Week 2: Random Variables

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Princeton

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[1] Many of these slides are adapted from Adam Glynn and Jens Hainmueller. Some additional individual slides come from Justin Grimmer, Erin Hartman, Ian Lundberg, and Matt Salganik. Many illustrations by Shay O'Brien.

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  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference with regression

Random Variable

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A function that maps outcomes of our random generative process to numbers.

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Formally:

Definition (Random Variable)

A random variable is a function $X : \Omega \to \mathbb{R}$ such that,
$\forall r \in \mathbb{R}, \{\omega \in \Omega : X(\omega) \leq r\} \in S$.

Event Space

Formally, a set S of subsets of Ω is an event space if it satisfies the following:

- Nonempty: $S \neq \emptyset$
- Closed under complements: if $A \in S$, then $A^C \in S$.
- Closed under countable unions: if $A_1, A_2, A_3, \dots \in S$, then $A_1 \cup A_2 \cup A_3 \cup \dots \in S$

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A technical note (beyond this course):

- The set that satisfies these properties is formally known as a σ-algebra or σ-field on Ω.
- For finite sample spaces, the event space is typically the power set (that is, the set of all subsets) of all events.
- But, this is not always true. For some sample spaces Ω, it is impossible to define the probability of every subset of Ω in a manner consistent with the axioms of probability.

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In more formal settings you will often see probability spaces defined in terms of a probability triple:

- The sample space Ω
- the σ-algebra
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We will (1) omit the discussion of σ-algebra and measurable spaces and (2) suppress the notation $X(\omega)$ working instead just with X.

Our First Example: NH Ballot Order

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
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$$X = \left\{ \phantom{\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}} \right.$$

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

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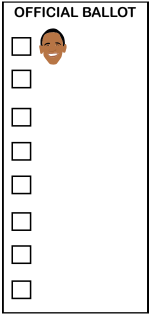

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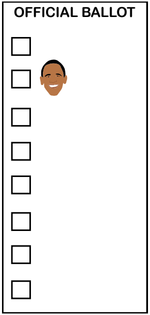

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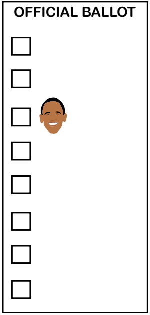

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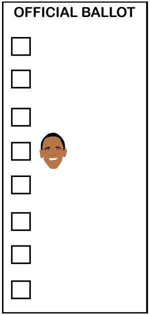

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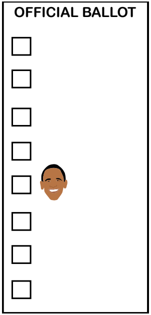

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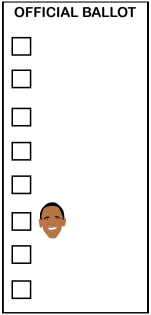

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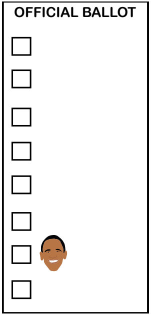

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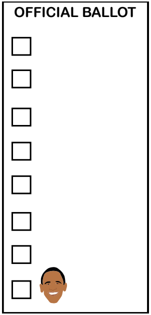

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Characterizing Distributions

Distributions have all kinds of wonky shapes. How do we characterize what they look like?

Operators and Functions on Random Variables

An operator A on a random variable maps the function X to a real number, denoted by $A[X]$.

- An operator on a random variable returns a value characterizing some feature of X. Because it is a descriptive characteristic, we will denote it with square brackets.

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The notation can look confusing, so ask questions if it becomes unclear! But we will return to the key subtleties of this point by the end of lecture!

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The expected value of a random variable X is denoted by $E[X]$ and is a measure of central tendency of X. The expectation is (roughly) the average of all of the values weighted by probability of occurrence.

The expected value of a *discrete* random variable X is defined as

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The expected value of a *discrete* random variable X is defined as

$$E[X] = \sum_{\text{all } x} x \cdot p_X(x).$$

The expected value of a *continuous* random variable X is defined as

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx.$$

Note that $E[\cdot]$ is an operator. It takes a random variable as an input and returns a number that describes the distribution (not a random variable!).

What did we expect for Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
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$4/26 \quad \times 1$

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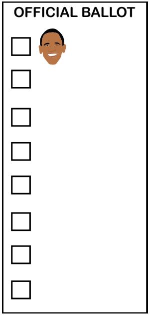

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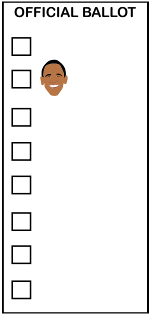

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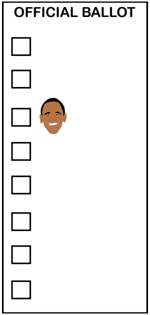

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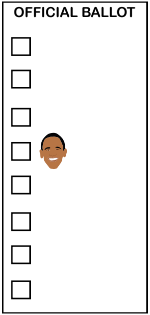

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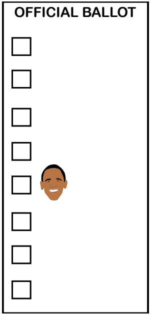

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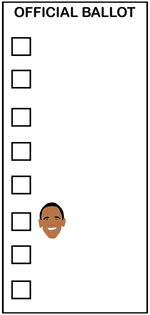

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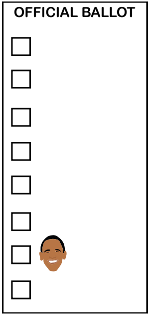

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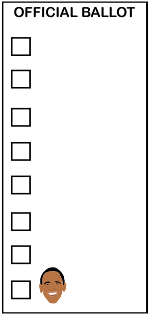

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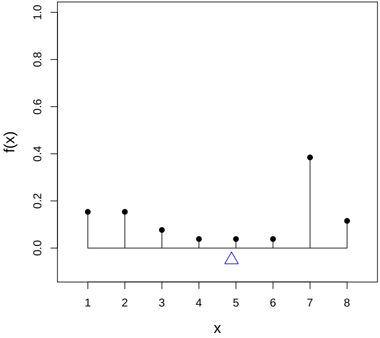
 & 2/26 & \times 3 \\
 & 1/26 & \times 4 \\
 & 1/26 & \times 5 \\
 & 1/26 & \times 6 \\
 & 10/26 & \times 7 \\
+ & 3/26 & \times 8 \\
\hline
 & & 4.88
\end{array}
$$

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

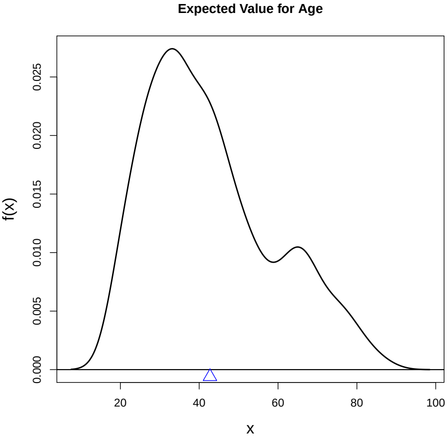
Interpreting Discrete Expected Value

The expected value for a discrete random variable is the balance point of the mass function.



Interpreting Continuous Expected Value

The expected value for a continuous random variable is the balance point of the density function.



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$$\bar{x} = \sum_{\text{all } x_i} x_i f_X(x_i), \text{ where } f_X(x_i) = \frac{1}{N}$$

Three properties of expectation:

- Additivity
- Homogenity
- LOTUS

Property 1 of Expected Value: Additivity

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Let X and Y be random variables with finite expectations. Then, $\forall a, b, c \in \mathbb{R}$,

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

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Why the name LOTUS? "because this can be done very easily and mechanically, perhaps in a state of unconsciousness." (Blitzstein and Hwang, Sec 4.5)

LOTUS: Artifical Example, evil dictator

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Distribution of kids per household

x	0	1	2	3
$P(X = x)$	0.3	0.2	0.2	0.3

- What is $E[X]$?
- What is $g(E[X])$?
- What is $E[g(X)]$?
- What does each mean?

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Notice that $E[g(x)]$ is like a weighted average of $g(X)$ where the density is the weight.

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- We can use random variables!

Identifying the Percent Angry

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So if we know $E[Y]$ and $E[X]$ we can get the expected proportion angered by our item without knowing the individual status of anyone!

Racial Prejudice Example (Kuklinski et al, 1997)

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$X = \#$ of angering items on the baseline list for Southerners:

x	0	1	2	3
$p_X(x)$	?	?	?	?
$\widehat{p}_X(x)$	0.02	0.27	0.43	0.28
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$\widehat{F}_Y(y)$	0.02	0.22	0.62	0.90	1.00

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$\widehat{p}_Y(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$y\widehat{p}_Y(y)$	0.00	0.20	0.80	0.84	0.40	2.24

Racial Prejudice Example

$X = \#$ of angering items on the baseline list for Southerners:

x	0	1	2	3	Sum
$\widehat{p}_X(x)$	0.02	0.27	0.43	0.28	1.00
$x\widehat{p}_X(x)$	0.00	0.27	0.86	0.84	1.97

$Y = \#$ of angering items on the treatment list for Southerners:

y	0	1	2	3	4	Sum
$\widehat{p}_Y(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$y\widehat{p}_Y(y)$	0.00	0.20	0.80	0.84	0.40	2.24

$$\widehat{E[A]} = 2.24 - 1.97 = 0.27$$

On List Experiments in Research

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When to Worry about Sensitivity Bias: A Social Reference Theory and Evidence from 30 Years of List Experiments

GRAEME BLAIR *University of California, Los Angeles*
ALEXANDER COPPOCK *Yale University*
MARGARET MOOR *Yale University*

Eliciting honest answers to sensitive questions is frustrated if subjects withhold the truth for fear that others will judge or punish them. The resulting bias is commonly referred to as social desirability bias, a subset of what we label sensitivity bias. We make three contributions. First, we propose a social reference theory of sensitivity bias to structure expectations about survey responses on sensitive topics. Second, we explore the bias-variance trade-off inherent in the choice between direct and indirect measurement technologies. Third, to estimate the extent of sensitivity bias, we meta-analyze the set of published and unpublished list experiments (a.k.a., the item count technique) conducted to date and compare the results with direct questions. We find that sensitivity biases are typically smaller than 10 percentage points and in some domains are approximately zero.

Variance: A Measure of Dispersion

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The expected value of a function $g()$ of the random variable X is denoted by $E[g(X)]$ and measures the central tendency of $g(X)$.

The variance is a special case of this, and the variance of a random variable X (a measure of its dispersion) is given by

$$\begin{aligned} V[X] &= E[(X - E[X])^2] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

It is the expectation of the squared distances from the mean.

For a discrete random variable X

$$V[X] = \sum_{\text{all } x} (x - E[X])^2 p_X(x)$$

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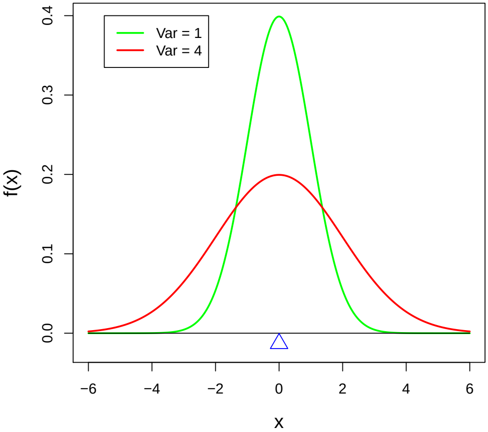
For a continuous random variable X

$$V[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx$$

Variance Measures the Spread of a Distribution



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- It is a reasonable measure for the "spread" of a distribution.
- The Normal distribution—more later—is completely determined by its expected value (location) and variance (spread).
- The square root of the variance is the standard deviation.
- The variance and standard deviation have some useful properties.

Property 1 of Variance: Behavior with Constants

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$$V[b] = 0$$
$$V[aX] = a^2 V[X]$$
$$V[aX + b] = a^2 V[X] + 0$$

Property 2 of Variance: Additivity for Independent Random Variables

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Suppose we have k independent random variables $X_1, \ldots, X_k$. If $V[X_i]$ exists for all $i = 1, \ldots, k$, then

$$V\left[\sum_{i=1}^{k} X_i\right] = V[X_1] + \cdots + V[X_k]$$

NB: independence is sufficient but not necessary.

What was the variance of Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$4/26 \quad \times (1 - 4.88)^2$$

$$+$$

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 & & \color{red}{6.79}
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Does variance matter for fairness?

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- Central moments: The expectation of a higher order polynomial of X that has been "centered" on $E[X]$. The j^{th} central moment of X is $E[(X - E[X])^j]$. Sometimes $X - E[X]$ is referred to as a "deviation" from $E[X]$.
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  - Generally provides more useful information about the spread and shape of a distribution than the raw moments.
  - Second central moment: variance
  - Third central moment (standardized): skew (asymmetry around the expectation)
  - Fourth central moment (standardized): kurtosis or "tailedness"

Another way to characterize distributions is with their moment-generating function.

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Let's prove the first result (see Blitzstein and Hwang 2014 Theorem 6.1.4 on pg 245 for this proof and the proof on mean absolute error).

Proof of Mean as Mean Squared Error Minimizer

Let X be a random variable and $E[X] = \mu$. We want to show that the value of c that minimizes the mean squared error $E\left[(X-c)^2\right]$ is the mean, μ (Blitzstein and Hwang Theorem 6.1.4).

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$$\begin{aligned} V[X] &= V[X-c] && \text{(Prop 1 of Variance)} \\ &= E\left[(X-c)^2\right] - (E[X-c])^2 && \text{(Defn of Variance)} \end{aligned}$$

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&= E\left[(X-c)^2\right] - (\mu - c)^2 && \text{(Linearity of Exp)} \\
V[X] + (\mu - c)^2 &= E\left[(X-c)^2\right]
\end{aligned}$$

Indicator Random Variables

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An indicator random variable I_A or $I(A)$ for an event A is defined to be 1 if A occurs and 0 otherwise.

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Definition (Indicator Random Variables)

Let A and B be events. Then the following properties hold:

1. $(I_A)^k = I_A$ for any positive integer k.
2. $I_{A^c} = 1 - I_A$
3. $I_{A \cap B} = I_A I_B$
4. $I_{A \cup B} = I_A + I_B - I_A I_B$

The Fundamental Bridge

The Fundamental Bridge

Theorem (Fundamental bridge between probability and expectation)

There is a one-to-one correspondence between events and indicator random variables, and the probability of an event A is the expected value of its indicator random variable I_A:

$$P(A) = E(I_A)$$

(Blitzstein and Hwang Theorem 4.4.2)

Summary

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Going Deeper:

Blitzstein, Joseph K., and Hwang, Jessica. (2019). *Introduction to Probability*. CRC Press. `http://stat110.net/`

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Next:

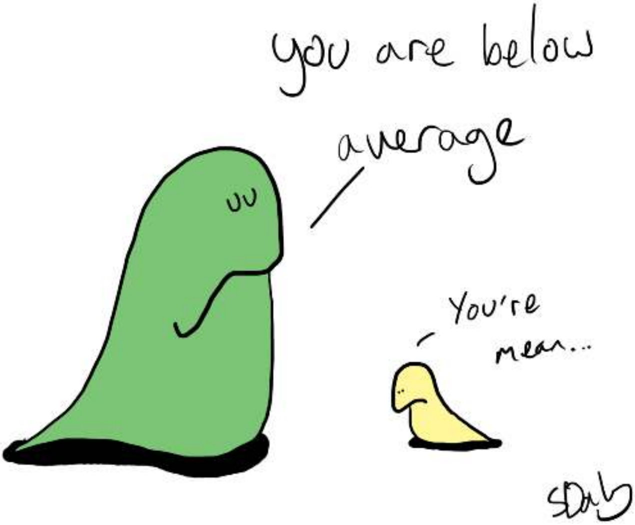
- properties of joint and conditional distributions
- famous distributions

Fun with

Fun with Averages

Fun with Averages

Central Tendency



The Story of Averages

Measurements

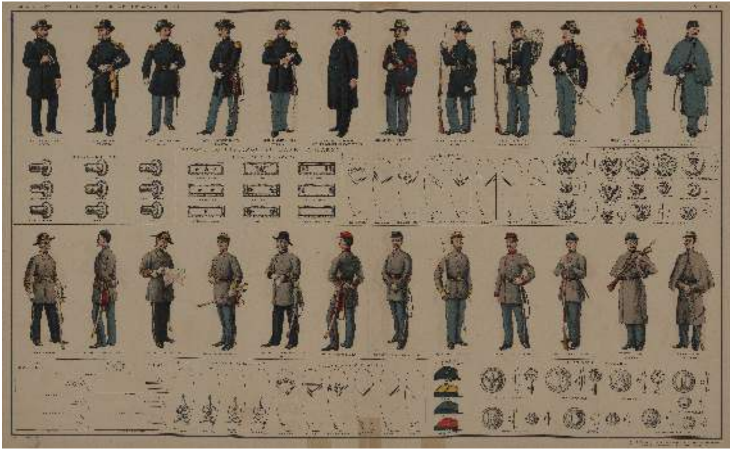
MESURES de la POITRINE.	NOMBRE d'hommes.	NOMBRE PROPORTIONNEL.	PROBABILITÉ d'après L'OBSERVATION.	RANG dans LA TABLE.	RANG d'après le CALCUL.	PROBABILITÉ d'après LA TABLE.	NOMBRE D'OBSERVATIONS calculé.
Pouces.							
33	3	5	0,5000			0,5000	7
34	18	31	0,4995	52	50	0,4993	29
35	81	141	0,4964	42,5	42,5	0,4964	110
36	185	322	0,4823	33,5	34,5	0,4854	323
37	420	732	0,4501	26,0	26,5	0,4531	732
38	749	1305	0,3769	18,0	18,5	0,3799	1333
39	1073	1867	0,2464	10,5	10,5	0,2466	1838
			0,0597	2,5	2,5	0,0628	
40	1079	1882	0,1285	5,5	5,5	0,1359	1987
41	934	1628	0,2913	13	13,5	0,3034	1675
42	658	1148	0,4061	21	21,5	0,4130	1096
43	370	645	0,4706	30	29,5	0,4690	560
44	92	160	0,4866	35	37,5	0,4911	221
45	50	87	0,4953	41	45,5	0,4980	69
46	21	38	0,4991	49,5	53,5	0,4996	16
47	4	7	0,4998	56	61,8	0,4999	3
48	1	2	0,5000			0,5000	1
	5738	1,0000					1,0000

Social Physics

Social Physics

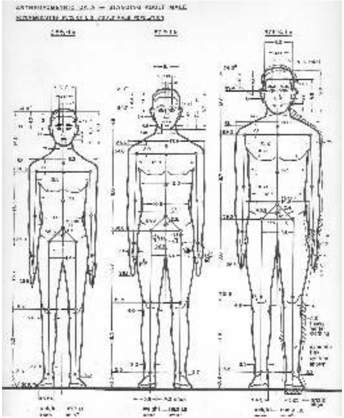
The determination of the average man is not merely a matter of speculative curiosity; it may be of the most important service to the science of man and the social system. It ought necessarily to precede every other inquiry into social physics, since it is, as it were, the basis. The average man, indeed, is in a nation what the centre of gravity is in a body; it is by having that central point in view that we arrive at the apprehension of all the phenomena of equilibrium and motion
- Quetelet, A treatise on man and the development of his faculties, 1842.

The Military Takes to the Idea



The Problem with Averages

The Average Man



Gilbert Daniels, 1950: On 10 dimensions that matter most for design of planes, how many are average on all 10 dimensions?

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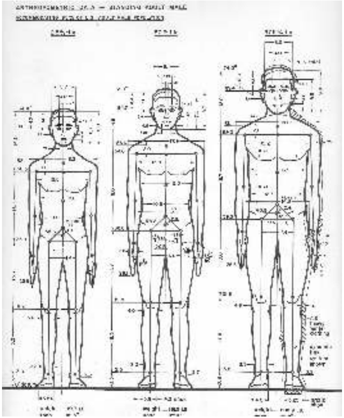


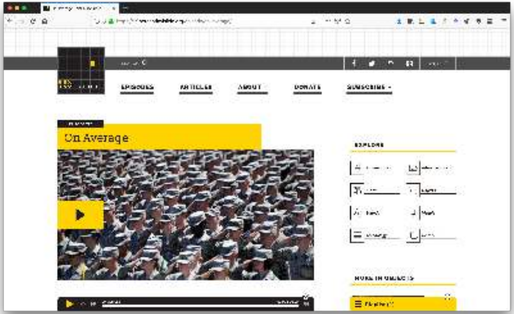
Gilbert Daniels, 1950: On 10 dimensions that matter most for design of planes, how many are average on all 10 dimensions? 0 of 4,063
One step deeper: high-dimensional statistics breaks lots of intuitions.
`https://arbital.com/p/5n1/`

Here's a pilot who is not average

Kim Campbell won the Distinguished Flying Cross for her heroics in Bagdad. She is not average.

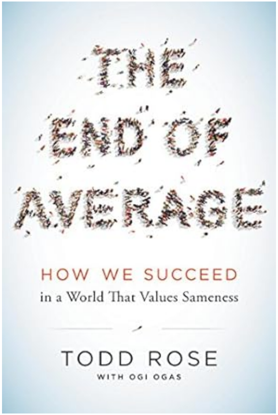
https://commons.wikimedia.org/wiki/File:Kim_campbell_a10.jpg

On averages

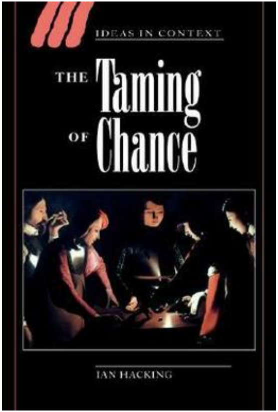


https://99percentinvisible.org/episode/on-average/

On averages



Fit creates opportunity



Intellectual history of averages

Where We've Been and Where We're Going...

- Last Week

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  - welcome and outline of course
  - described uncertain outcomes with probability

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  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference with regression

Joint Distributions

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- The joint distribution of two (or more) variables describes the pairs of observations that we are more or less likely to see.
- The conditional distribution describes one random variable given knowledge of another.
- We will start with a visual preview, then step back to go through the math more concretely.

Understanding Joint Distributions Mathematically

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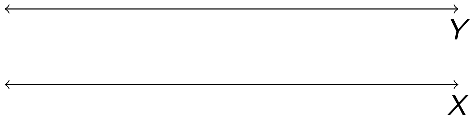
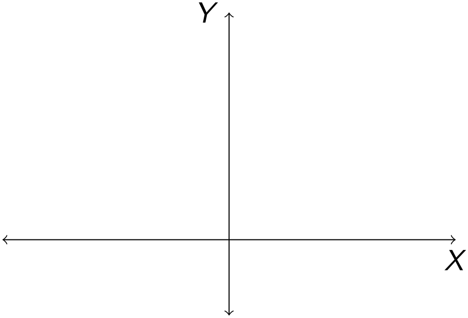


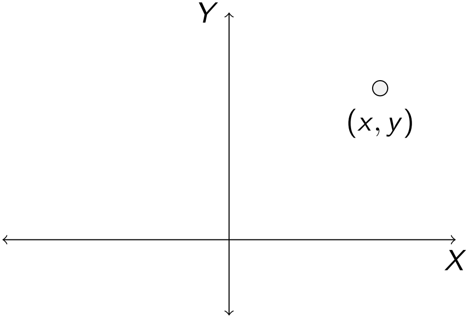
Understanding Joint Distributions Mathematically

- Consider two random variables now, X and Y, each on the real line, $\mathbb{R}$.
- The pair form a two-dimensional space, or $\mathbb{R} \times \mathbb{R}$



Understanding Joint Distributions Mathematically

- Consider two random variables now, X and Y, each on the real line, $\mathbb{R}$.
- The pair form a two-dimensional space, or $\mathbb{R} \times \mathbb{R}$
- One realization of the random variable is a point in that space



Example: Racial Prejudice

- Recall the list experiment about racial prejudice. Suppose we define $X = 0$ (Non-southern), 1 (Southern) and $Y =$ "number of angering items" for a randomly selected respondent receiving the treatment list.

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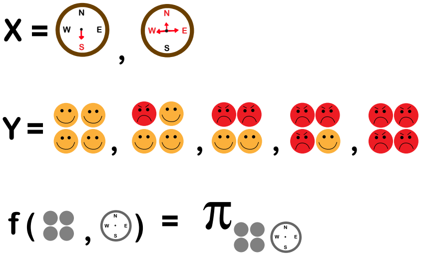

Example Joint Distribution: Binary X, Discrete Y

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$f_{X,Y}(x,y)$	x		
y	0	1	$f_Y(y)$
0	π_{00}	π_{01}	$\pi_{00} + \pi_{01}$
1	π_{10}	π_{11}	$\pi_{00} + \pi_{01}$
2	π_{20}	π_{21}	$\pi_{00} + \pi_{01}$
3	π_{30}	π_{31}	$\pi_{00} + \pi_{01}$
4	π_{40}	π_{41}	$\pi_{00} + \pi_{01}$
$f_X(x)$	$\sum_{y=0}^{4} \pi_{y0}$	$\sum_{y=0}^{4} \pi_{y1}$	

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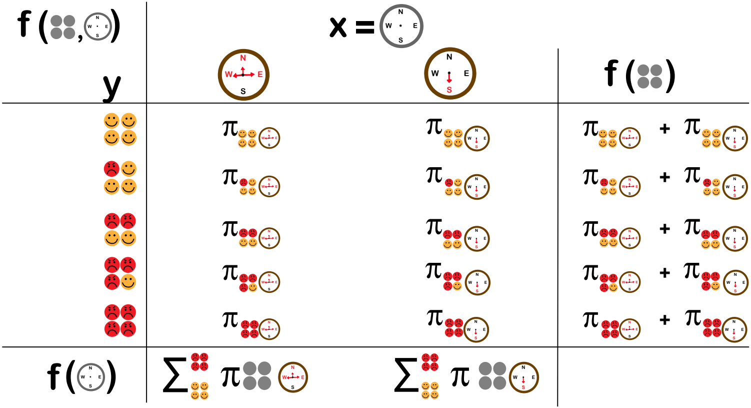

Discrete Conditional Distribution

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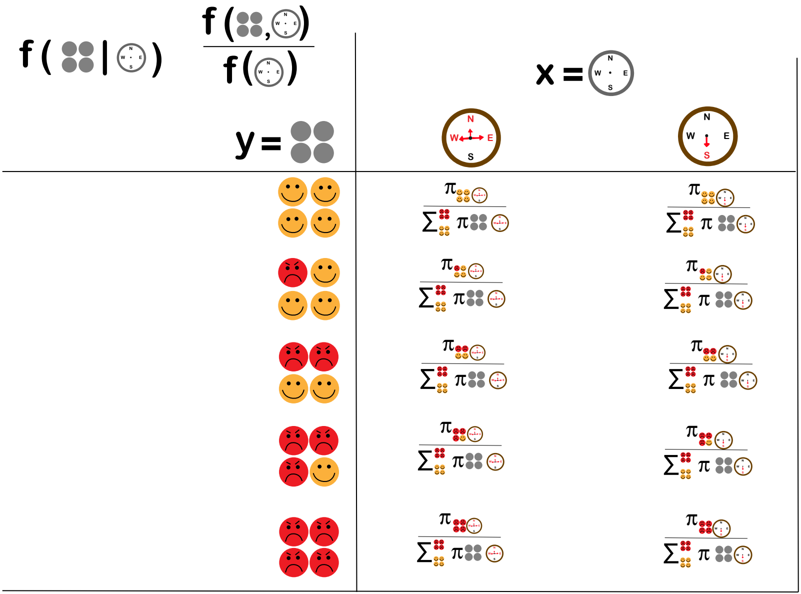

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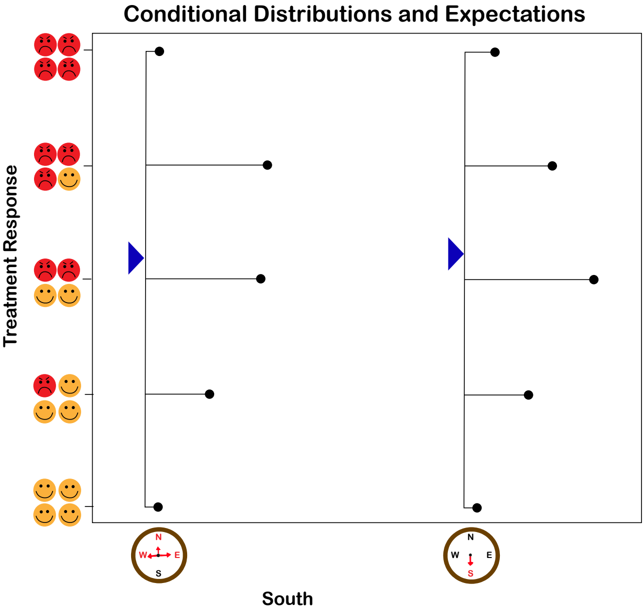

Example: Continuous Conditional Distribution

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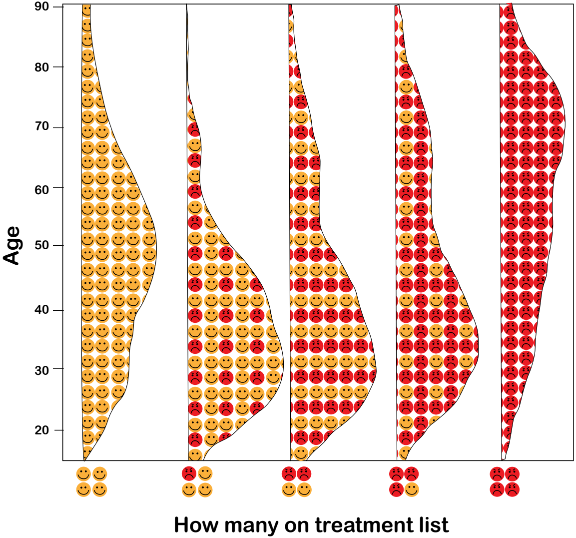
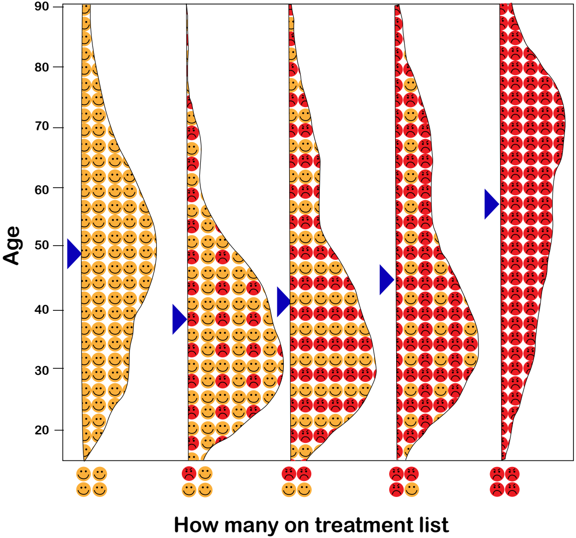


Conditional Expectation Function (coming soon!)



Joint Probability Mass Function

Definition

For two discrete random variables X and Y the joint Probability Mass Function (PMF) $P_{X,Y}(x, y)$ gives the probability that $X = x$ and $Y = y$ for all x and y:

$$p_{X,Y}(x, y) \ = \ P(X = x \text{ and } Y = y)$$

Restrictions:

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- $p_{X,Y}(x,y) \geq 0$ and $\sum_x \sum_y p_{X,Y}(x,y) = 1$.

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Should the U.S. allow more immigrants to come and live here?

		X: Education			
		less HS	HS	College	BA
Y: Support	oppose	0.07	0.22	0.18	0.15
	neutral	0.03	0.06	0.05	0.05
	favor	0.01	0.03	0.04	0.11

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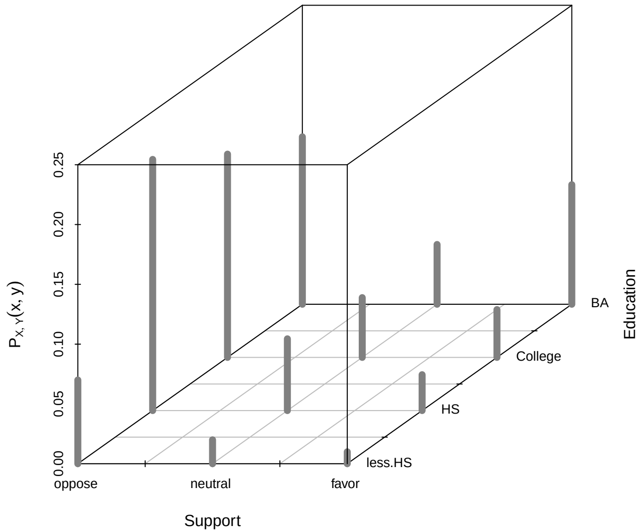
With discrete random variables this is very similar to thinking about a cross-tab, with frequencies/ probabilities in the cells instead of raw numbers.

Joint Probability Mass Function



From Joint to Marginal PMF

Given the joint PMF $p_{X,Y}(x, y)$ can we recover the marginal PMF $p_Y(y)$ (distribution over a single variable)?

		X: Education				
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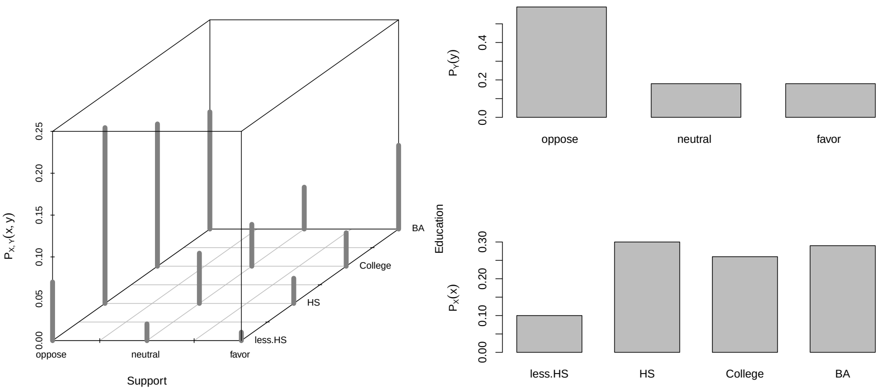
		X: Education				
		less HS	HS	College	BA	$p_Y(y)$
Y: Support	oppose	0.07	0.21	0.17	0.14	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.10	0.19

To obtain $p_Y(y)$ we marginalize the joint probability function $p_{X,Y}(x,y)$ over X:

$$p_Y(y) = \sum_x p_{X,Y}(x,y) = \sum_x P(X = x, Y = y)$$

Joint and Marginal Probability Mass Functions



Why Does Marginalization Work?

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Define the conditional mass function $P(X = x|Y = y)$ as,

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Then it follows that:

$$p_{X,Y}(x, y) = p_{X|Y}(x|y)p_Y(y)$$

Marginalizing over y to get $p_X(x)$ is then,

$$p_X(x_j) = \sum_{i=1}^{N} p_{X|Y}(x_j|y_i)p_Y(y_i)$$

A Table

	$Y = 0$	$Y = 1$	
$X = 0$	$p(0,0)$	$p(0, 1)$	$p_X(0)$
$X = 1$	$p(1,0)$	$p(1,1)$	$p_X(1)$
	$p_Y\ (0)$	$p_Y\ (1)$	

A Table

	Y = 0	Y = 1	
X = 0	0.01	0.05	?
X = 1	0.25	0.69	?
	0.26	0.74	

$$
\begin{aligned}
p_X(0) &= P(X=0|Y=0)P(Y=0) + P(X=0|Y=1)P(Y=1) \\
&= \frac{0.01}{0.26} \times 0.26 + \frac{0.05}{0.74} \times 0.74 \\
&= 0.06
\end{aligned}
$$

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$$

$$
\begin{aligned}
p_X(1) &= P(X=1|Y=0)P(Y=0) + P(X=1|Y=1)P(Y=1) \\
&= \frac{0.25}{0.26} \times 0.26 + \frac{0.69}{0.74} \times 0.74 \\
&= 0.94
\end{aligned}
$$

Conditional PMF

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Key relationships:

- $p_{X,Y}(x, y) = p_{Y|X}(y|x)p_X(x)$ (multiplicative rule)
- $p_{Y|X}(y|x) = p_{X|Y}(x|y)p_Y(y)/p_X(x)$ (Bayes' rule)

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Conditional distributions are key in statistical modeling because they inform us how the distribution of Y varies across different levels of X.

From Joint to Conditional: $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$

Table: Joint PMF $p_{X,Y}(x,y)$ and Marginal PMFs $p_X(x), p_Y(y)$

		Education				
	$p_{X,Y}(x,y)$	less HS	HS	College	BA	$p_Y(y)$
Support	oppose	0.07	0.22	0.18	0.15	0.62
	neutral	0.03	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.11	0.19
	$p_X(x)$	0.11	0.32	0.27	0.31	1.00

From Joint to Conditional: $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$

Table: Joint PMF $p_{X,Y}(x,y)$ and Marginal PMFs $p_X(x), p_Y(y)$

		Education				
	$p_{X,Y}(x,y)$	less HS	HS	College	BA	$p_Y(y)$
Support	oppose	0.07	0.22	0.18	0.15	0.62
	neutral	0.03	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.11	0.19
	$p_X(x)$	0.11	0.32	0.27	0.31	1.00

Table: Conditional PMF $p_{Y|X}(y|x)$

		Education				
	$p_{Y|X}(y|x)$	less HS	HS	College	BA	
Support	oppose	0.70	0.70	0.65	0.48	0.62
	neutral	0.20	0.20	0.19	0.17	0.19
	favor	0.10	0.10	0.15	0.34	0.19
		1.00	1.00	1.00	1.00	1.00

Joint and Conditional Probability Mass Functions

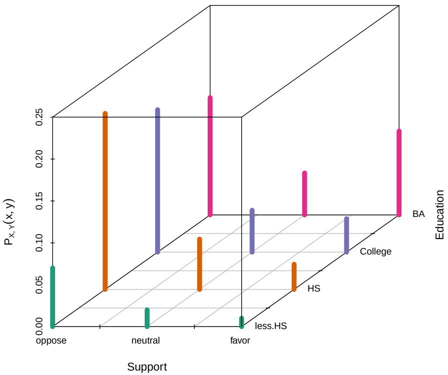

Figure: Joint

Joint and Conditional Probability Mass Functions

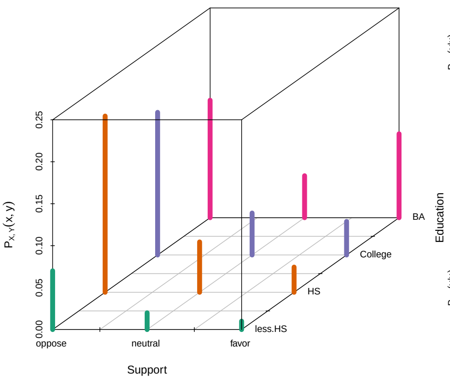

Figure: Joint

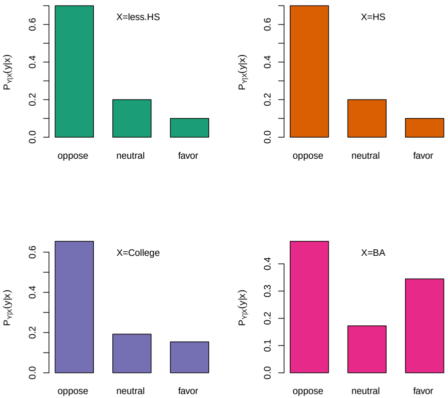

Figure: Conditional

Joint Probability Density Function

Definition

For two continuous random variables X and Y the joint PDF $f_{X,Y}(x, y)$ gives the density height where $X = x$ and $Y = y$ for all x and y.

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The multiplicative rule:

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)f_X(x)$$

where

- $f_{Y|X}(y|x)$: Conditional PDF of Y given $X = x$
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Restrictions:

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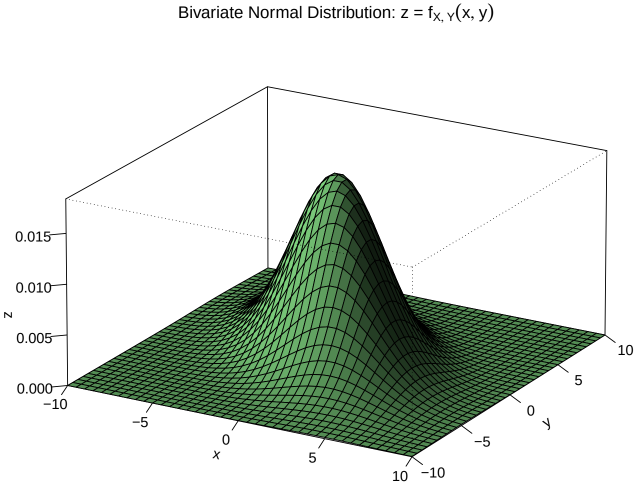
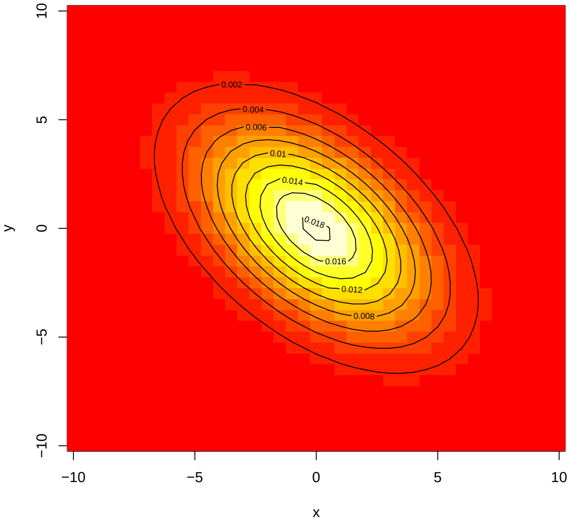
Restrictions:

- $\int_x \int_y f_{X,Y}(x, y)\, dy\, dx = 1$

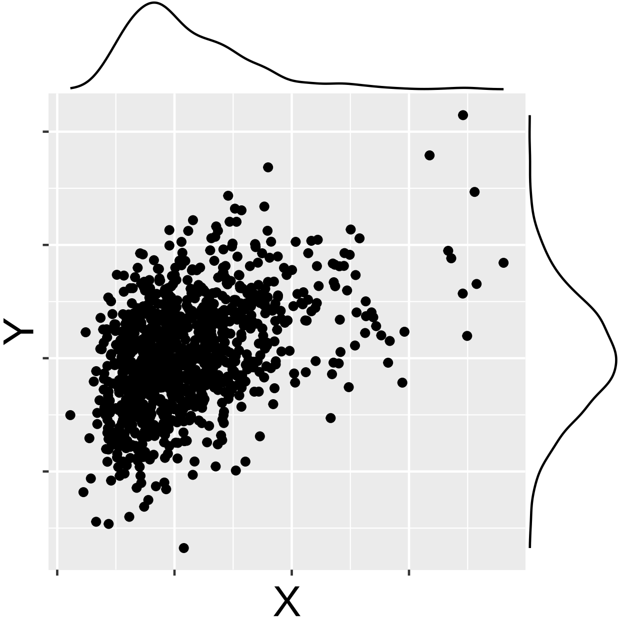
3D Plot of a Joint Probability Density Function



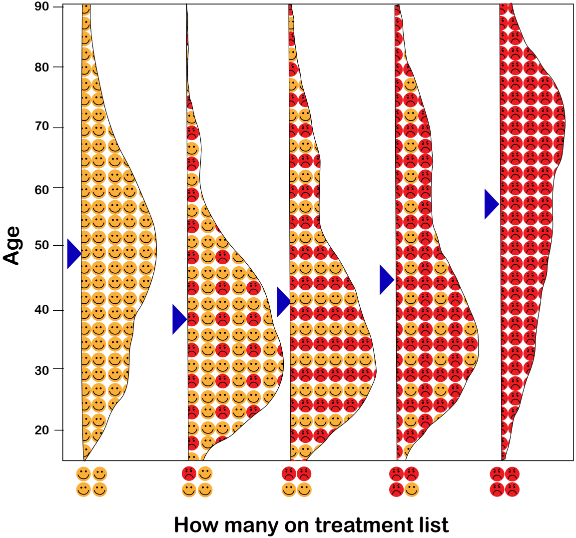
Contour Plot of a Joint Probability Density Function



From Joint to Marginal PDF



Remember this?



Conditioning on X

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- Typically, we summarize the conditional distributions with a few parameters such as the conditional mean of $E[Y|X = x]$ and the conditional variance $V[Y|X = x]$

- Moreover, we are often interested in estimating $E[Y|X]$, i.e. the conditional expectation function that describes how the conditional mean of Y varies across all possible values of X.

Conditional Expectation

Definition (Conditional Expectation (Discrete))

Let Y and X be discrete random variables. The conditional expectation of Y given $X = x$ is defined as:

$$E[Y|X = x] = \sum_y y\, P(Y = y|X = x) = \sum_y y\, p_{Y|X}(y|x)$$

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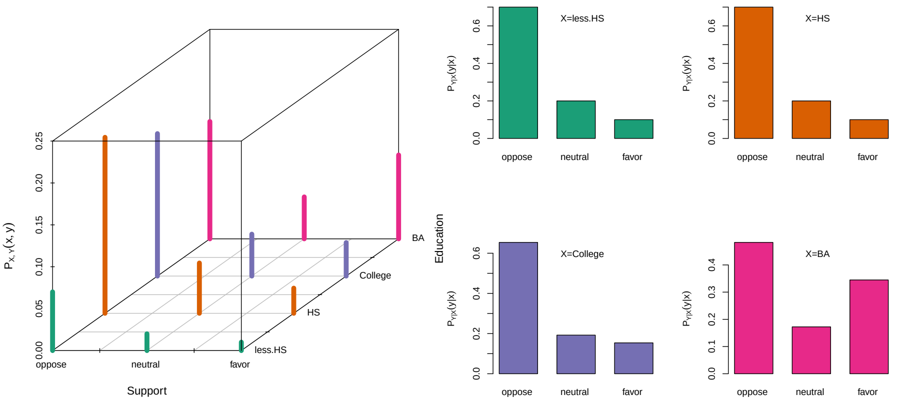
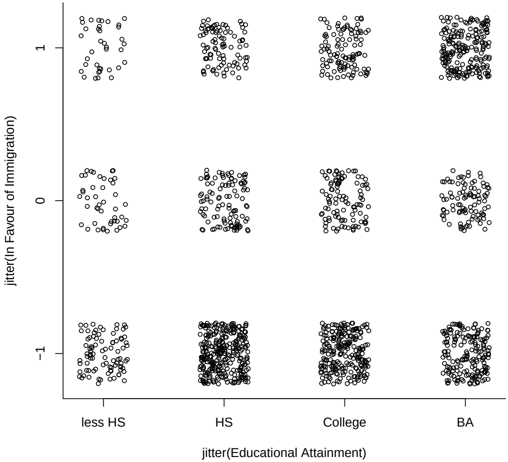
Let Y and X be continuous random variables. The conditional expectation of Y given $X = x$ is given by:

$$E[Y|X = x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x) dy$$

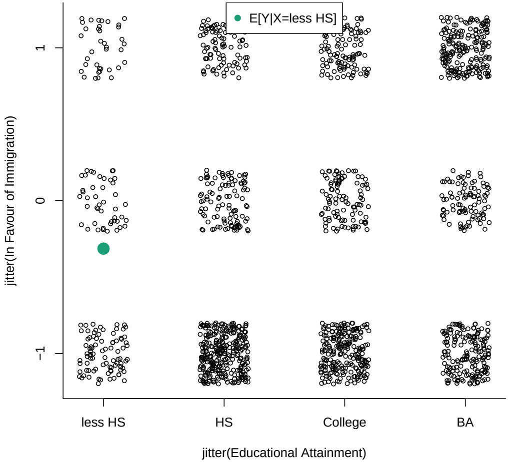
Joint and Conditional Probability Mass Functions



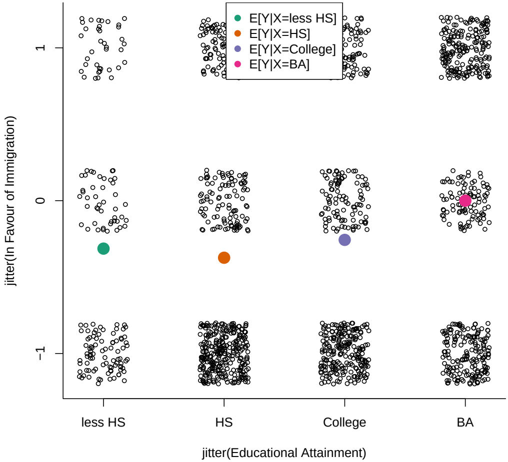
Conditional PMF $P_{Y|X}(y|x)$



Conditional Expectation $E[Y|X = 1]$



Conditional Expectation Function $E[Y|X]$



Law of Iterated Expectations

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Theorem (Law of Iterated Expectations/Adam's Law)

For two random variables X and Y,

$$E[Y] = E[E[Y|X]] = \begin{cases} \displaystyle\sum_{\text{all } x} E[Y|X=x] \cdot p_X(x) & (\textit{discrete } X) \\ \displaystyle\int_{-\infty}^{\infty} E[Y|X=x] \cdot f_X(x)dx & (\textit{continuous } X) \end{cases}$$

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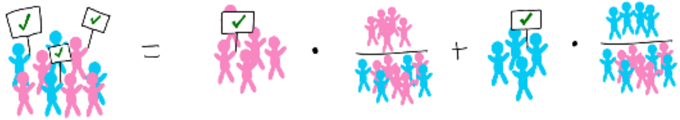
Note that the outer expectation is taken with respect to the distribution of X.

Example: Y (support) and $X \in \{1, 0\}$ (AfAm). Then, the LIE tells us:

$$\begin{aligned} \underbrace{E[Y]}_{\text{Average Support}} &= E[E[Y|X]] \\ &= \underbrace{E[Y|X=1]}_{\text{Average Support}|\text{AfAm}^c} \cdot \underbrace{p_X(1)}_{P(\text{AfAm}^c)} + \underbrace{E[Y|X=0]}_{\text{Average Support}|\text{AfAm}} \cdot \underbrace{p_X(0)}_{P(\text{AfAm})} \end{aligned}$$



Properties of Conditional Expectation

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   - This says that the conditional expectation is the function of X that minimizes the squared prediction error for Y across any possible function of X.
   - This is analogous to the result we saw about the mean.

Conditional Variance

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Remember, the conditional distribution of $Y|X$ is basically like any other probability distribution, so we are going to want to summarize the center *and* spread.

Conditional Variance

Definition

The conditional variance of Y given $X = x$ is defined as:

$$V[Y|X = x] = \begin{cases} \displaystyle\sum_{\text{all } y} (y - E[Y|X = x])^2 P_{Y|X}(y|x) & \text{(discrete } Y\text{)} \\ \displaystyle\int_{-\infty}^{\infty} (y - E[Y|X = x])^2 f_{Y|X}(y|x) dy & \text{(continuous } Y\text{)} \end{cases}$$

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A useful related result is the law of total variance (Eve's Law):

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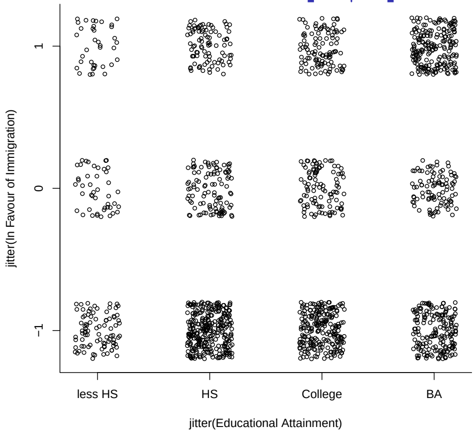
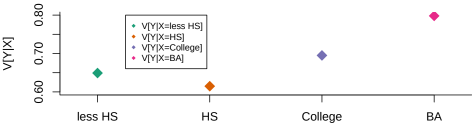
Example: Y (support) and $X \in \{1, 0\}$ (group). The LTV says that the total variance in support can be decomposed into two parts:

1. On average, how much support varies within groups (within variance)
2. How much average support varies between groups (between variance)

Conditional Variance Function $V[Y|X]$





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Let's look at this in pictures.
(If you want to know more: Blitzstein and Hwang pg 392-393)

Important Subtleties in Pictures

Sample space

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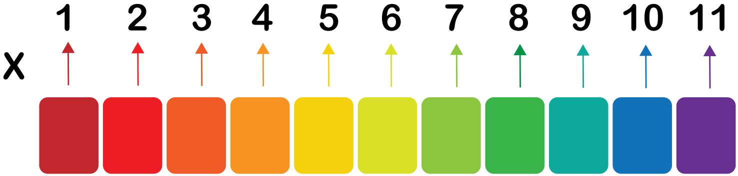
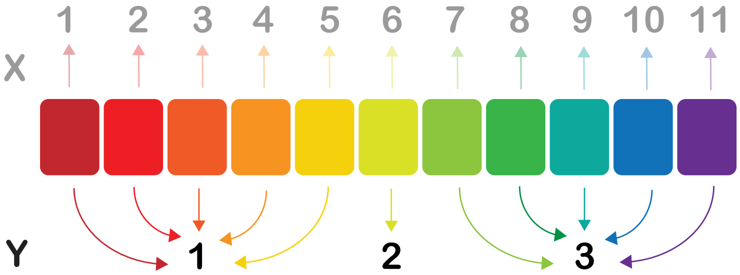


Random variable

Important Subtleties in Pictures



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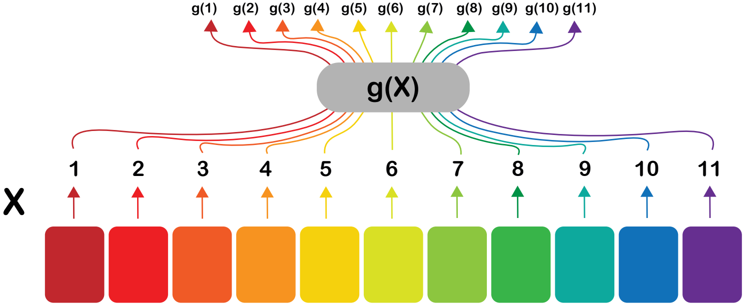
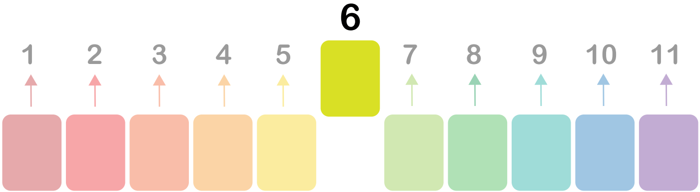


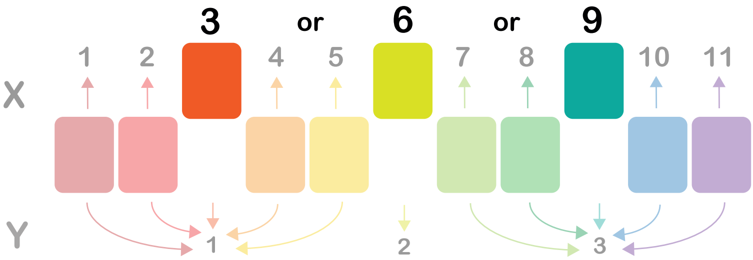
Function of a random variable is a random variable

Important Subtleties in Pictures



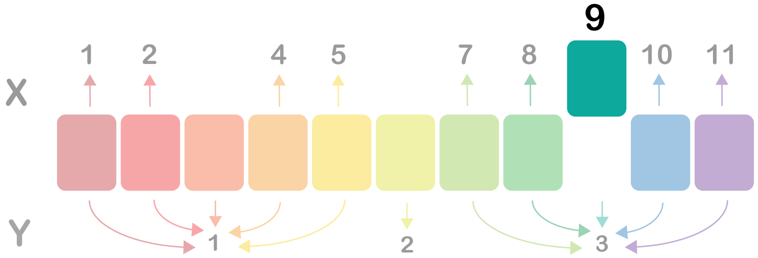
$E[X]$

Important Subtleties in Pictures



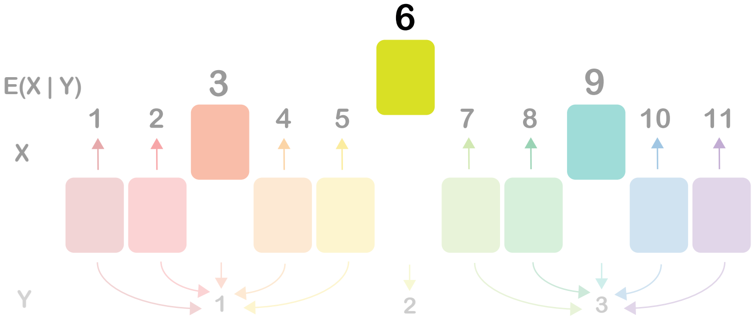
$$E[X|Y]$$

Important Subtleties in Pictures



$$E[X|Y = 3]$$

Important Subtleties in Pictures



$$E[E[X|Y]] = E[X]$$

Independence

Definition (Independence of Random Variables)

Two random variables Y and X are independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for all x and y. We write this as $Y \perp\!\!\!\perp X$.

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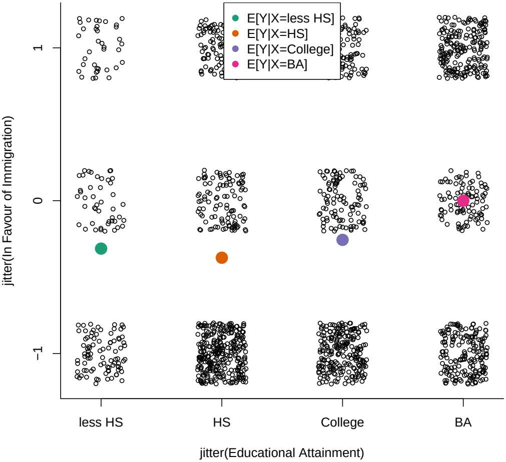
Is $Y \perp\!\!\!\perp X$?



Expected Values with Independent Random Variables

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\end{aligned}$$

We can prove the continuous case by following the same steps, with $\sum$ replaced by $\int$.

Covariance

Covariance

Definition

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$$\begin{aligned} Cov[X, Y] &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

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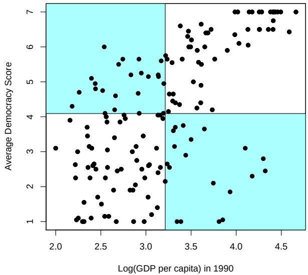
  - Points in upper right and lower left quadrants (relative to the means) add to the covariance.
  - Points in the upper left and lower right quadrants subtract from the covariance.



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Therefore, $X \perp\!\!\!\perp Y \implies Cov[X, Y] = 0$, but not vice versa.

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Proof: Plug in to the definition of variance and expand (try it yourself!)

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- Always satisfies: $-1 \leq Cor[X, Y] \leq 1$.

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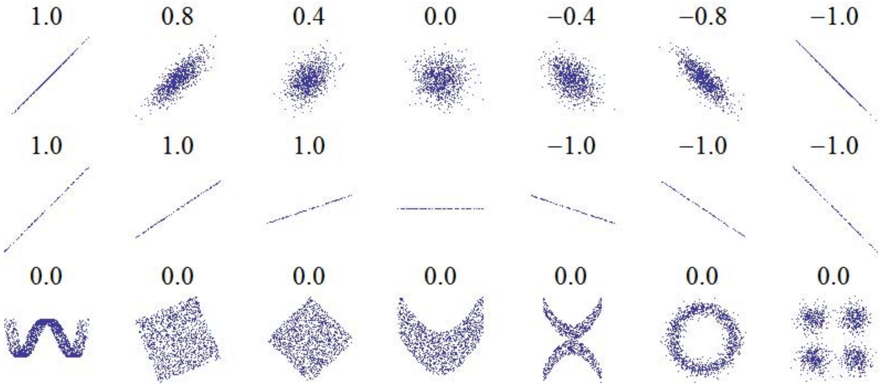

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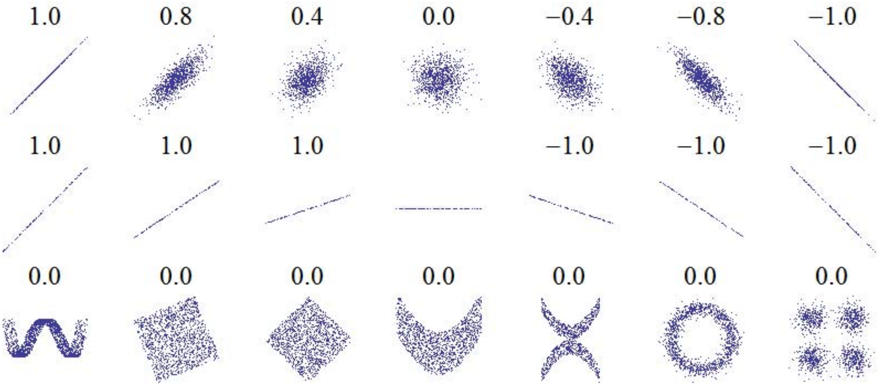

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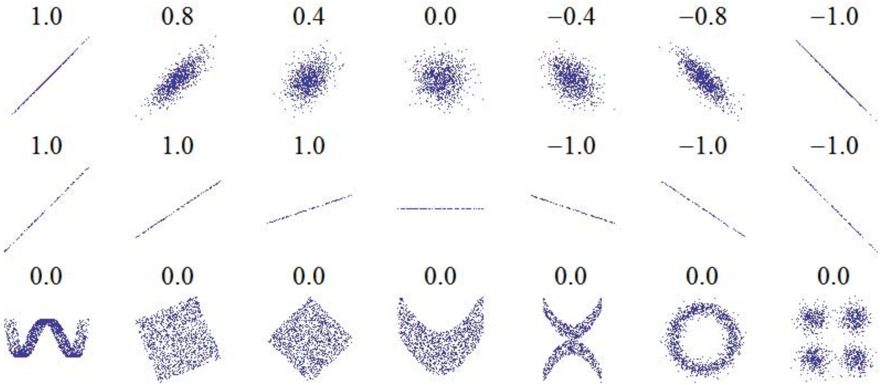

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Definition (Conditional Independence of Random Variables)

Random variables Y and X are conditionally independent given Z iff

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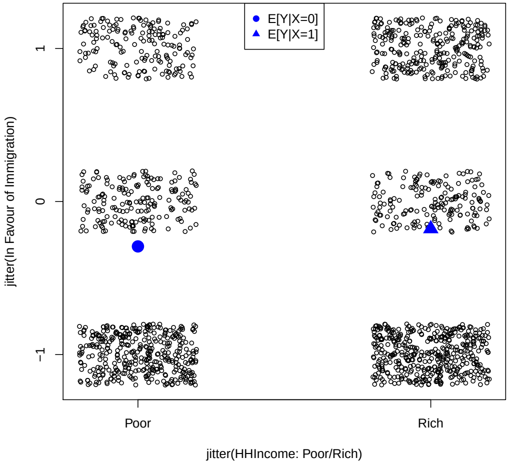
Is $Y \perp\!\!\!\perp X$?

Example: X = wealth, Y = support for immigration, Z = education.



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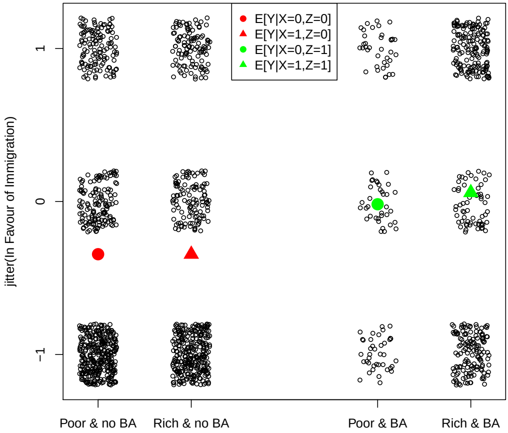
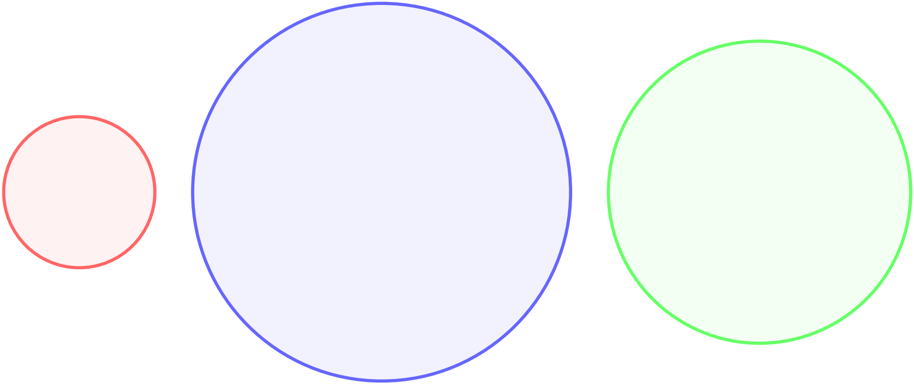
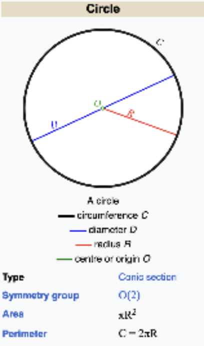
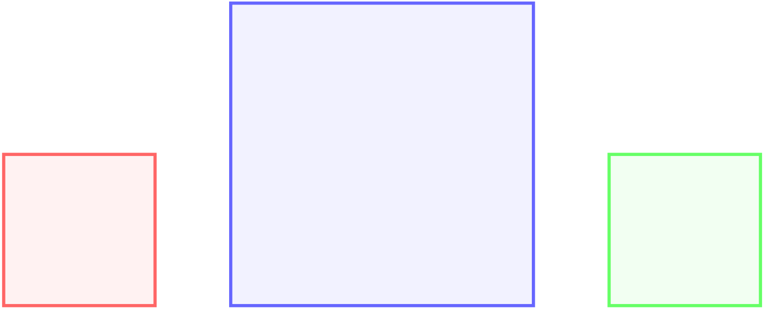




These circles are all different, but they are from the same “family”

- The family of circles is defined by one parameter: usually the radius
- If something is a circle with radius r, then you know other things about it





These squares are all different, but they are from the same “family”.

- The family of squares is defined by one parameter: usually the length
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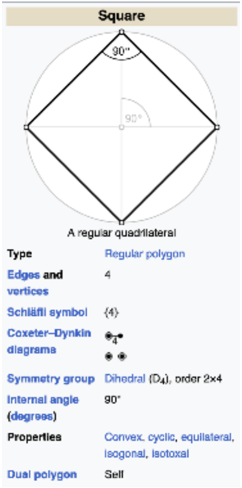
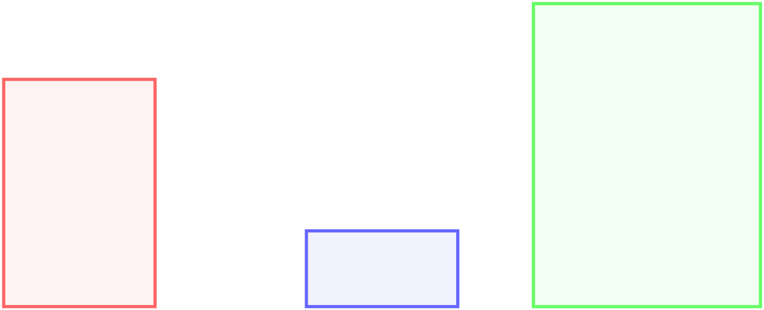
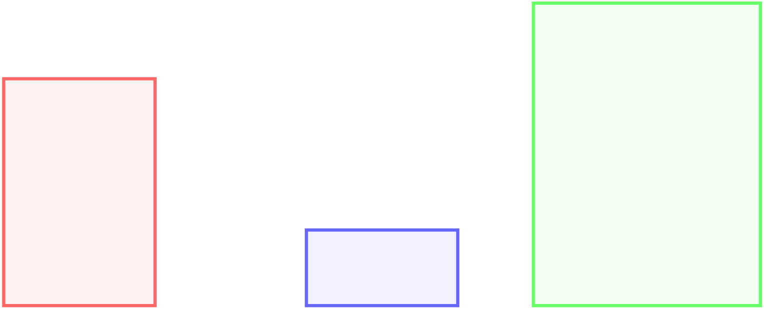
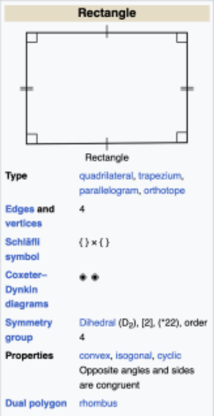






These rectangles are all different, but they are from the same “family”.

- The family of rectangle is defined by two parameters: length and width
- If something is a rectangle with length l and width w then you know other things about it



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- Examples: Bernoulli, Binomial, Gamma, Normal, Poisson, t-distribution

Bernoulli Random Variable

Definition

Suppose X is a random variable, with $X \in \{0, 1\}$ and $P(X = 1) = \pi$.
Then we will say that X is Bernoulli random variable,

$$P(X = x) \;=\; \pi^x (1 - \pi)^{1-x}$$

for $x \in \{0, 1\}$ and $P(X = x) = 0$ otherwise.
We will (equivalently) say that

$$X \;\sim\; \text{Bernoulli}(\pi)$$

$\sim$ means equality in distribution (not values!). Often $X \sim \text{Bernoulli}(\pi)$ would be read 'X is distributed Bernoulli with parameter π'

Bernoulli Random Variable Mean and Variance

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$E[X] = \pi$
$\text{var}(X) = \pi(1 - \pi)$

Importantly, we can also just look this up!

Normal/Gaussian Random Variables

Definition

Suppose X is a random variable with $X \in \mathbb{R}$ and density

$$f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Then X is a normally distributed random variable with parameters μ and σ^2.
Equivalently, we'll write

$$X \quad \sim \quad \text{Normal}(\mu, \sigma^2)$$

Expected Value/Variance of Normal Distribution

Z is a standard normal distribution if

Expected Value/Variance of Normal Distribution

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Proposition

Scale/Location. If $Z \sim N(0,1)$, then $X = aZ + b$ is,

$$X \sim Normal(b, a^2)$$

Intuition

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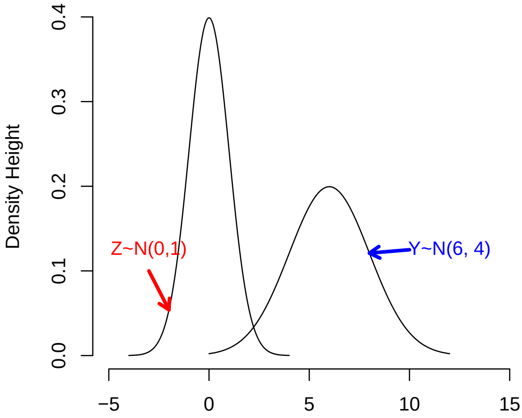
$Y \sim \text{Normal}(6, 4)$



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Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var[Z] &= 1 \end{aligned}$$

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Multivariate Normal

Definition

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$ is a vector of random variables. If $\boldsymbol{X}$ has pdf

$$f_{X_1,X_2}(\boldsymbol{x}) \quad = \quad (2\pi)^{-N/2}\det(\boldsymbol{\Sigma})^{-1/2}\exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

Then we will say $\boldsymbol{X}$ has a Multivariate Normal Distribution,

$$\boldsymbol{X} \quad \sim \quad \text{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Multivariate Normal Distribution

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$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$f(x_1, x_2) = (2\pi)^{-2/2} 1^{-1/2} \exp\left(-\frac{1}{2}\left((\boldsymbol{x} - \mathbf{0})' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\boldsymbol{x} - \mathbf{0})\right)\right)$$

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$\rightsquigarrow$ product of univariate standard normally distributed random variables

Properties of the Multivariate Normal Distribution

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$

$$\begin{aligned} E[\boldsymbol{X}] &= \boldsymbol{\mu} \\ \text{cov}(\boldsymbol{X}) &= \boldsymbol{\Sigma} \end{aligned}$$

So that,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \ldots & \text{cov}(X_1, X_N) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \ldots & \text{cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \ldots & \text{var}(X_N) \end{pmatrix}$$

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\text{cov}(X_2, X_1) & \text{var}(X_2) & \ldots & \text{cov}(X_2, X_N) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \ldots & \text{var}(X_N)
\end{pmatrix}
$$

A nice extra property: the marginals are normal!

Universality of the Uniform

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Theorem (Universality of the Uniform)

- *Regardless of the distribution of* X*,* $F(X) \sim \textit{Uniform}(0, 1)$
- *For a random variable* X *with CDF* F *and a Uniform random variable* U*,* $F^{-1}(U) \sim X$

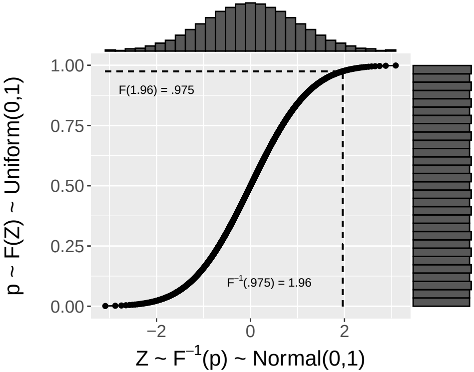
Blitzstein and Hwang Theorem 5.3.1

Universality of the Uniform

Theorem (Universality of the Uniform)

- *Regardless of the distribution of X, $F(X) \sim Uniform(0,1)$*
- *For a random variable X with CDF F and a Uniform random variable U, $F^{-1}(U) \sim X$*

Blitzstein and Hwang Theorem 5.3.1

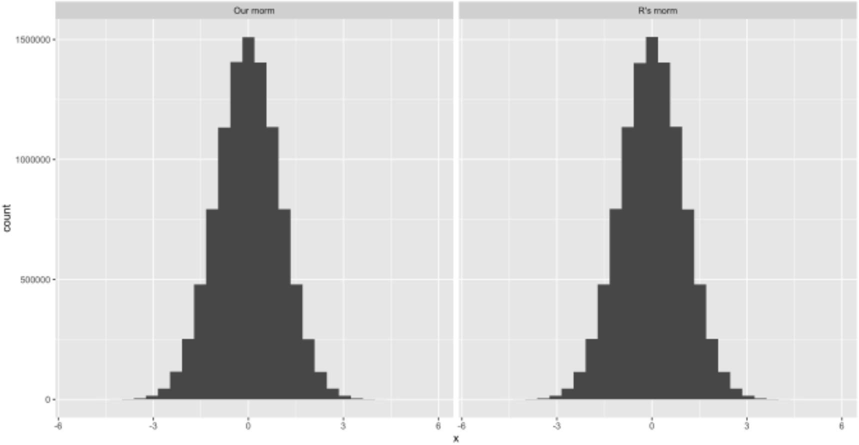


Self-written `rnorm`

Self-written `rnorm`

```
draw.rnorm <- function(n, mean = 0, sd = 1) {
  u <- runif(n)
  z <- qnorm(u, mean=mean, sd=sd)
  return(z)
}
```

Did we succeed?



This Week in Review

- Random Variables!
- Expectation and Variance!
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Next week: inference!

Fun with

Fun with Connections

Fun with Connections

Exponential: $-\log$ Uniform

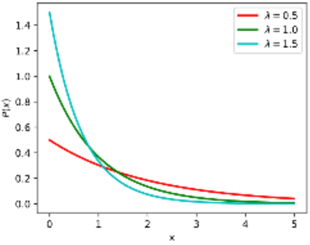
Figure credit: Wikipedia

The Uniform distribution is defined on the interval $(0, 1)$. Suppose we wanted a distribution defined on all positive numbers.

Definition

X follows an **exponential** distribution with rate parameter λ if

$$X \sim -\frac{1}{\lambda} log(U)$$



Exponential: $-\log$ Uniform

The exponential is often used for wait times. For instance, if you're waiting for shooting stars, the time until a star comes might be exponentially distributed.
Key properties:

- Memorylessness: Expected remaining wait time does not depend on the time that has passed
- $E(X) = \frac{1}{\lambda}$
- $V(X) = \frac{1}{\lambda^2}$

Exponential-uniform connection

Suppose $X_1, X_2 \overset{iid}{\sim} Expo(\lambda)$. What is the distribution of $\frac{X_1}{X_1+X_2}$?

Exponential-uniform connection

Suppose $X_1, X_2 \overset{iid}{\sim} Expo(\lambda)$. What is the distribution of $\frac{X_1}{X_1+X_2}$? The proportion of the wait time that is represented by X_1 is uniformly distributed over the interval, so

$$\frac{X_1}{X_1 + X_2} \sim Uniform(0, 1)$$

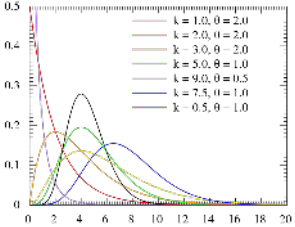
Gamma: Sum of independent Exponentials

Figure credit: Wikipedia

Definition

Suppose we are waiting for a shooting stars, with the time between stars $X_1, \ldots, X_a \overset{iid}{\sim} Expo(\lambda)$. The distribution of time until the ath shooting star is

$$G \sim \sum_{i=1}^{a} X_i \sim Gamma(a, \lambda)$$



Gamma: Properties

Properties of the *Gamma*(a, λ) distribution include:

- $E(G) = \frac{a}{\lambda}$
- $V(G) = \frac{a}{\lambda^2}$

Beta: Uniform order statistics

Suppose we draw $U_1, \ldots, U_k \sim \mathit{Uniform}(0,1)$, and we want to know the distribution of the jth order statistic, $U_{(j)}$. Using the Uniform-Exponential connection, we could also think of these $U_{(j)}$ as being the location of the jth Exponential in a series of $k+1$ Exponentials. Thus,

$$U_{(j)} \sim \frac{\sum_{i=1}^{j} X_i}{\sum_{i=1}^{j} X_i + \sum_{i=j+1}^{k+1} X_i} \sim \mathit{Beta}(j, k-j+1)$$

This defines the **Beta distribution**.
Can we name the distribution at the top of the fraction?

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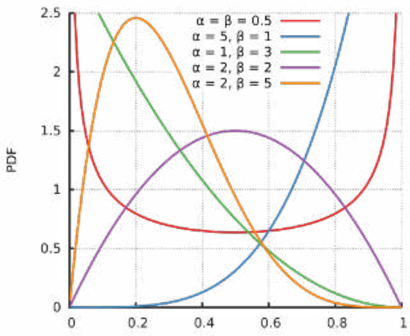
$$\sim \frac{G_j}{G_j + G_{k-j+1}}$$

What do Betas look like?

Figure credit: Wikipedia



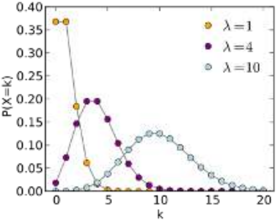
Poisson: Number of Exponential events in a time interval

Figure credit: Wikipedia

Definition

Suppose the time between shooting stars is distributed $X \sim Expo(\lambda)$. Then, the number of shooting stars in an interval of time t is distributed

$$Y_t \sim Poisson(\lambda t)$$



Poisson: Number of Exponential events in a time interval

Properties of the Poisson:

- If $Y \sim Pois(\lambda t)$, then $V(Y) = E(Y) = \lambda t$
- Number of events in disjoint intervals are independent

χ_n^2: A particular Gamma

Definition

We define the **chi-squared distribution** with n degrees of freedom as

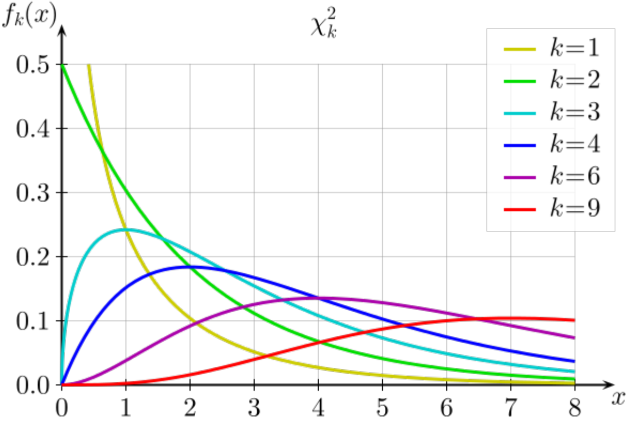
$$\chi_n^2 \sim Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

More commonly, we think of it as the sum of a series of independent squared Normals, $Z_1, \ldots, Z_n \overset{iid}{\sim} Normal(0, 1)$:

$$\chi_n^2 \sim \sum_{i=1}^{n} Z_i^2$$

χ^2_n: A particular Gamma

Figure credit: Wikipedia

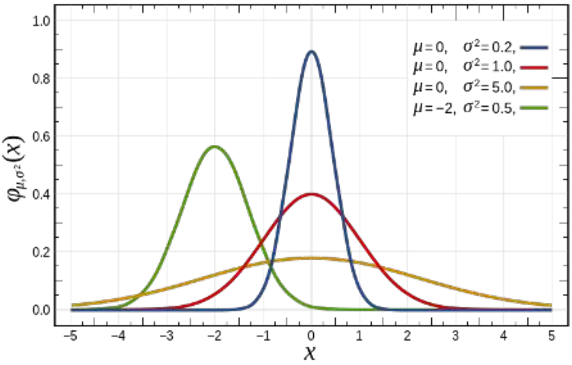


Normal: Square root of χ^2_1 with a random sign

Figure credit: Wikipedia

Definition

Z follows a **Normal** distribution if $Z \sim S\sqrt{\chi^2_1}$, where S is a random sign with equal probability of being 1 or -1.



Normal: An alternate construction

Note: This is far above and beyond what you need to understand for the course!

Box-Muller Representation of the Normal

Let $U_1, U_2 \overset{iid}{\sim} Uniform$. Then

$$Z_1 \equiv \sqrt{-2\log U_2}\cos(2\pi U_1)$$

$$Z_2 \equiv \sqrt{-2\log U_2}\sin(2\pi U_1)$$

so that $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

What is this?

- **Inside the square root:** $-2\log U_2 \sim Expo\left(\frac{1}{2}\right) \sim Gamma\left(\frac{2}{2}, \frac{1}{2}\right) \sim \chi^2_2$
- **Inside the cosine and sine:** $2\pi U_1$ is a uniformly distributed angle in polar coordinates, and the cos and sin convert this to the cartesian x and y components, respectively.
- **Altogether in polar coordinates:** The square root part is like a radius distributed $\chi \sim |Z|$, and the second part is an angle.

One Step Deeper: Exponential Family

Nearly every distribution we will discuss is in the exponential family. An exponential family distribution has the density of the following form:

$$f_Y(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

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Example: Poisson(μ):

$$P(Y_i = y \mid \mu) \;=\; \exp\left\{y\log\mu - \exp(\log\mu) - \log y!\right\}$$

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Many other examples, including: Normal, Bernoulli/binomial, Gamma, multinomial, exponential, negative binomial, beta, uniform, chi-squared, etc.

This slide and the following based on material from Teppei Yamamoto

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  $\Rightarrow \mathbb{E}(Y_i) = \frac{db(\theta_i)}{d\theta_i} = \exp(\theta_i) = \mu_i$ and $\mathbb{V}(Y_i) = \frac{d^2 b(\theta_i)}{d\theta_i^2} = \exp(\theta_i) = \mu_i$