$, but any other exponent will work

x	y	z
0	1	0
1	2	.69
2	4	
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
10	1024	



What happens when we take the log of y?

$$\log 4 = 1.39 \qquad e^{1.39} = 4$$

We're going to use $y = 2^x$, but any other exponent will work

x	y	z
0	1	0
1	2	.69
2	4	1.39
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
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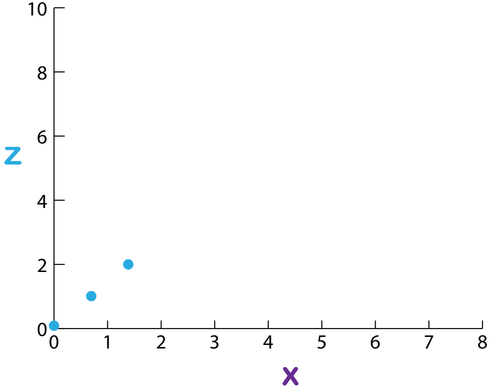

What happens when we take the log of y?

$\log 8 = 2.08 \qquad e^{2.08} = 8$

We're going to use $y = 2^x$, but any other exponent will work

x	y	z
0	1	0
1	2	.69
2	4	1.39
3	8	2.08
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
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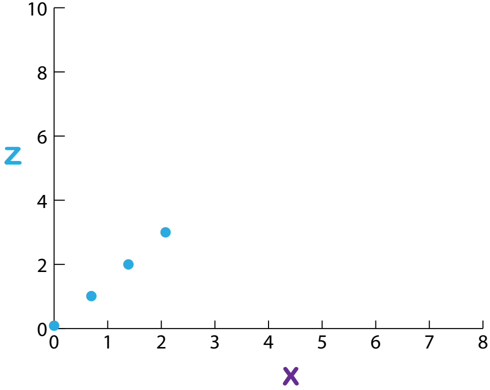

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0	1	0
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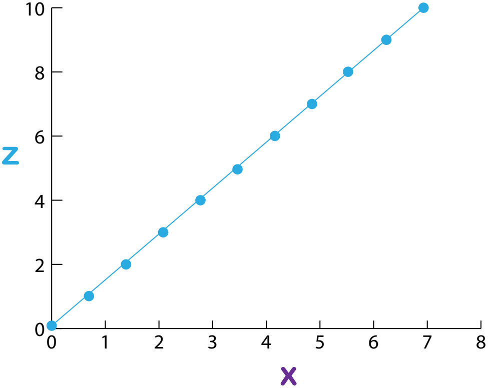
2	4	1.39
3	8	2.08
4	16	2.77
5	32	3.47
6	64	4.16
7	128	4.85
8	256	5.55
9	512	6.24
10	1024	6.93



Interpretation

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- Regress $\log(Y)$ on $X \longrightarrow \beta_1$ approximates percent increase in our prediction of Y associated with one unit increase in X.

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The log transformation changes the interpretation of β_1:

- Regress $\log(Y)$ on $X \longrightarrow \beta_1$ approximates percent increase in our prediction of Y associated with one unit increase in X.
- Regress Y on $\log(X) \longrightarrow \beta_1$ approximates increase in Y associated with a percent increase in X.

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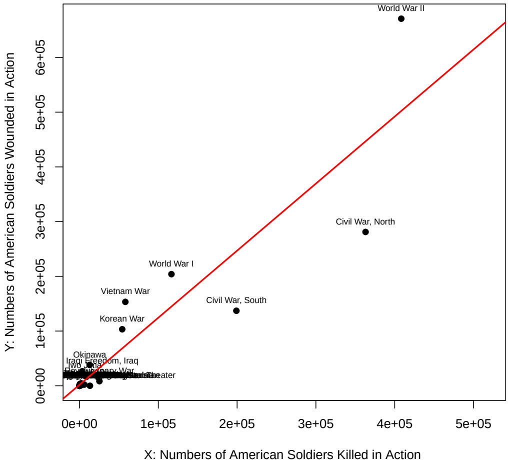
- Note that these approximations work only for small increments.
- In particular, they do not work when X is a discrete random variable.

Example from the American War Library



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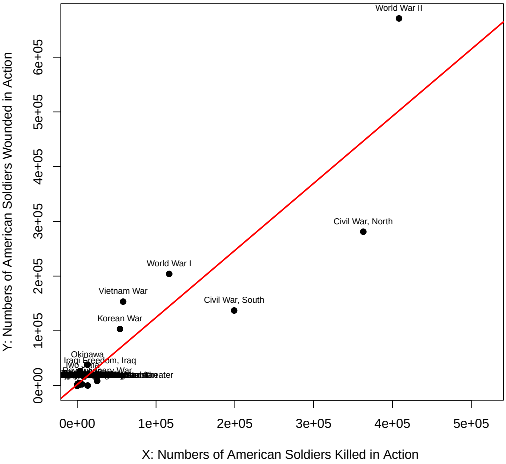
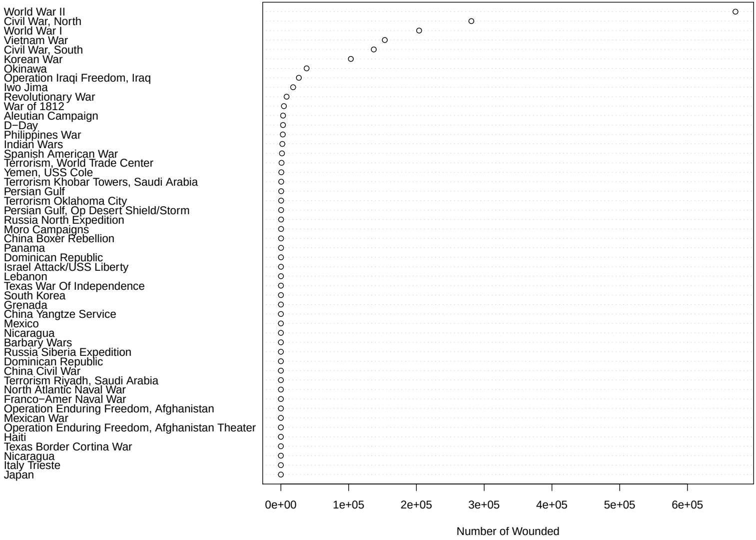
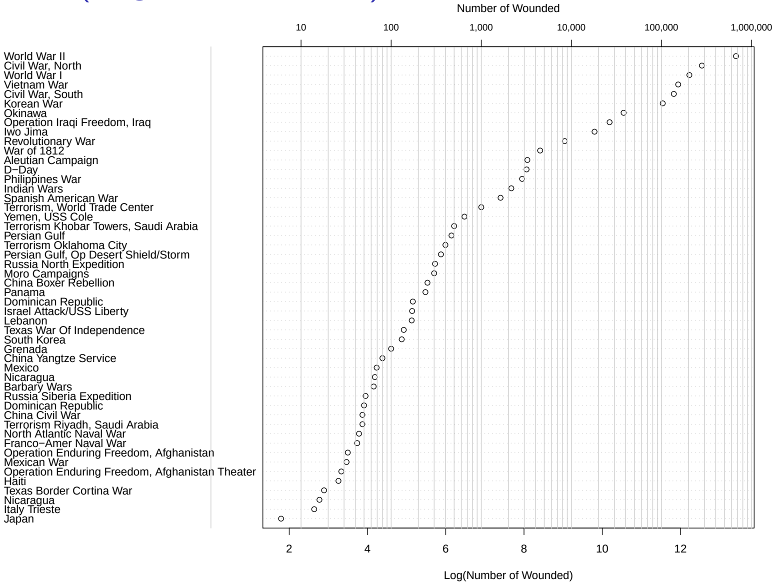


$\hat{\beta}_1 = 1.23 \longrightarrow$ One additional soldier killed predicts 1.23 additional soldiers wounded

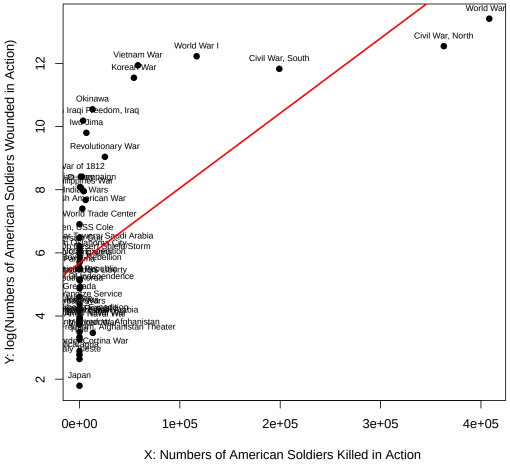
Wounded (Scale in Levels)



Wounded (Logarithmic Scale)



Regression: Log-Level



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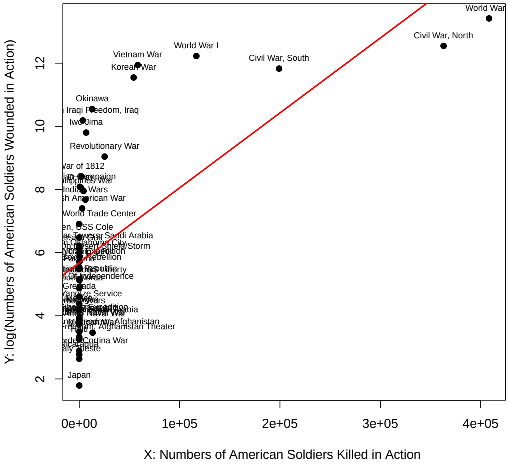
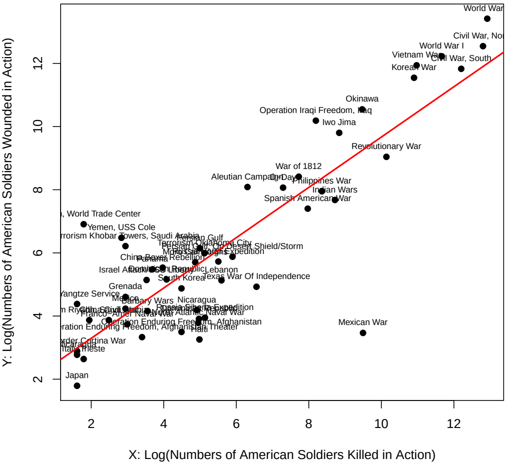


$\hat{\beta}_1 = 0.0000237 \longrightarrow$ One additional soldier killed predicts 0.0023 percent increase in the number of soldiers wounded

Regression: Log-Log



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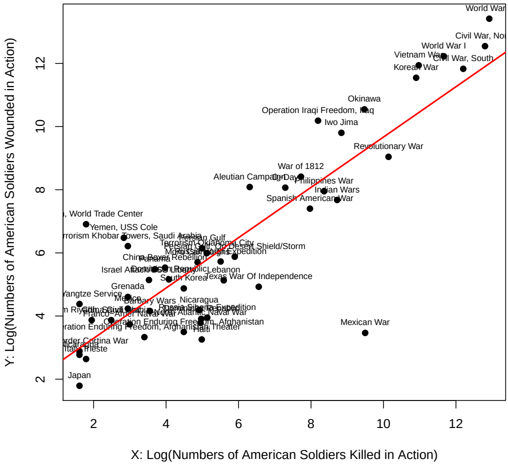

$\hat{\beta}_1 = 0.797 \longrightarrow$ A percent increase in deaths predicts 0.797 percent increase in the wounded

Four Most Commonly Used Models

Model	Equation	β_1 Interpretation
Level-Level	$Y = \beta_0 + \beta_1 X$	$\Delta Y = \beta_1 \Delta X$
Log-Level	$log(Y) = \beta_0 + \beta_1 X$	$\% \Delta Y = 100 \beta_1 \Delta X$
Level-Log	$Y = \beta_0 + \beta_1 log(X)$	$\Delta Y = (\beta_1 / 100) \% \Delta X$
Log-Log	$log(Y) = \beta_0 + \beta_1 log(X)$	$\% \Delta Y = \beta_1 \% \Delta X$

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This can be derived from a series expansion of the log function. Numerically, when $|x| \leq .1$, the approximation is within 0.001.

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Applying our approximation and multiplying by 100 we find,

$$p \approx 100\left(\log(a) - \log(b)\right)$$

Be Careful: Log-Level with binary X

Assume we have: $log(Y) = \beta_0 + \beta_1 X$ where X is binary with values 1 or 0. Assume $\beta_1 > .2$. What is the problem with saying that a one unit increase in X is associated with a $\beta_1 \cdot 100$ percent change in Y?

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A one unit change in X (ie. going from 0 to 1) creates $100(exp(\hat{\beta}_1) - 1)$ percent increase in our prediction $\hat{Y}$.

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- What are we characterizing? The geometric mean.

Geometric Mean

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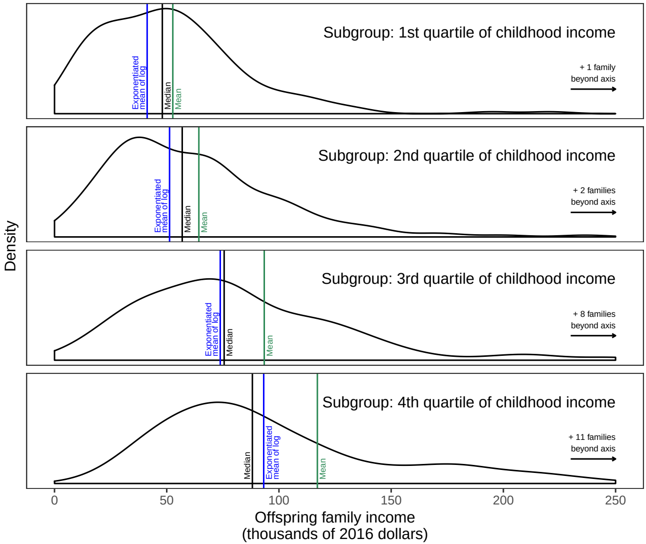
The geometric mean is a robust measure of central tendency.

Geometric Mean is Closer to the Median Than the Mean



(Lundberg and Stewart, 2020)

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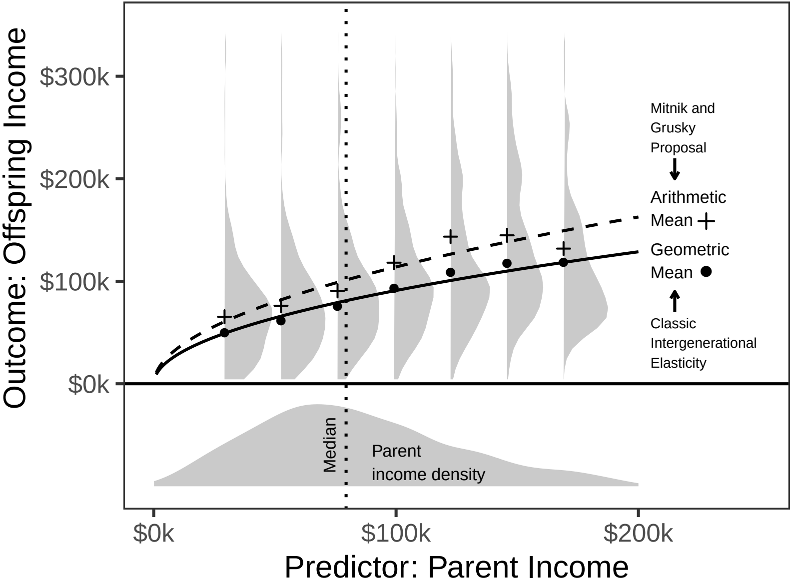

Bootstrapped Sampling Distributions

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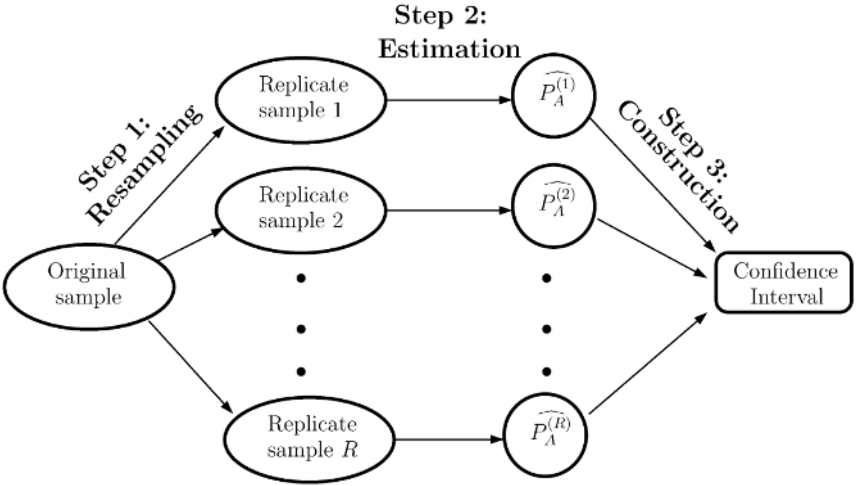
The plug-in principle gives us a way forward.



Source: Salganik (2006)

This works for almost* any estimator

*basically it works when plug-in estimation works

Statistical Science
1986, Vol. 1, No. 1, 54–77

Bootstrap Methods for Standard Errors, Confidence Intervals, and Other Measures of Statistical Accuracy

B. Efron and R. Tibshirani

Efron and Tibshirani (1986), `http://www.jstor.org/stable/2245500`

The Bootstrap

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Bootstrap: Use the eCDF as a plug-in for the CDF, and resample from that. I.e. we are pretending our sample eCDF looks sufficiently close to our true CDF, and so we're sampling from the eCDF as an approximation to repeated sampling from the true CDF. This is called a resampling method.

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3. Repeat steps 1 and 2 many (B) times.

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1. Take a with replacement sample of size n from our sample.
2. Calculate our would-be estimate using this bootstrap sample.
3. Repeat steps 1 and 2 many (B) times.
4. Using the resulting collection of bootstrap estimates, calculate the standard deviation of the bootstrap distribution of our estimator. This serves our estimate of the standard deviation of the sampling distribution

Example of a Bootstrap

```r
samp <- c(9.7, 4.99, 5.9, 3.58, 8.15, 5.54, 4.77, 5.01, 4.89,
          3.42, 8.63, 7.17, 8.93, 7.5, 4.93, 8.6, 6.26, 7.31,
          8.96, 3.95)

obs_mean = mean(samp)

obs_mean
```

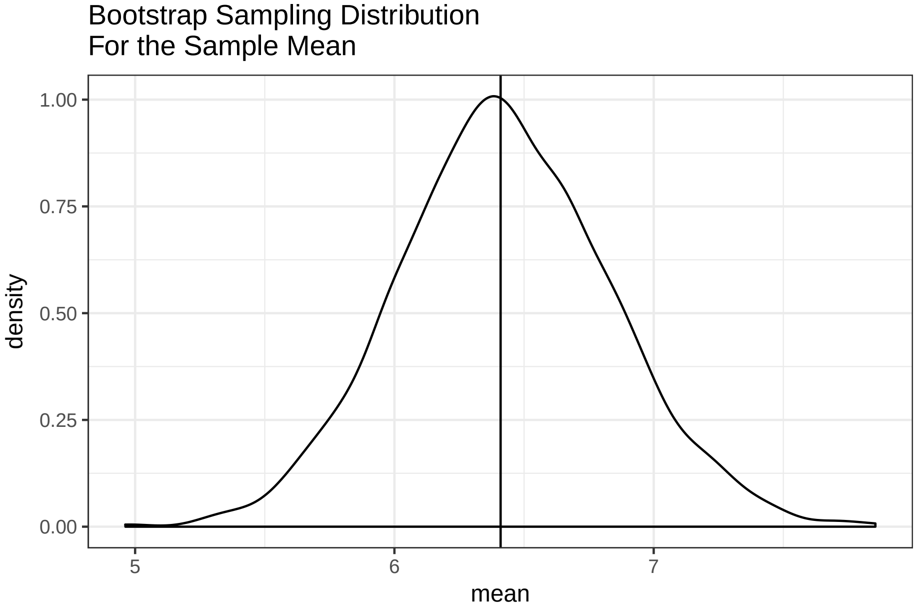
```
## [1] 6.4095
```

Example of a Bootstrap

```r
# resample WITH REPLACEMENT reps times
# recalculate the mean within each bootstrap replicate
boot_samp_dist <- replicate(2000, {
      mean(samp[sample.int(length(samp), replace = TRUE)])
    })
```

```r
ggplot(tibble(boot_samp_dist = boot_samp_dist),
       aes(x = boot_samp_dist)) +
  geom_density() +
  geom_vline(xintercept = obs_mean) +
  theme_bw() + ggtitle("Bootstrap Sampling Distribution
                        For the Sample Mean") +
  xlab("mean")
```

Example of a Bootstrap



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1) Using normal approximation intervals, use the estimates from step 4.

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2) Percentile method for the CI: Sort B bootstrap estimates from smallest to largest. α interval is constructed as

$$CI_{1-\alpha} = \left[\alpha/2 * B \text{ sample}, (1-\alpha/2) * B \text{ sample}\right]$$

- Percentile method does not rely on normal approximation, and behaves better with small n.

This Week in Review

- CEFs!
- OLS!
- Additional Techniques You Will See!

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Going Deeper:

Aronow and Miller (2019) *Foundations of Agnostic Statistics*. Cambridge University Press. Chapter 4.

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Next week: Statistical Theory for Linear Regression