Week 8: What Can Go Wrong and How To Fix It, Diagnostics and Solutions

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Princeton

October 31–November 2, 2022

[1]These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller, Erin Hartman and Kevin Quinn.

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  - multiple regression

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- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

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Regrettably we won't have time to cover many topics that are important like missing data, sensitivity analysis, etc.

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   - Polynomials
   - LOESS
   - Log Transformations
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   - Hat Matrix
   - Standardized and Studentized Residuals
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Y &= \beta_0 + (\beta_1 + \beta_2) X_1 + \beta_3 X_1 X_1 + u \\
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- A third order polynomial is given by:
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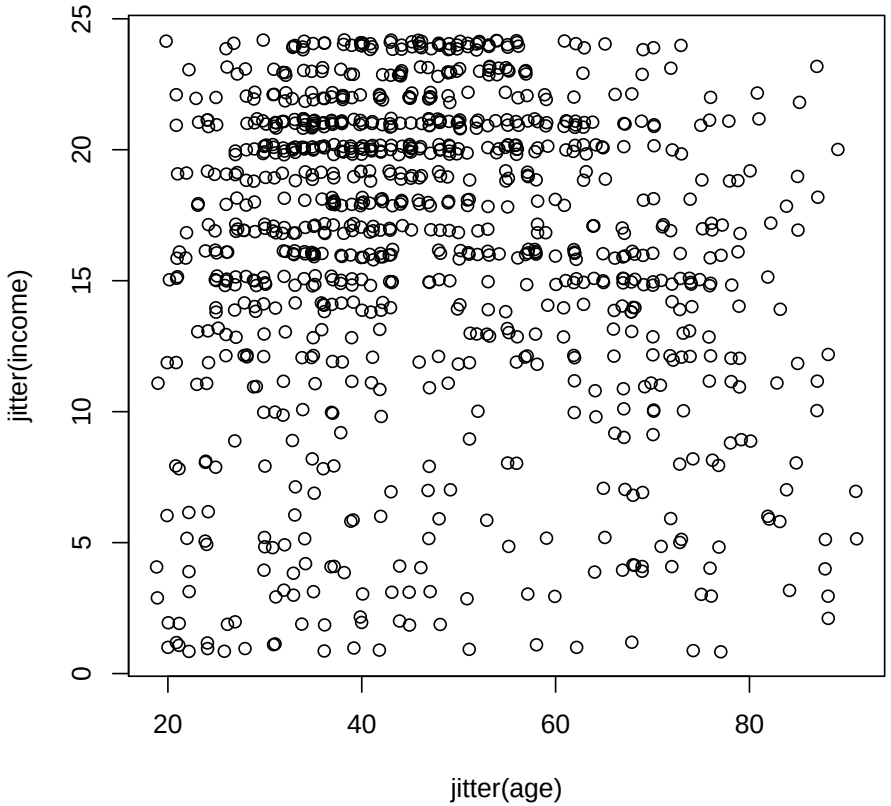
Polynomial Example: Income and Age

- Let's look at data from the U.S. and examine the relationship between Y: income and X: age



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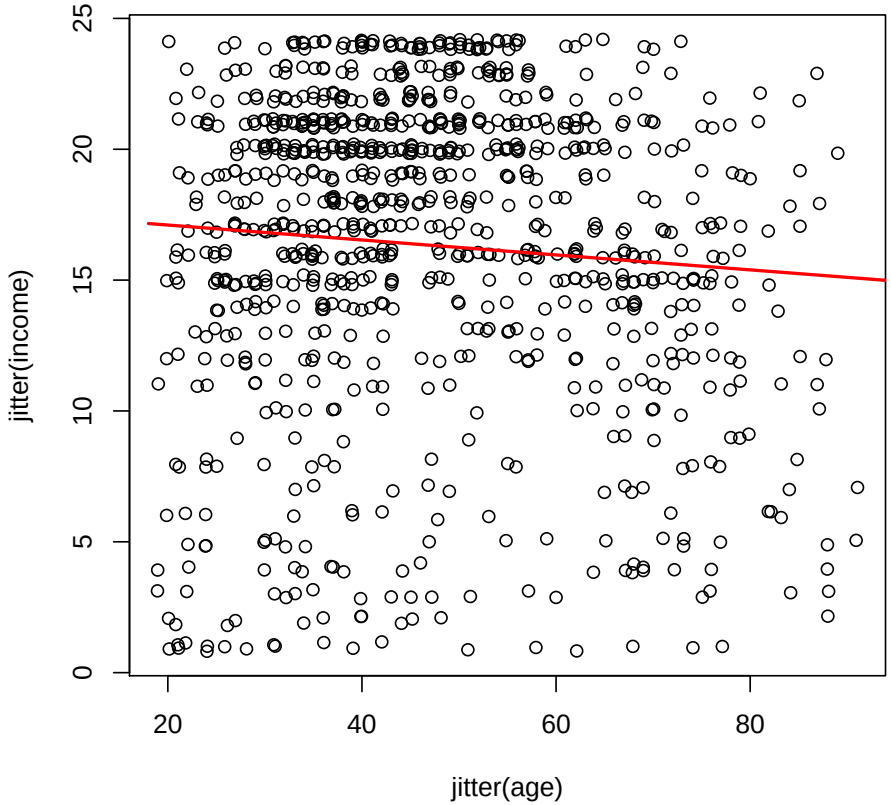

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- We see that a simple linear specification does not fit the data very well:
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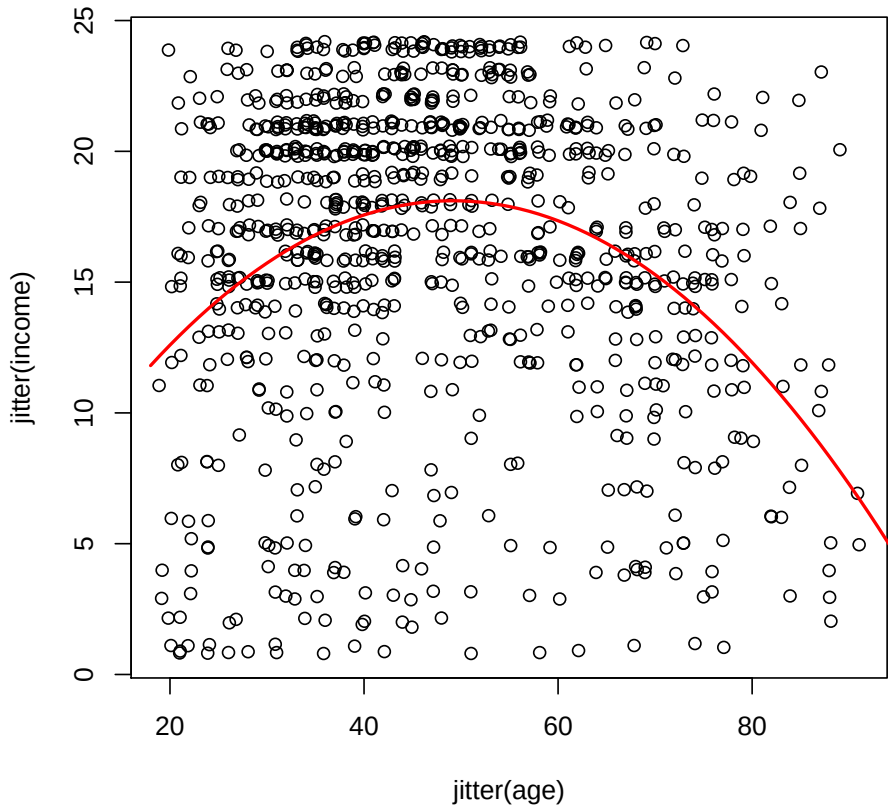

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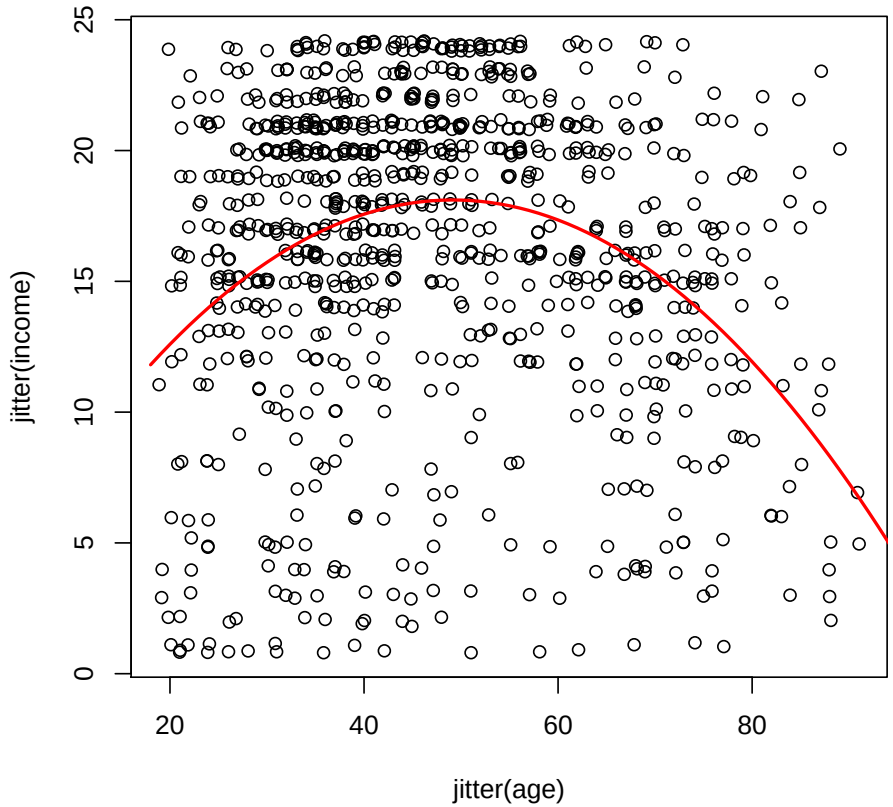

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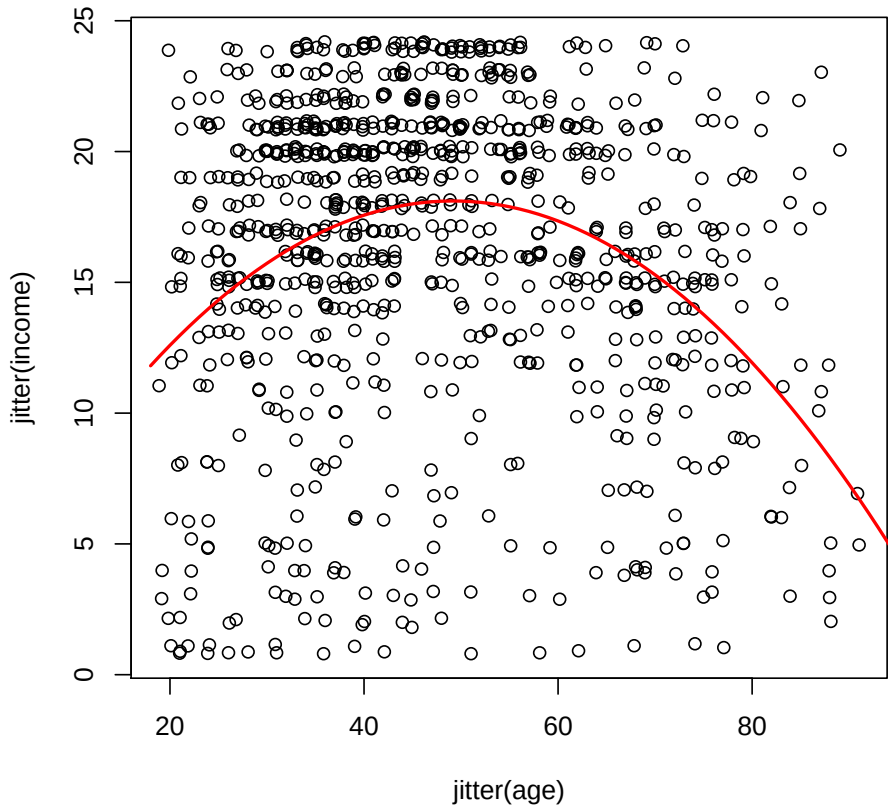

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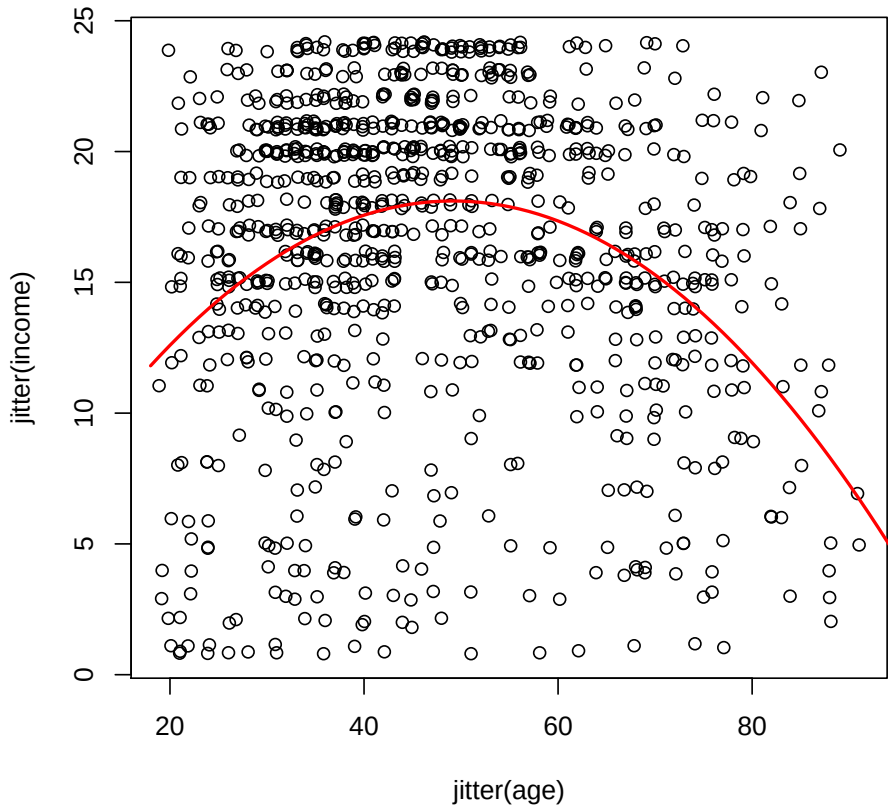

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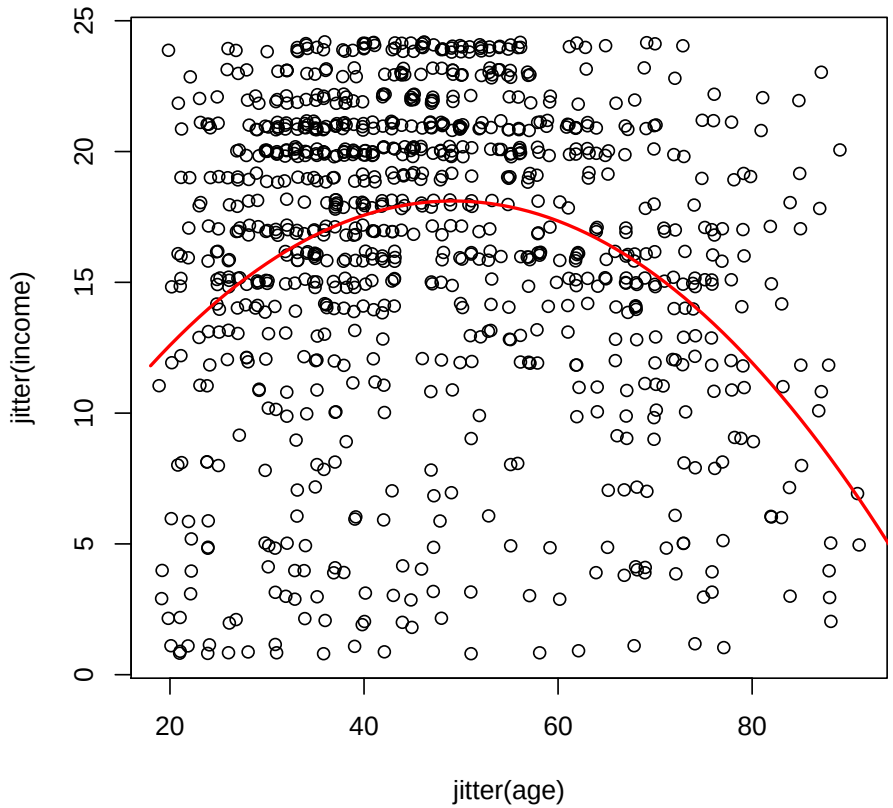

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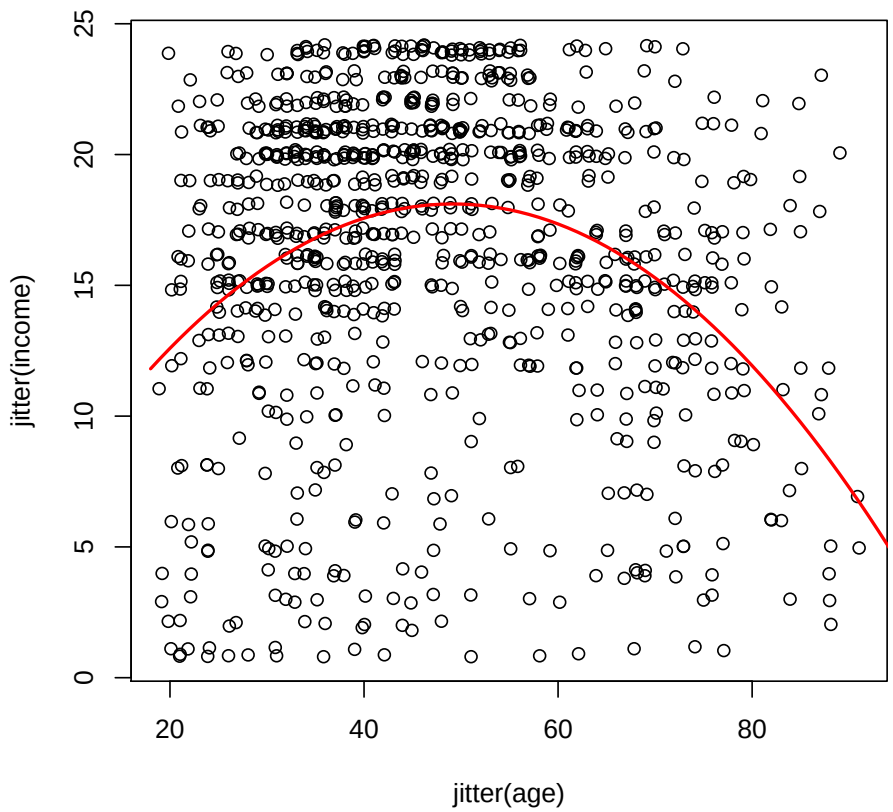
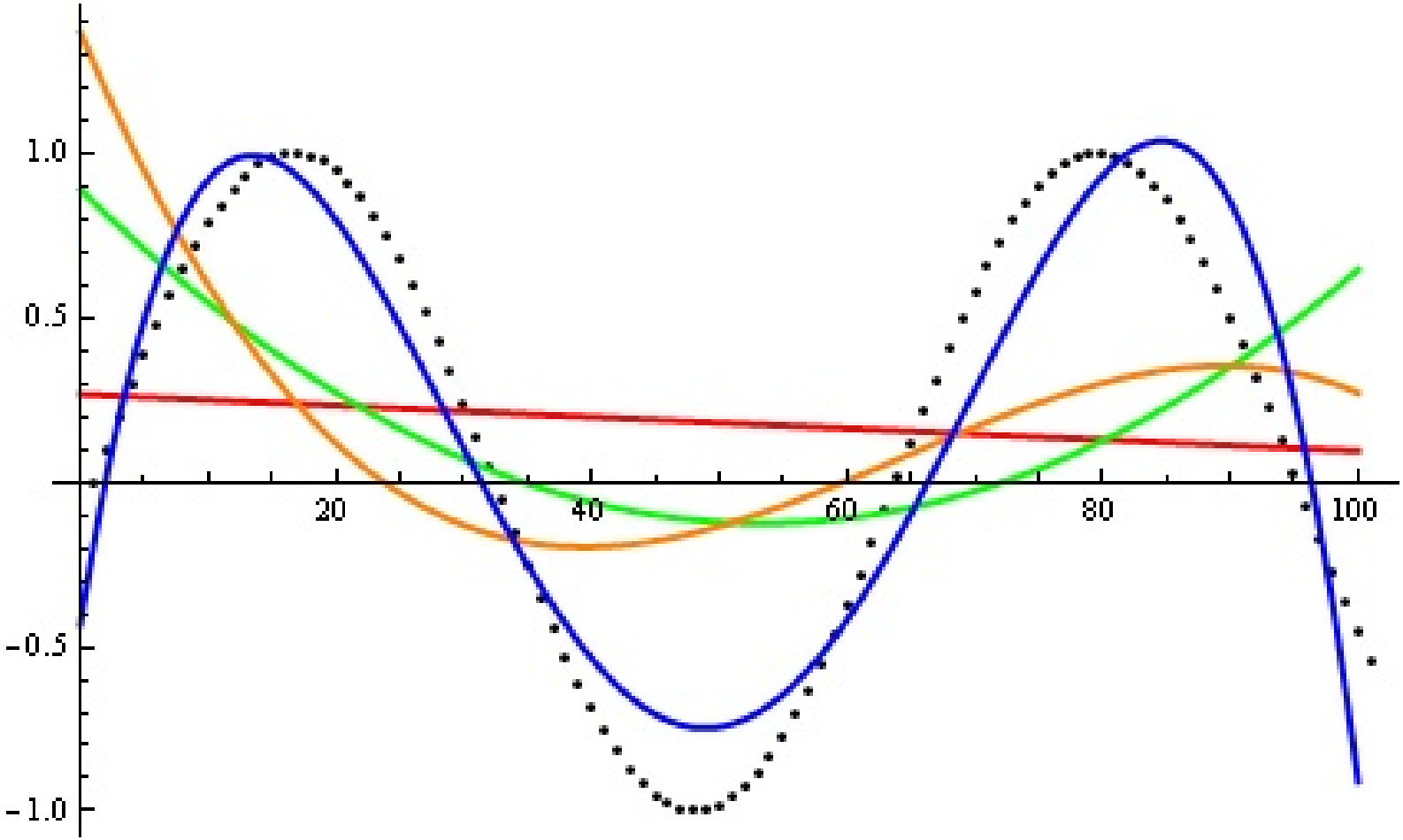
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.
- Maximum/Minimum occurs at $|\frac{\beta_1}{2\beta_2}|$. Here turning point is at $X_1 = 50$.



Higher Order Polynomials



Approximating data generated with a sine function. Red line is a first degree polynomial, green line is second degree, orange line is third degree and blue is fourth degree

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- Getting uncertainty on this quantity is a great use case for the bootstrap!

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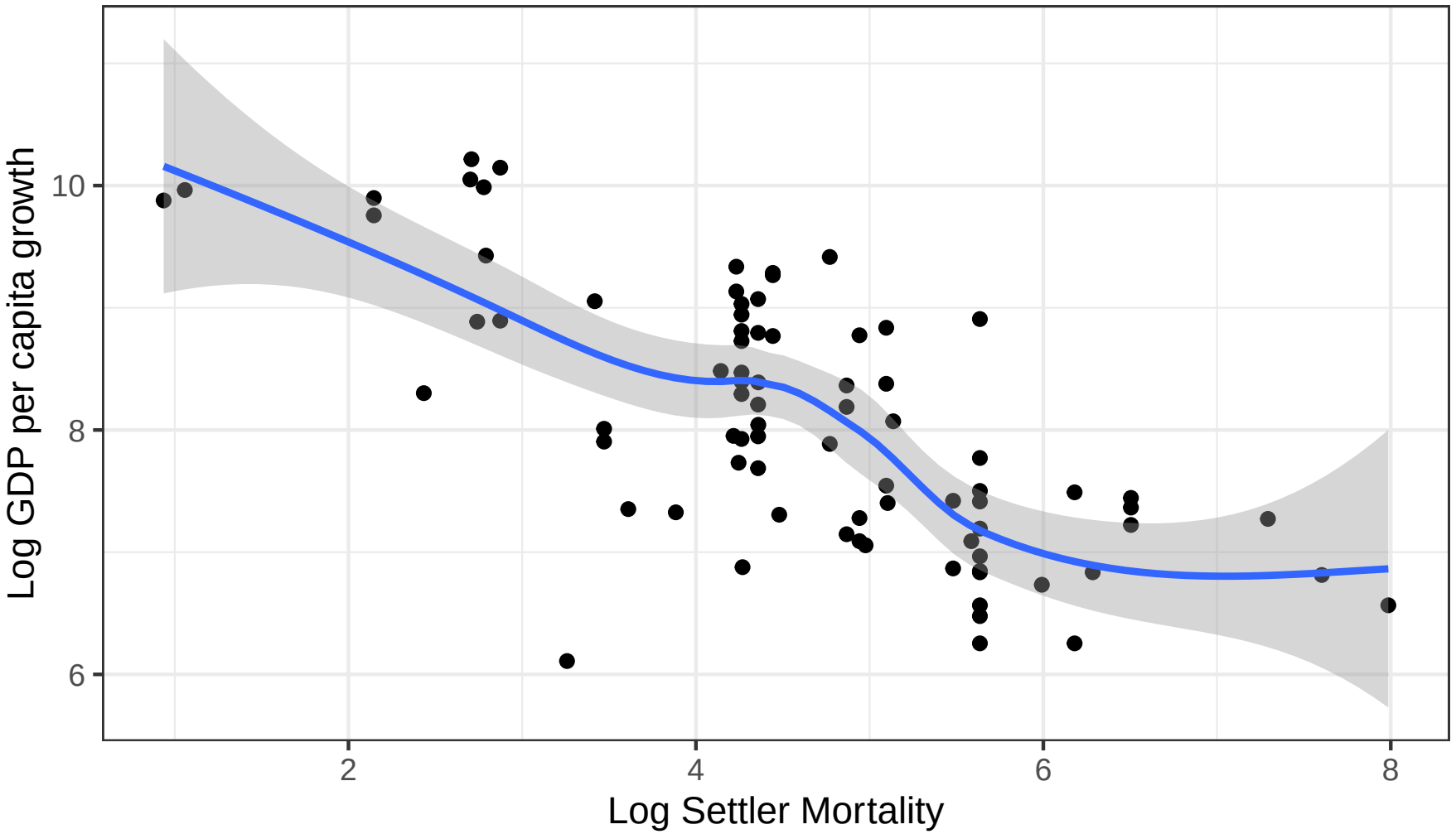
So what is ggplot2 doing?



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  3. Use the fitted regression line to predict the expected value of $E[Y|X = x_0]$

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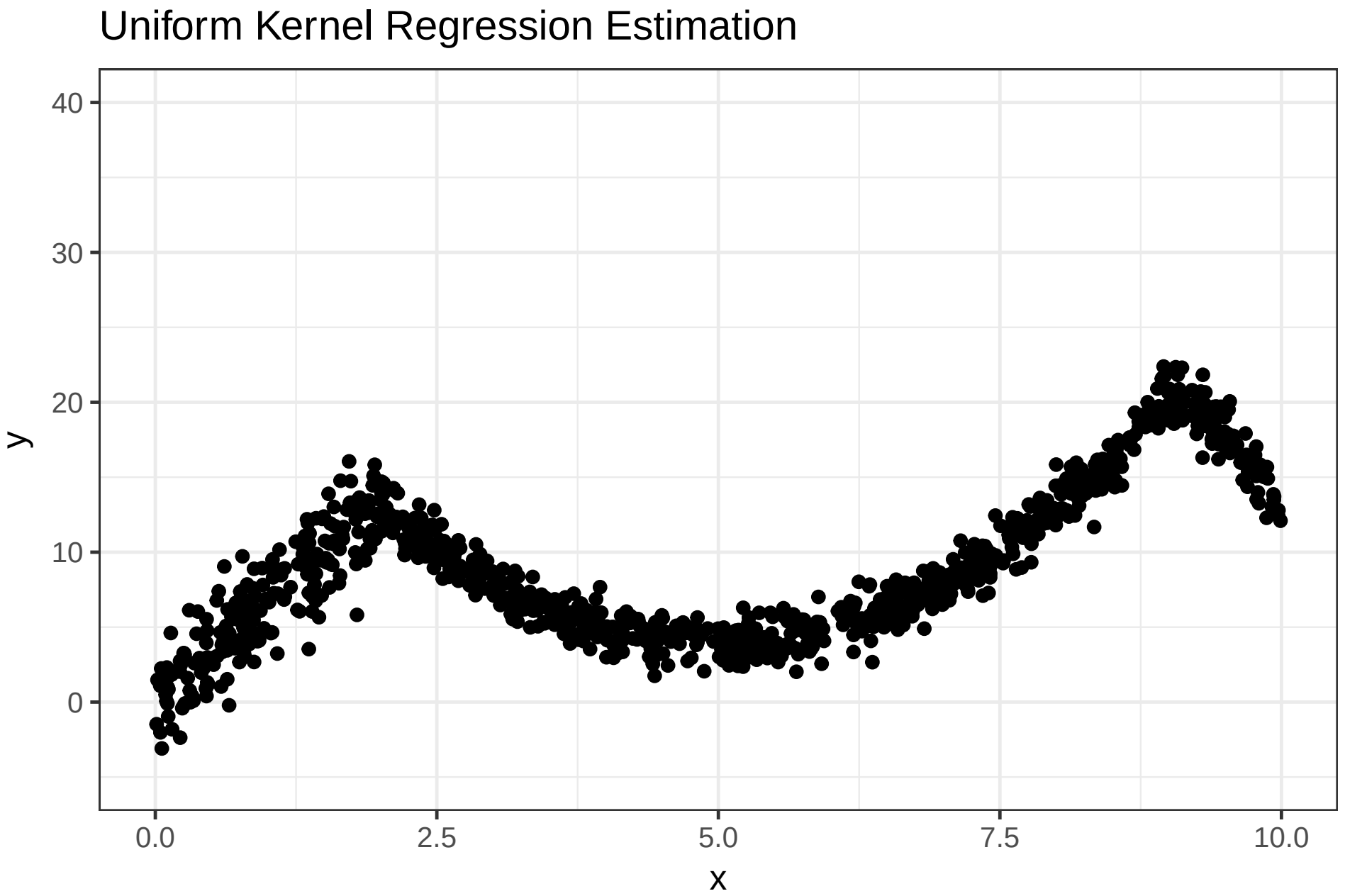

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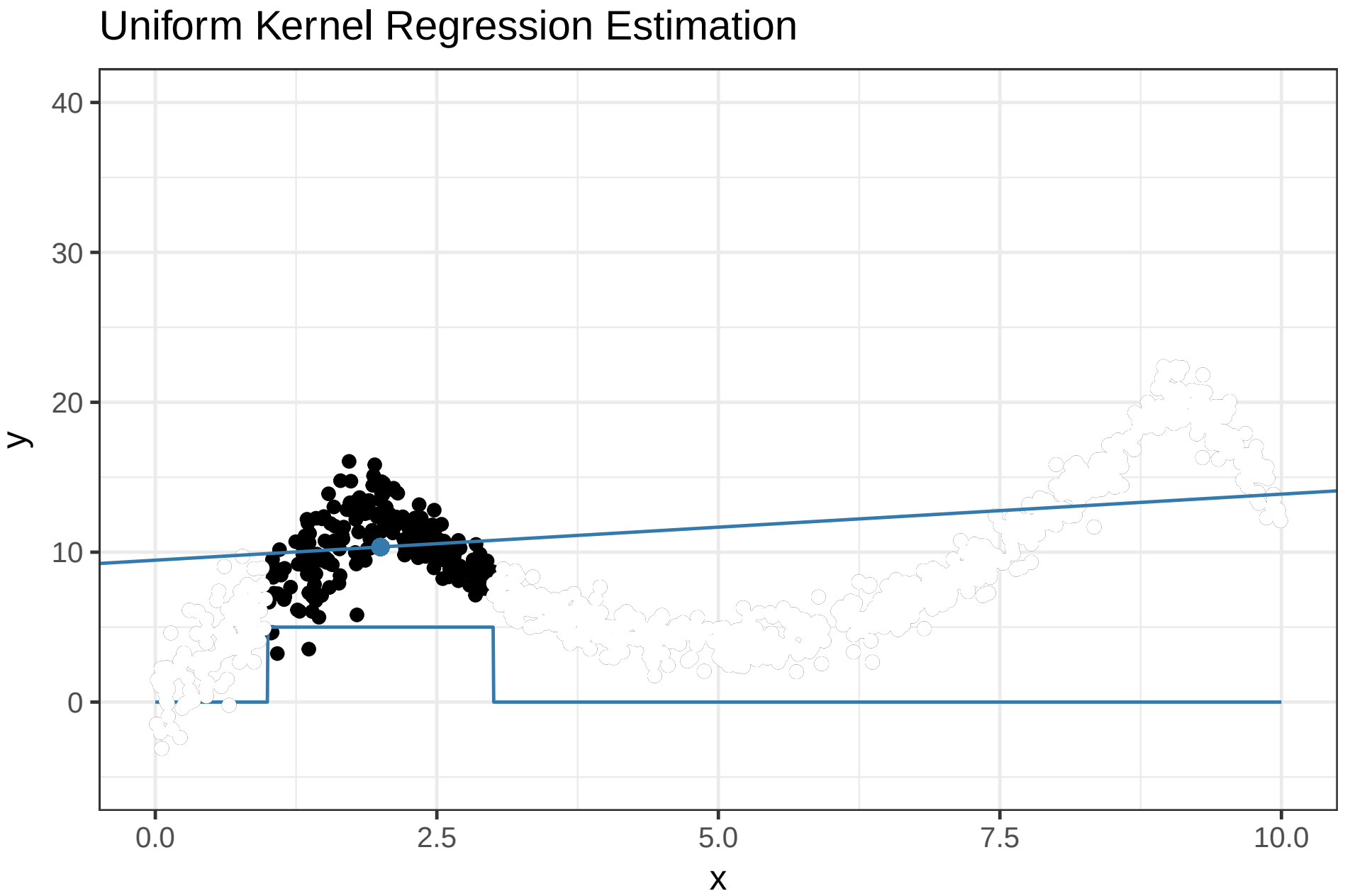

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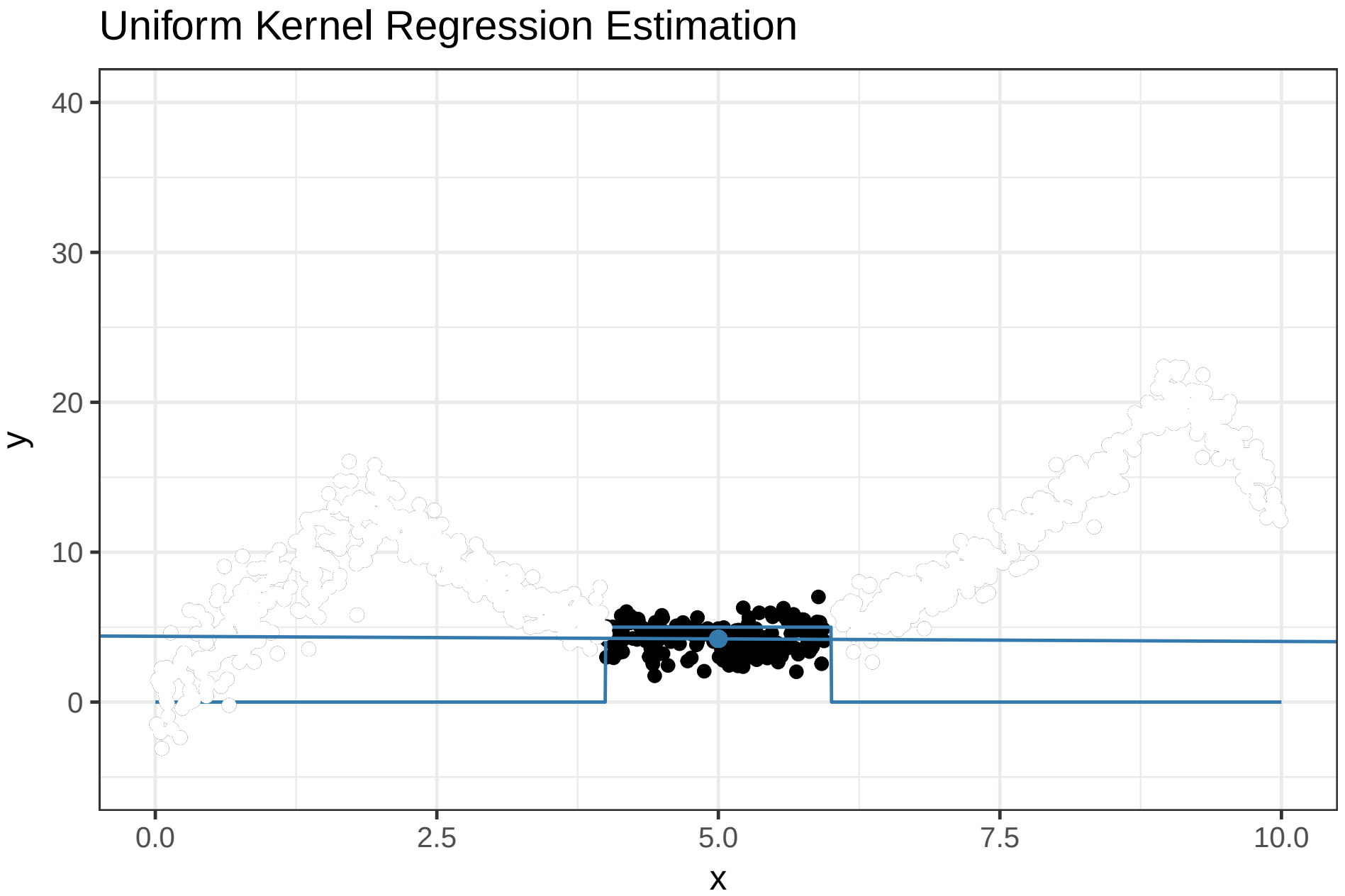

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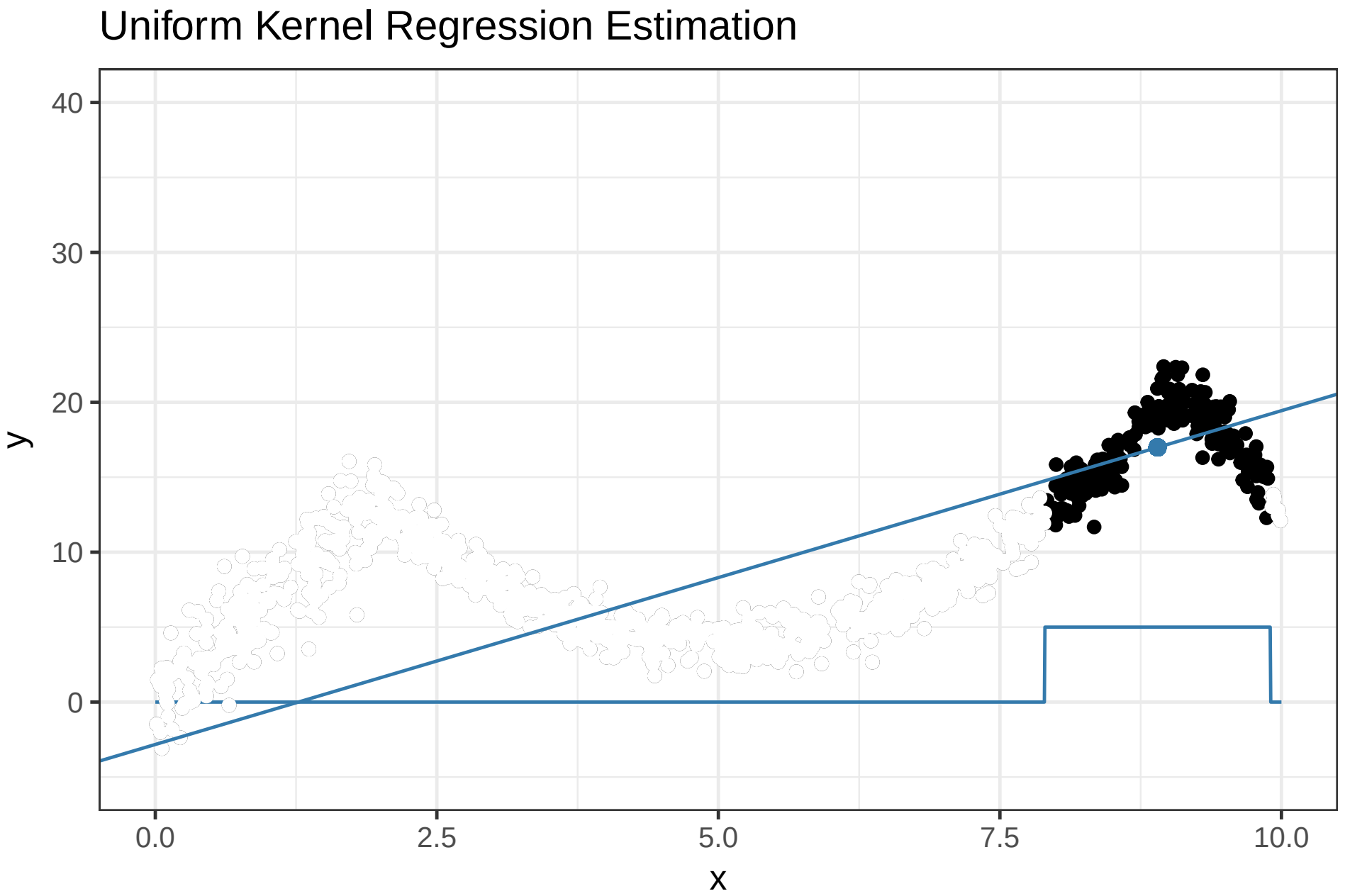

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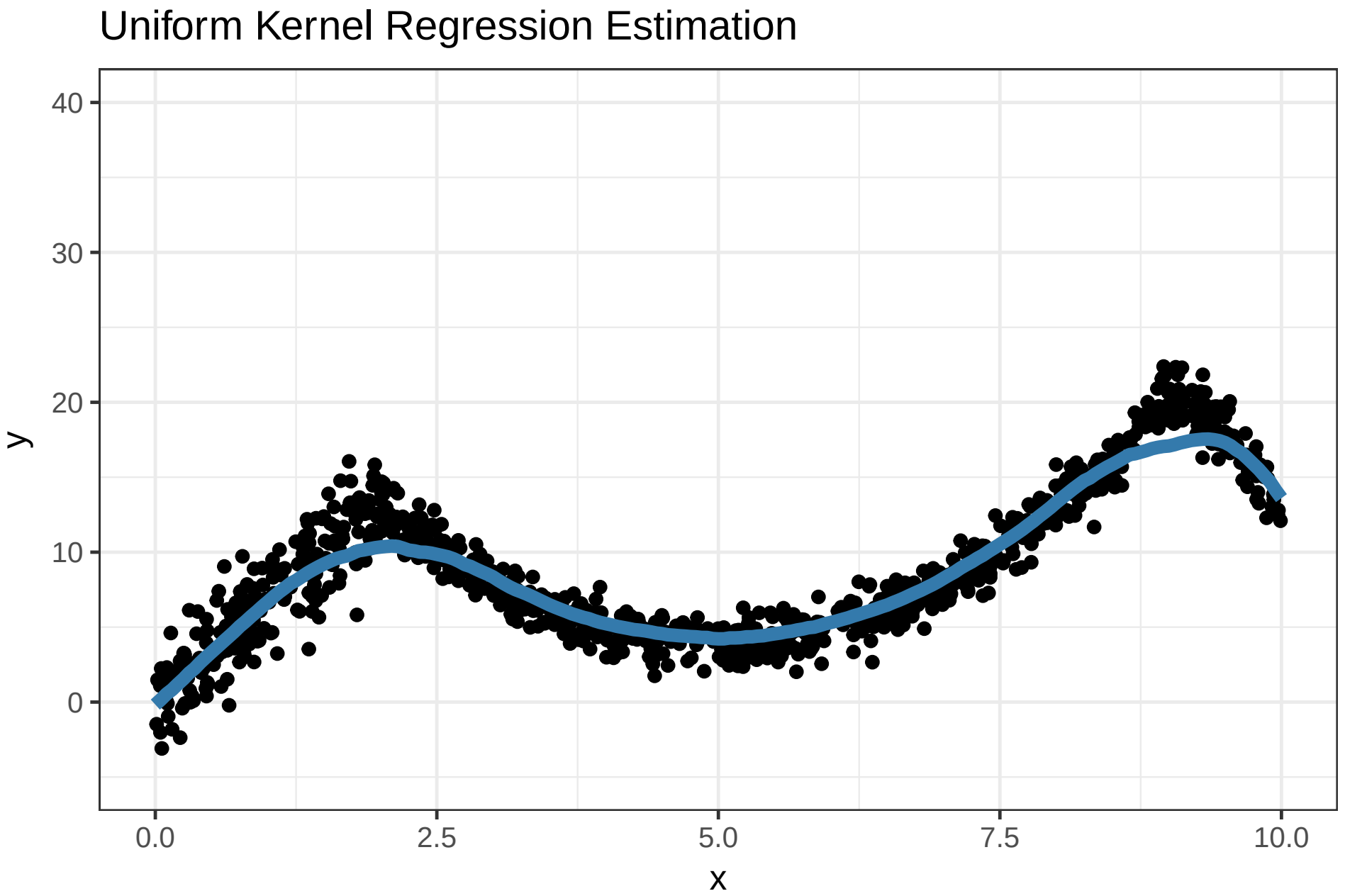

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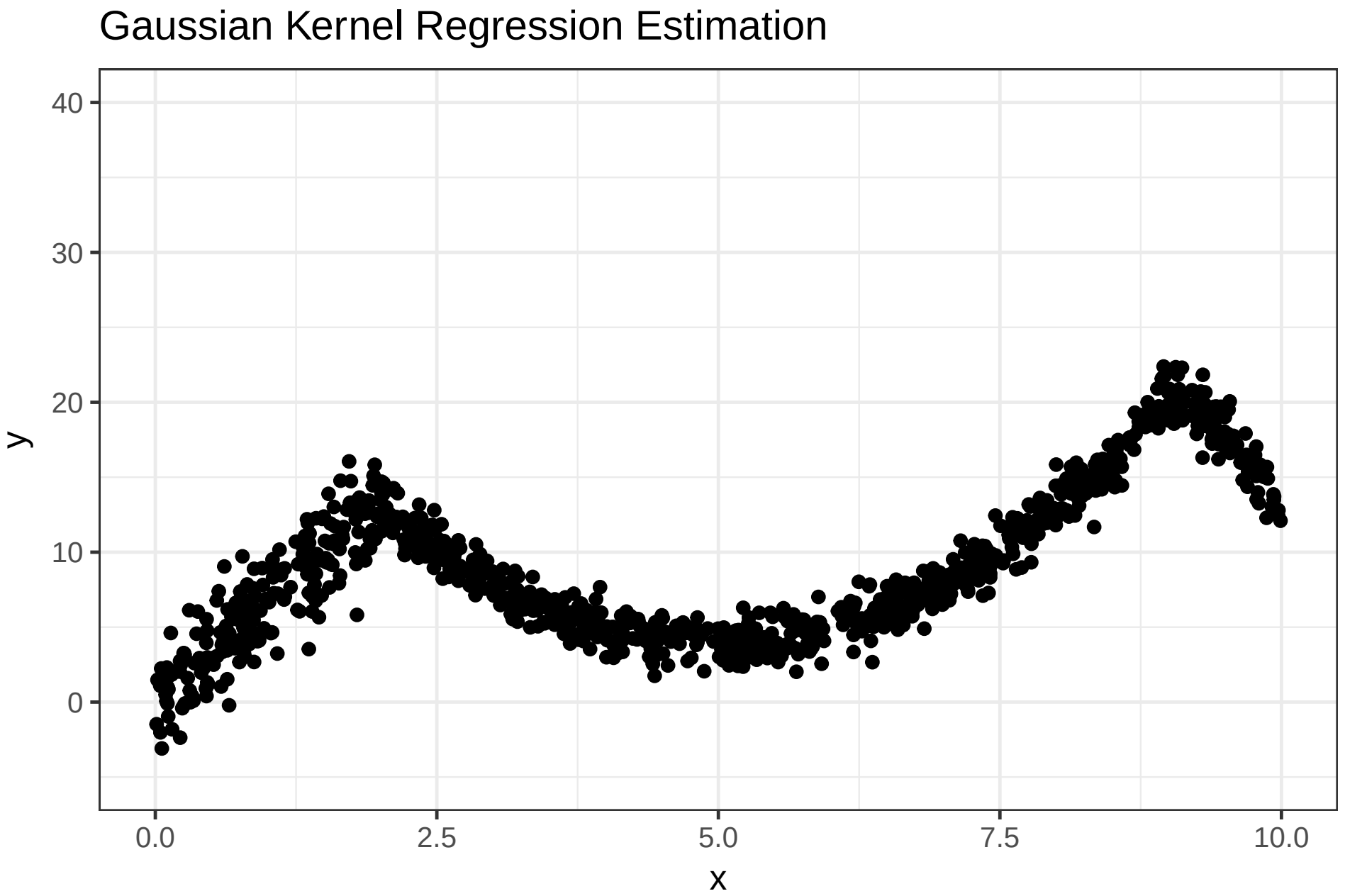

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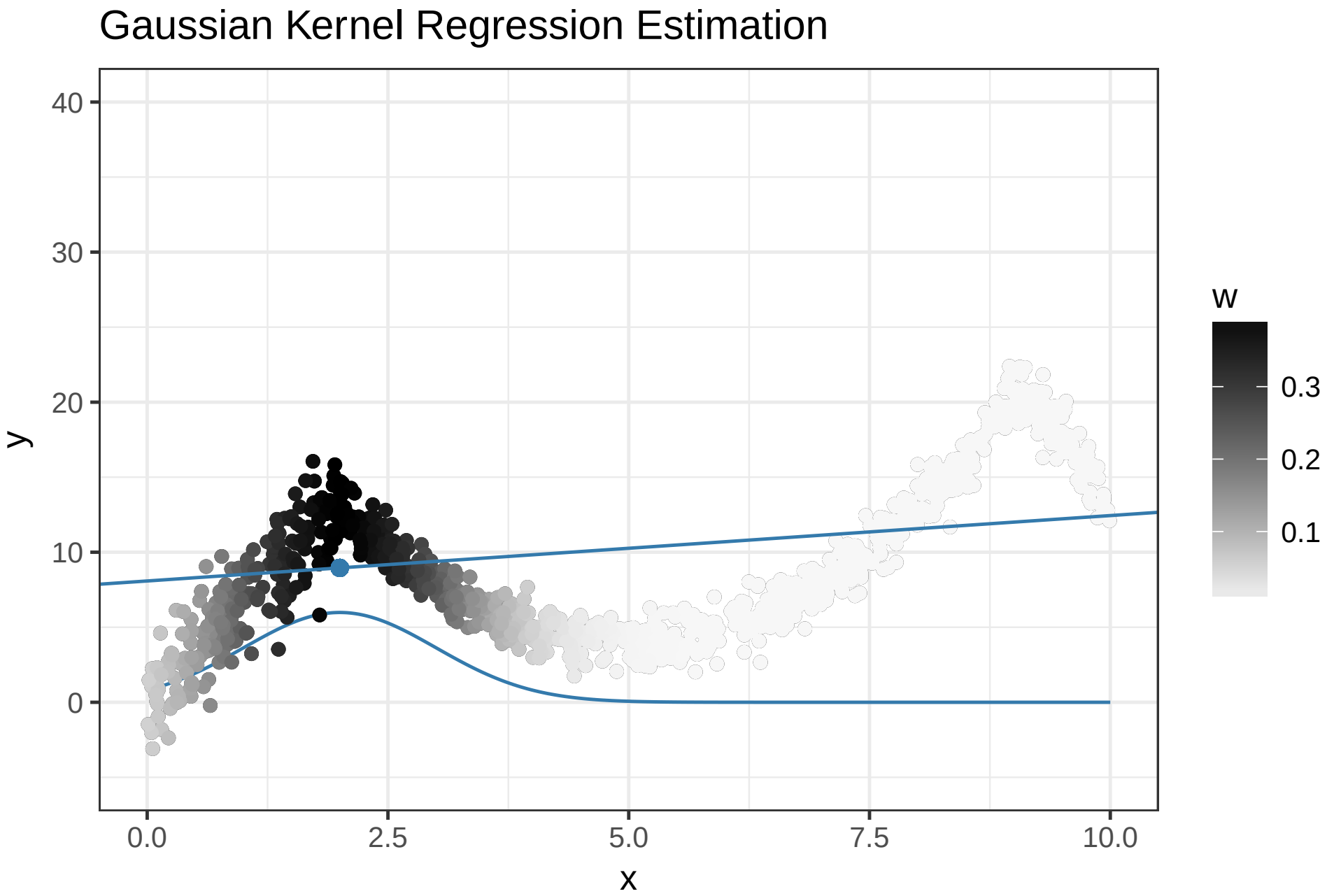

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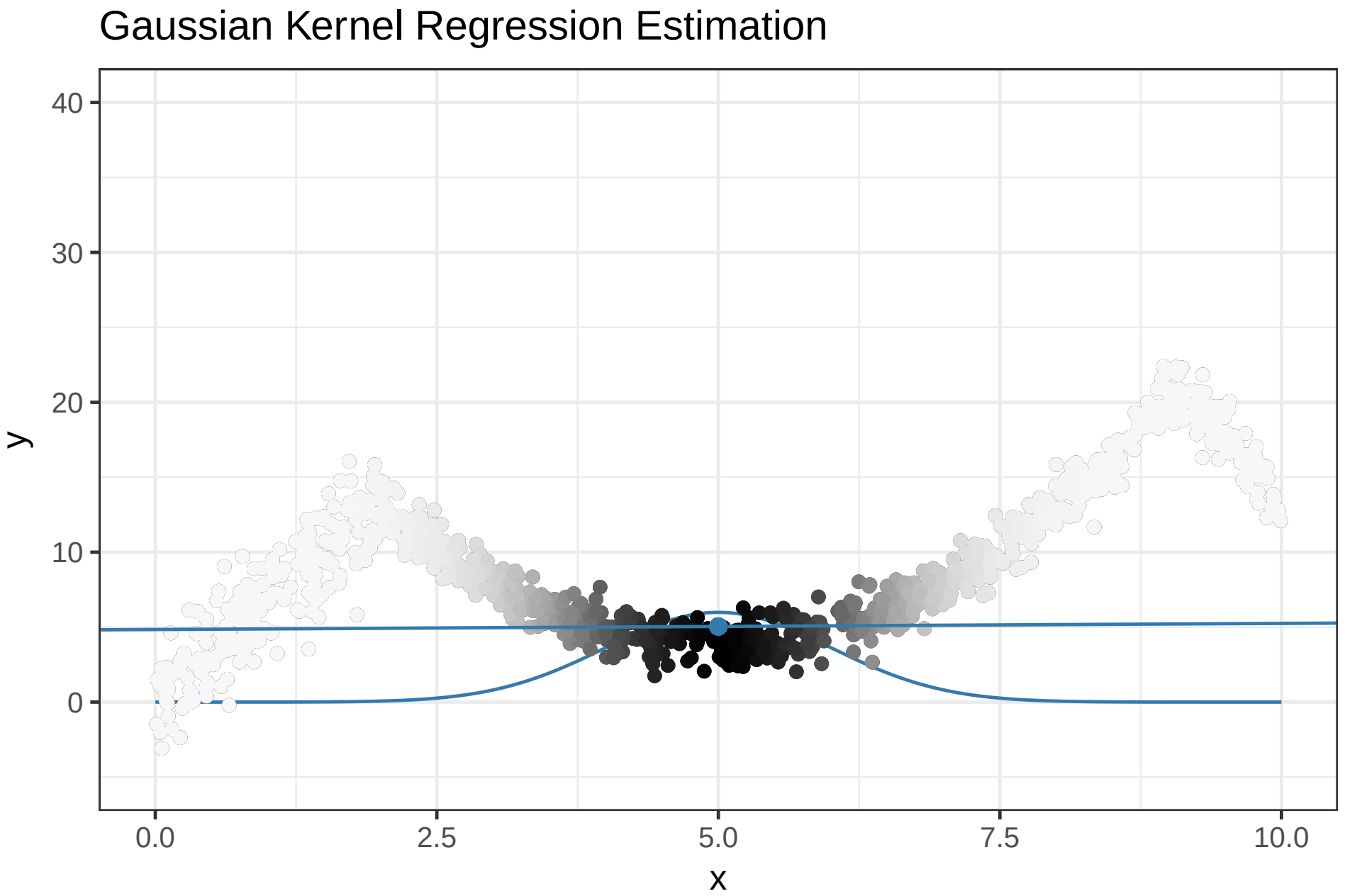

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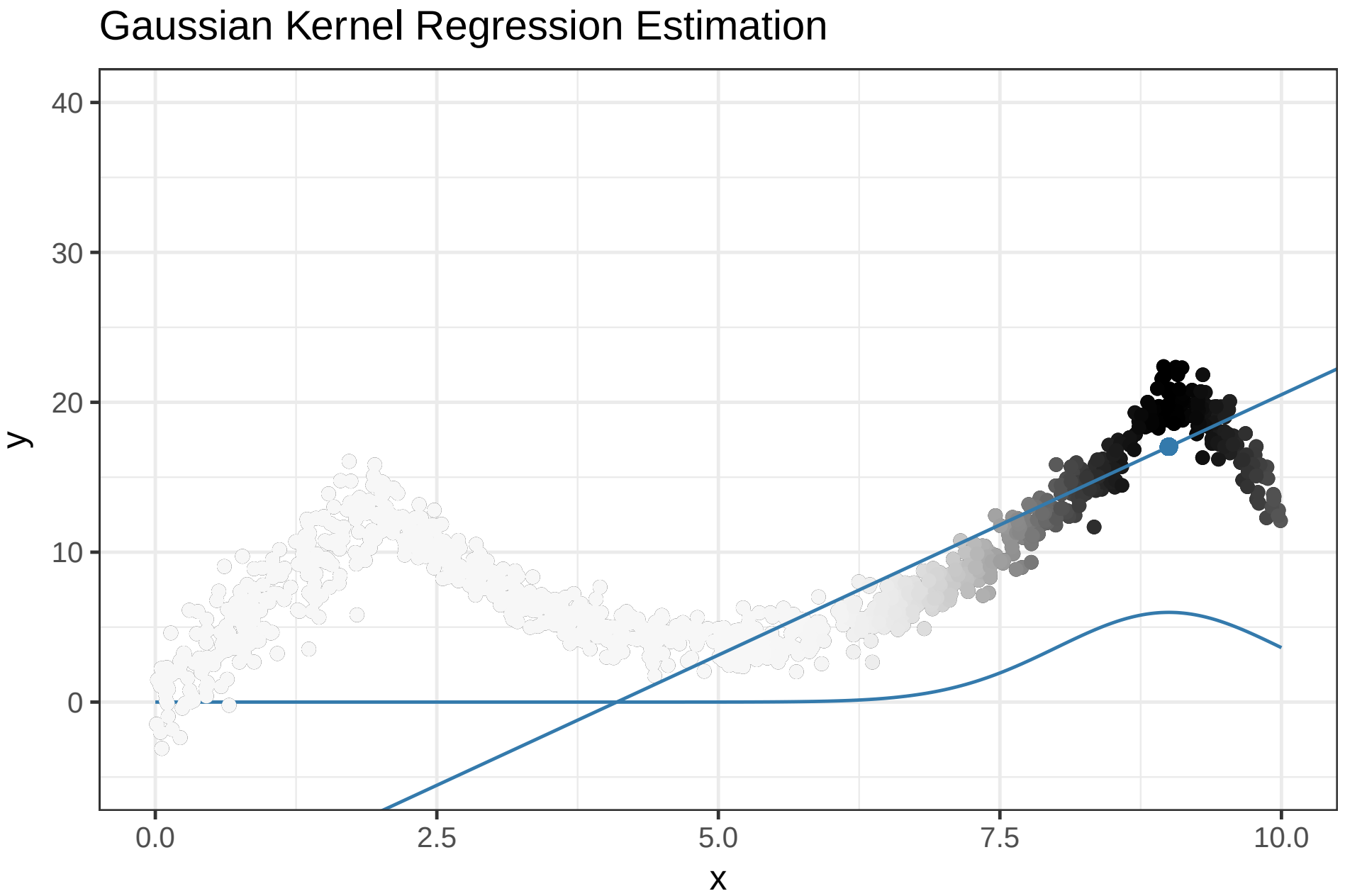

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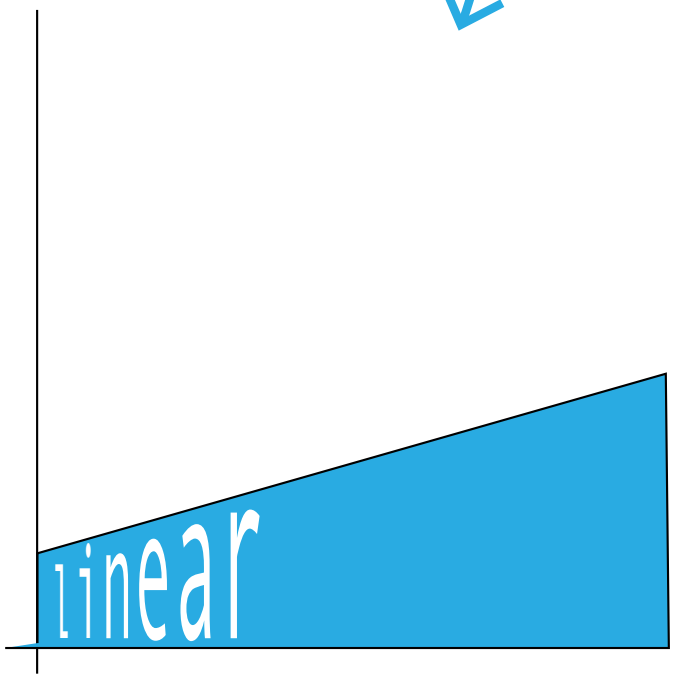
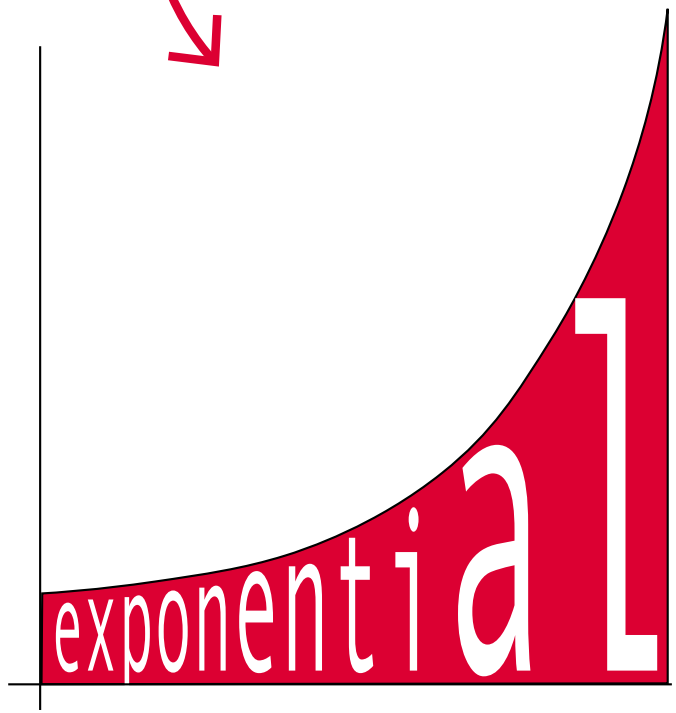
- Many of these non-linear transformations are made by creating multiple variables out of a single X and so will have to wait for future weeks.
- The function $\log(\cdot)$ is one common transformation that has only one parameter.
- This is particularly useful for positive and right-skewed variables.

Why does everyone keep logging stuff??

Logs linearize exponential growth.



grows by a fixed amount.



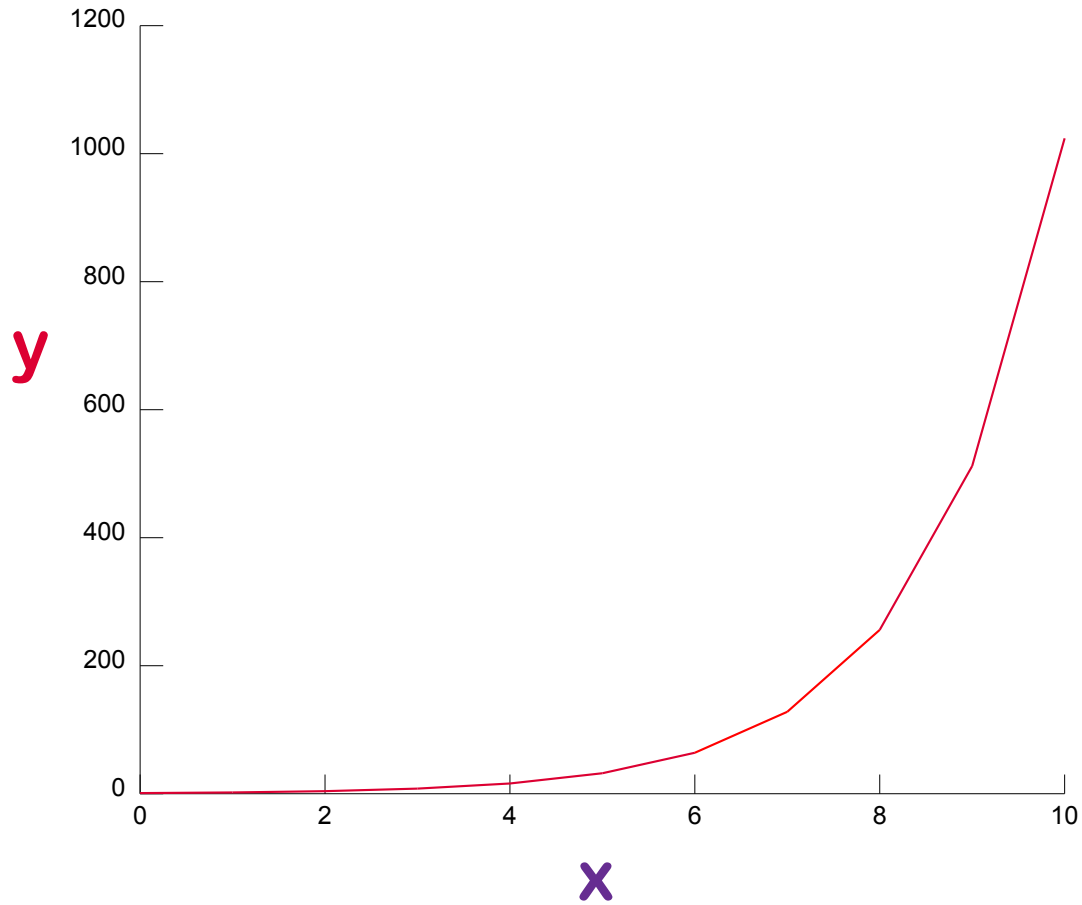
grows by a fixed percent.

How? Let's look.

First, here's a graph showing exponential growth.

We're going to use $y = 2^x$, but any other exponent will work

x	$y=(2^x)$
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024



What happens when we take the log of y?

$$\log y = z \qquad e^z = y$$

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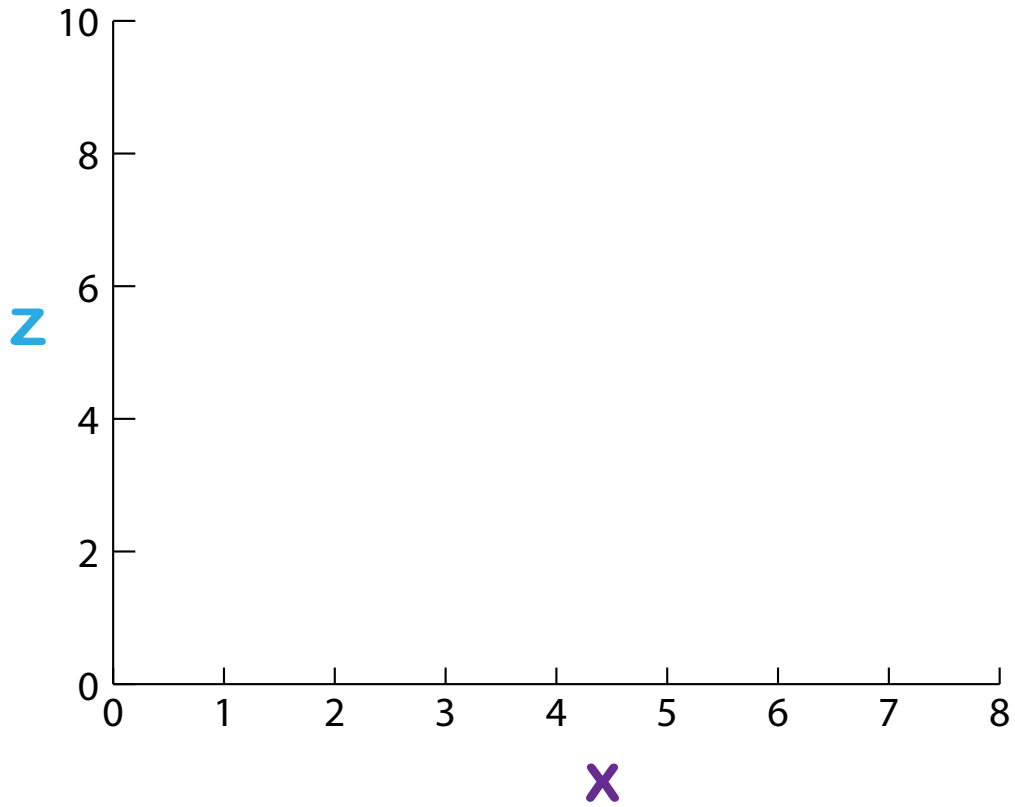

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$\log 1 = 0$ $\quad$ $e^0 = 1$

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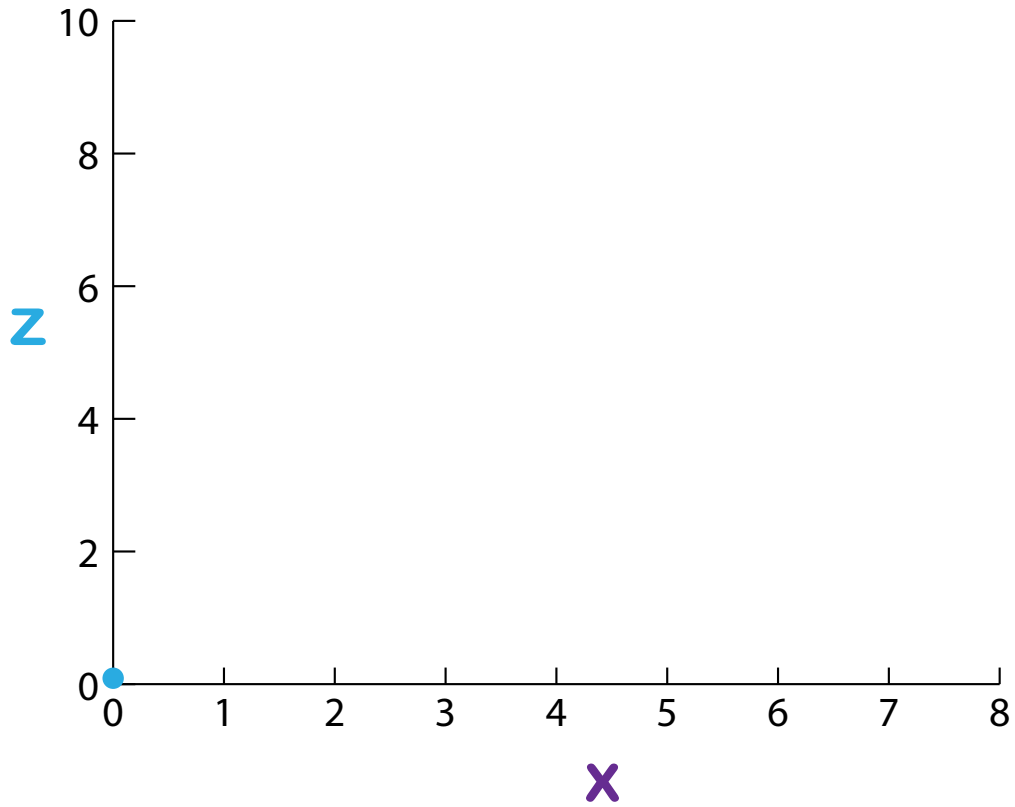

What happens when we take the log of y?

$\log 2 = .69 \qquad e^{.69} = 2$

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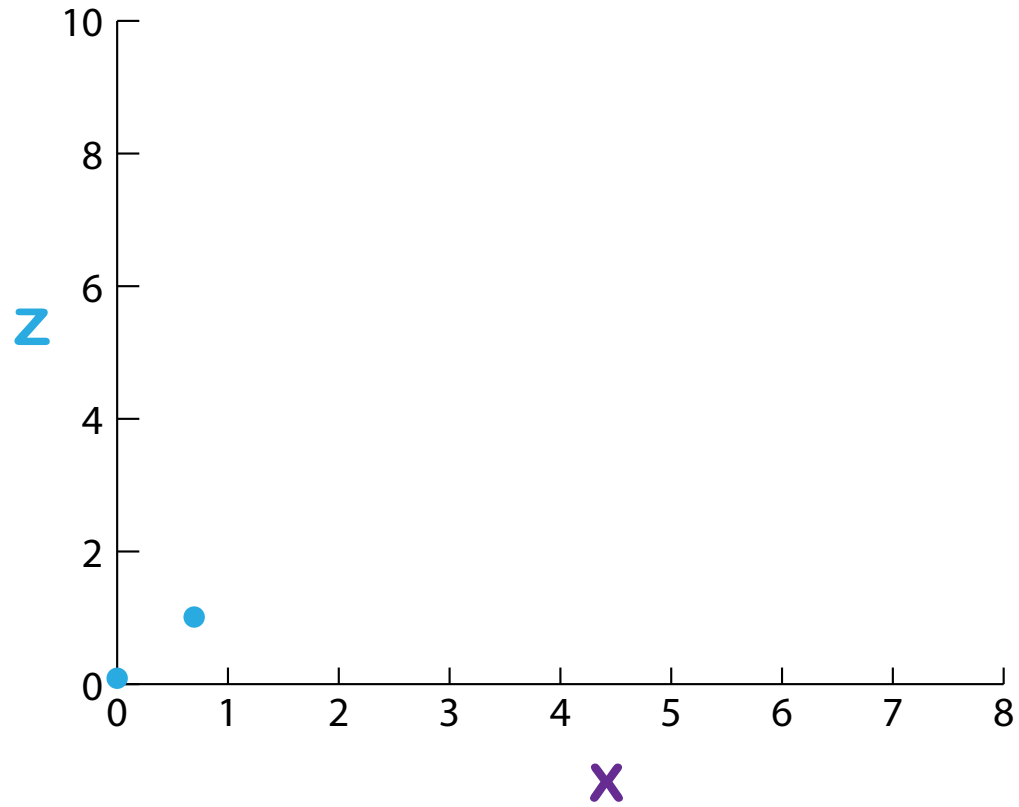

What happens when we take the log of y?

$$\log 4 = 1.39 \qquad e^{1.39} = 4$$

We're going to use $y = 2^x$, but any other exponent will work

x	y	z
0	1	0
1	2	.69
2	4	1.39
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
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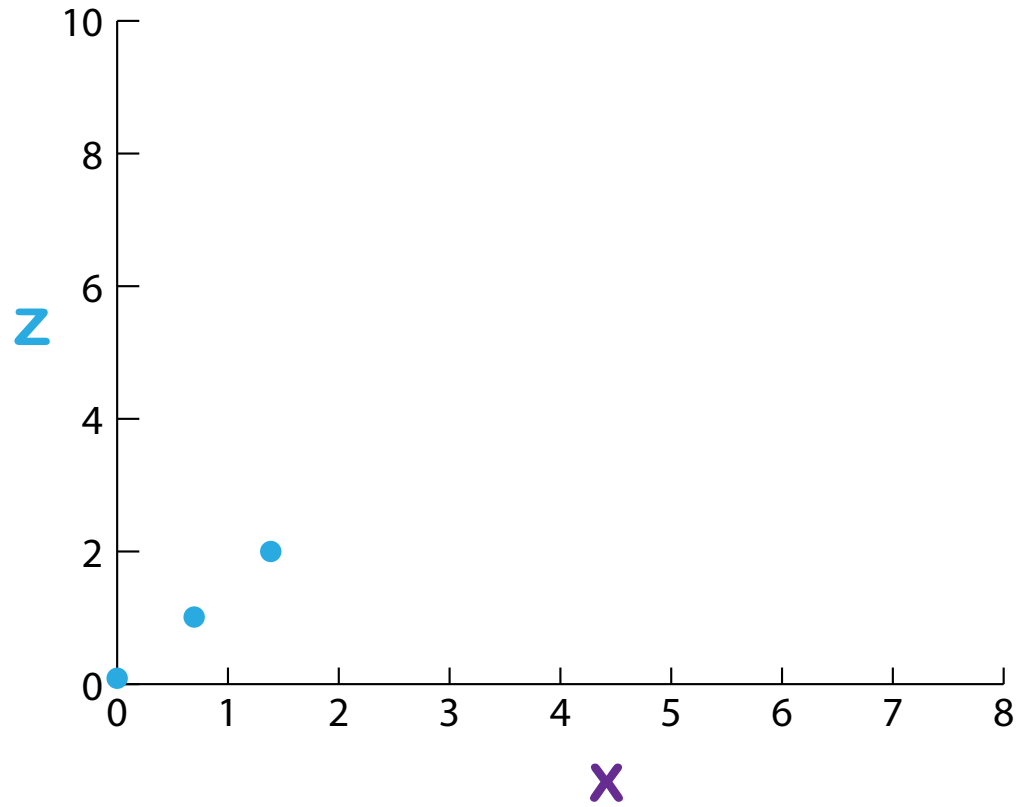

What happens when we take the log of y?

$$\log 8 = 2.08 \qquad e^{2.08} = 8$$

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0	1	0
1	2	.69
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3	8	2.08
4	16	
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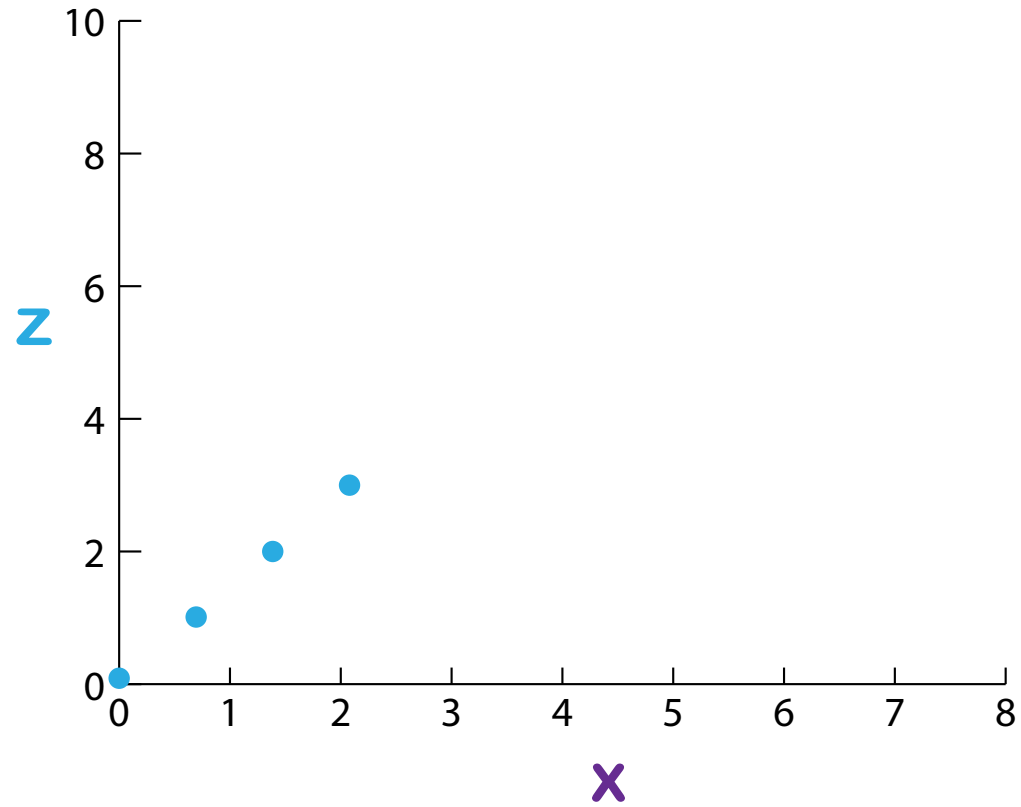

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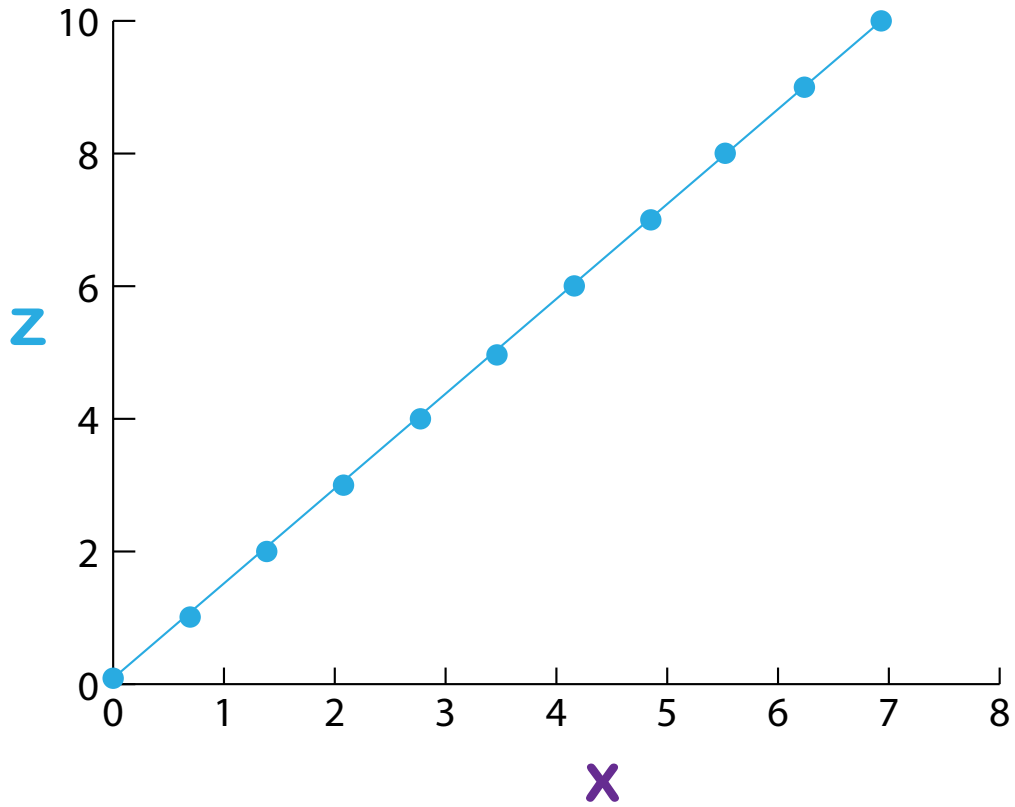
0	1	0
1	2	.69
2	4	1.39
3	8	2.08
4	16	2.77
5	32	3.47
6	64	4.16
7	128	4.85
8	256	5.55
9	512	6.24
10	1024	6.93



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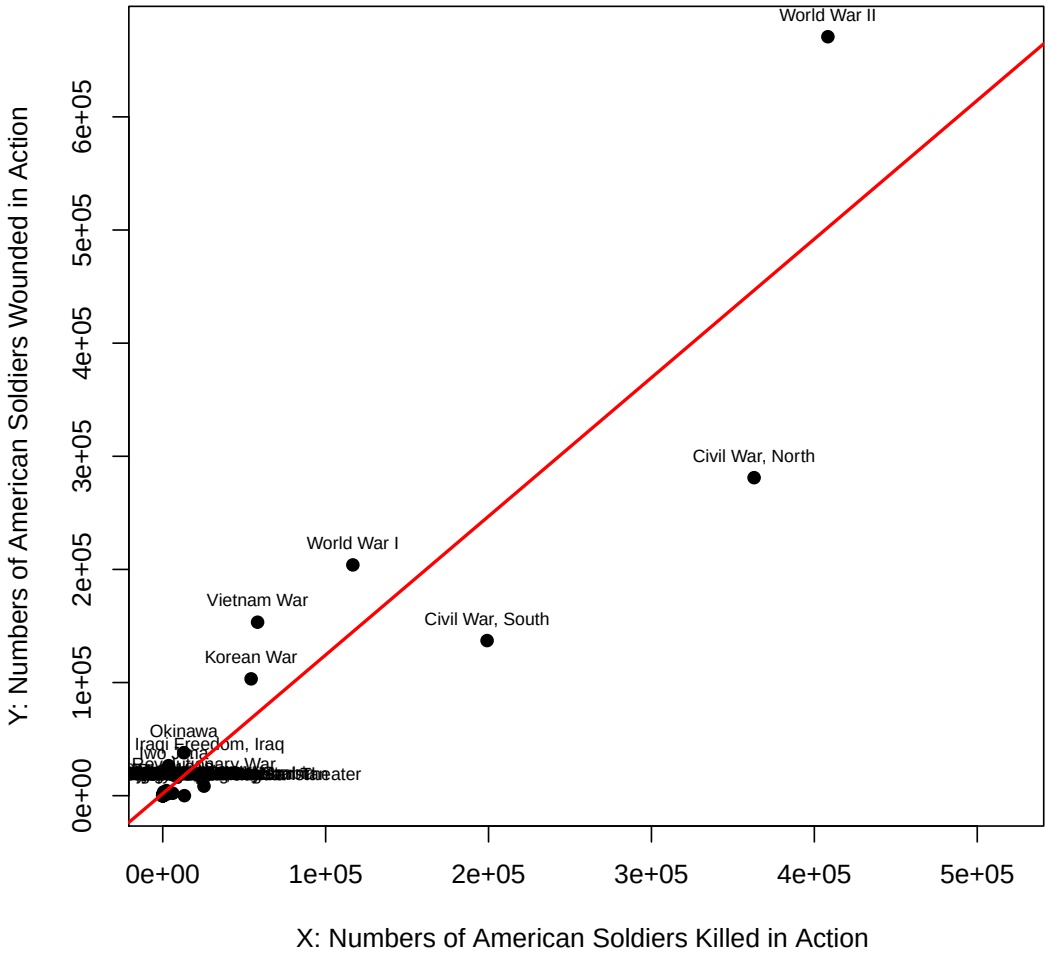
- Note that these approximations work only for small increments.
- In particular, they do not work when X is a discrete random variable.

Example from the American War Library



$\hat{\beta}_1 = 1.23 \longrightarrow$

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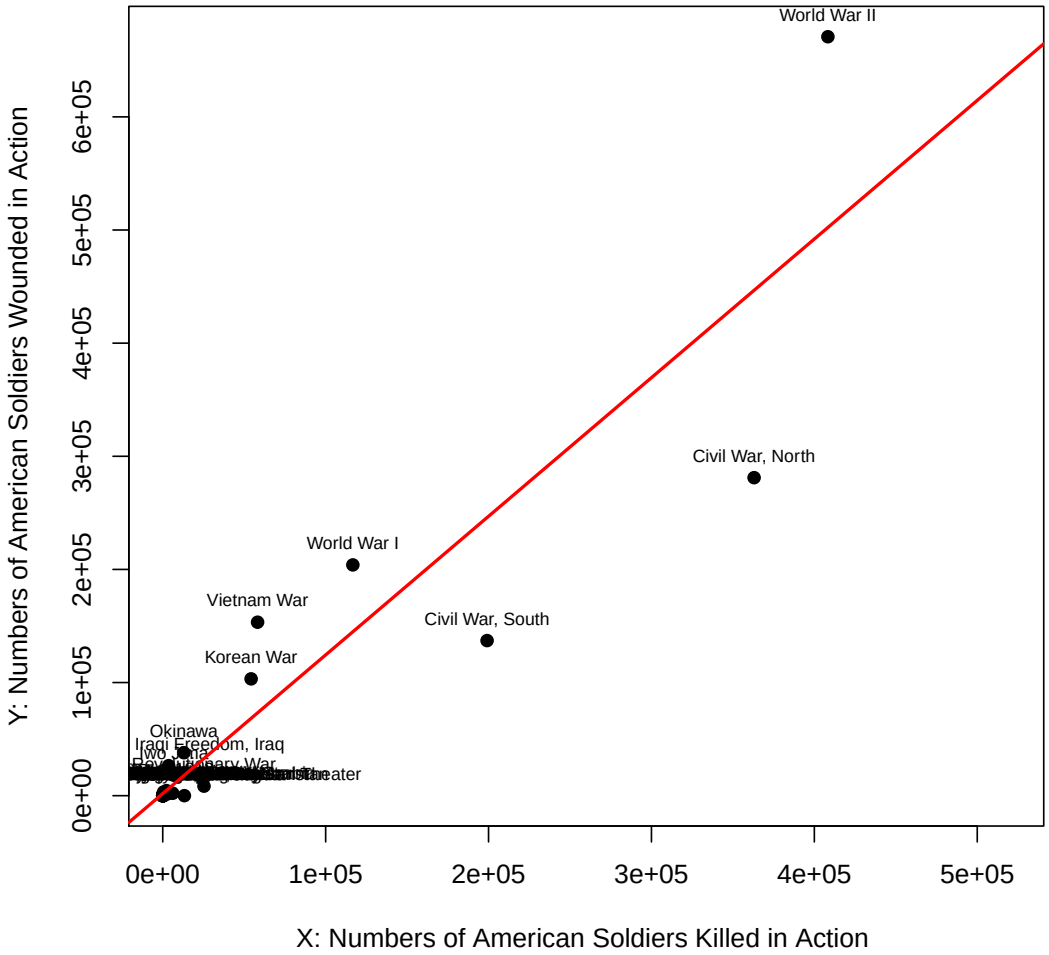
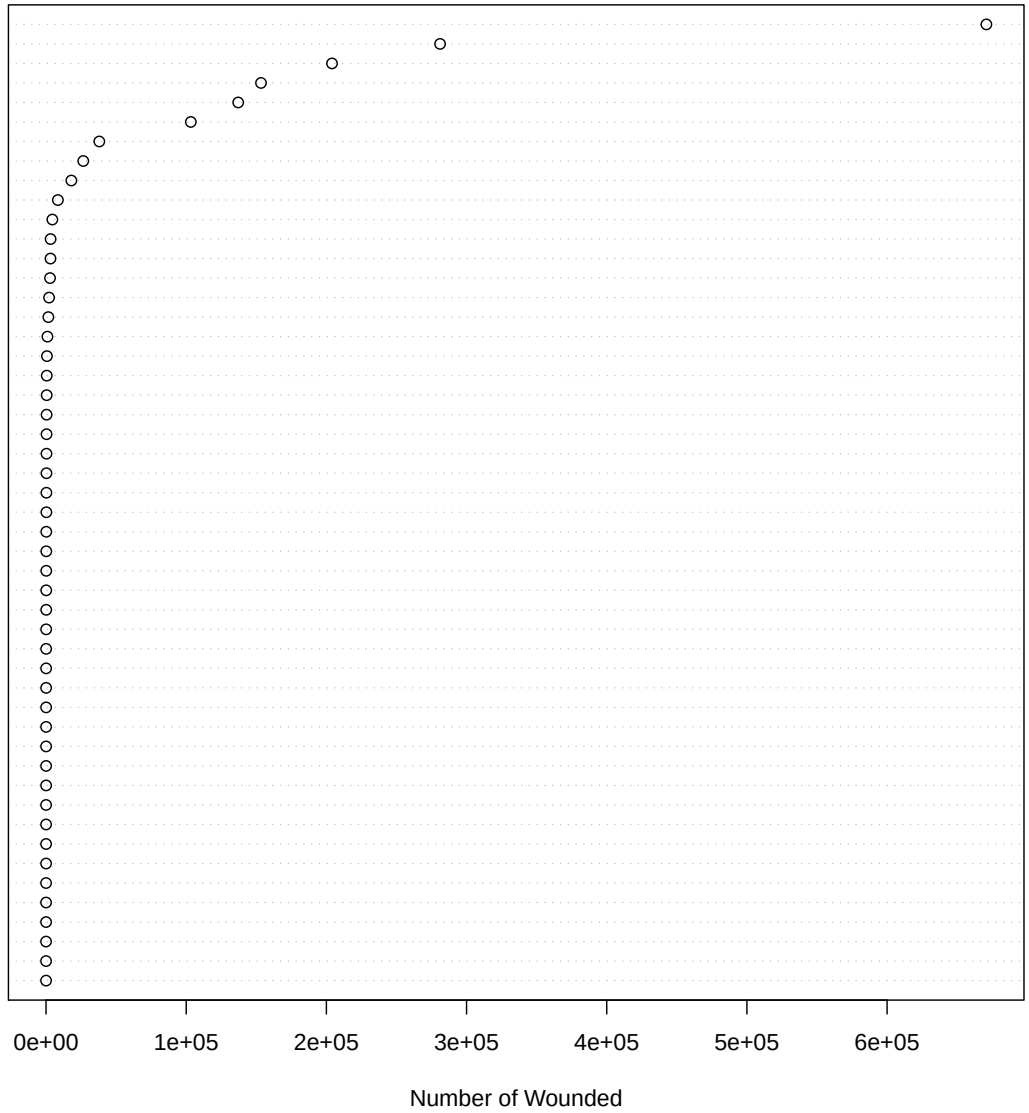


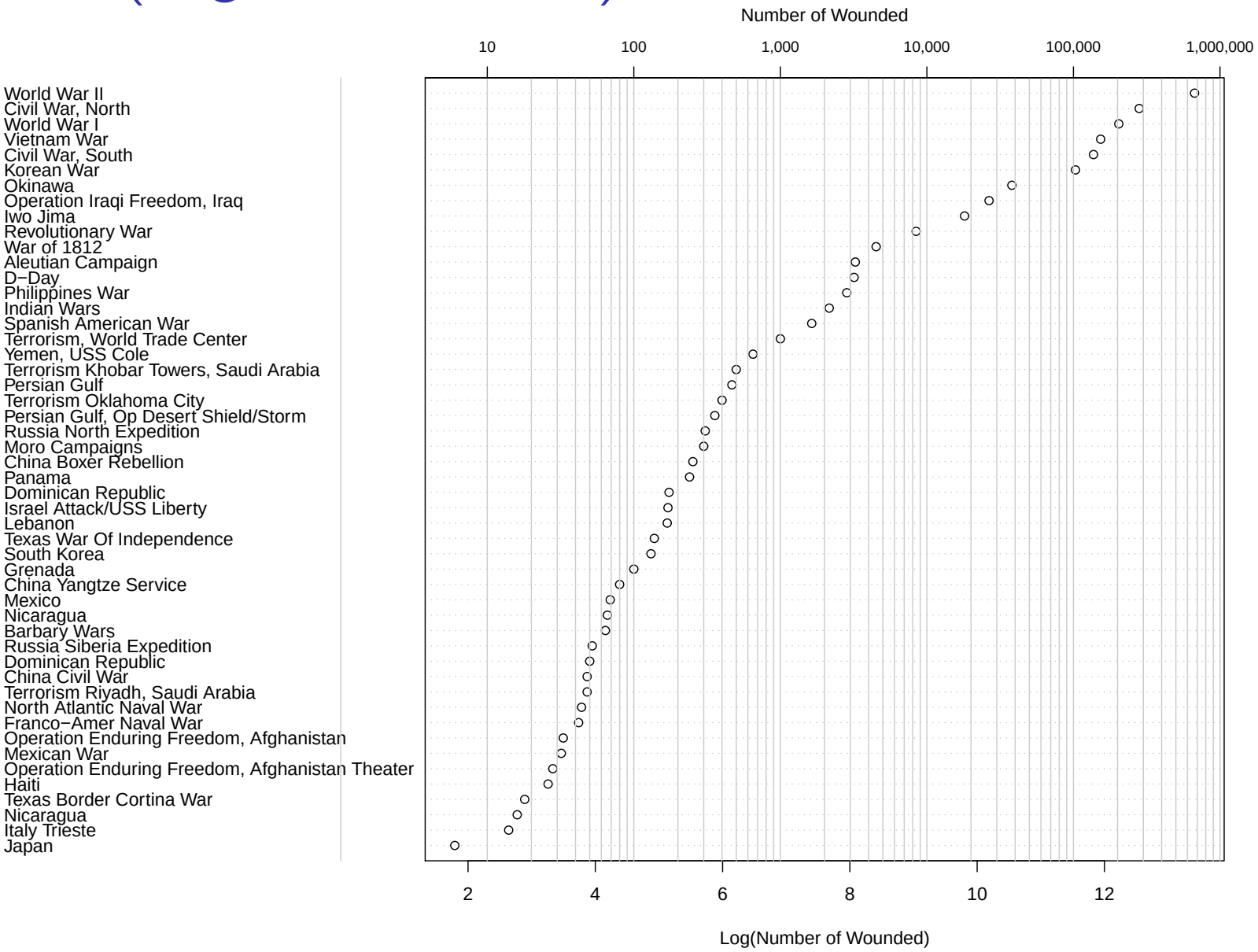
$\hat{\beta}_1 = 1.23 \longrightarrow$ One additional soldier killed predicts 1.23 additional soldiers wounded

Wounded (Scale in Levels)

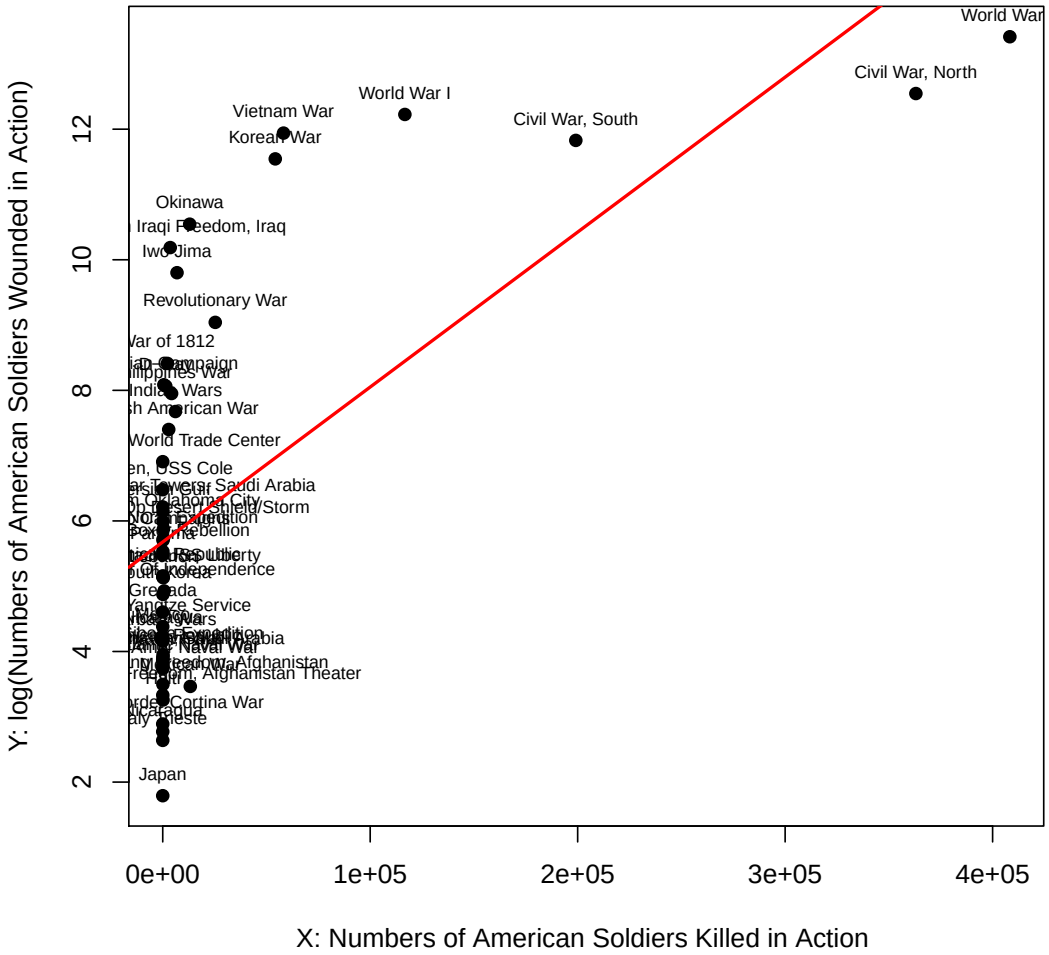
World War II
Civil War, North
World War I
Vietnam War
Civil War, South
Korean War
Okinawa
Operation Iraqi Freedom, Iraq
Iwo Jima
Revolutionary War
War of 1812
Aleutian Campaign
D−Day
Philippines War
Indian Wars
Spanish American War
Terrorism, World Trade Center
Yemen, USS Cole
Terrorism Khobar Towers, Saudi Arabia
Persian Gulf
Terrorism Oklahoma City
Persian Gulf, Op Desert Shield/Storm
Russia North Expedition
Moro Campaigns
China Boxer Rebellion
Panama
Dominican Republic
Israel Attack/USS Liberty
Lebanon
Texas War Of Independence
South Korea
Grenada
China Yangtze Service
Mexico
Nicaragua
Barbary Wars
Russia Siberia Expedition
Dominican Republic
China Civil War
Terrorism Riyadh, Saudi Arabia
North Atlantic Naval War
Franco−Amer Naval War
Operation Enduring Freedom, Afghanistan
Mexican War
Operation Enduring Freedom, Afghanistan Theater
Haiti
Texas Border Cortina War
Nicaragua
Italy Trieste
Japan



Wounded (Logarithmic Scale)



Regression: Log-Level



$\hat{\beta}_1 = 0.0000237 \longrightarrow$

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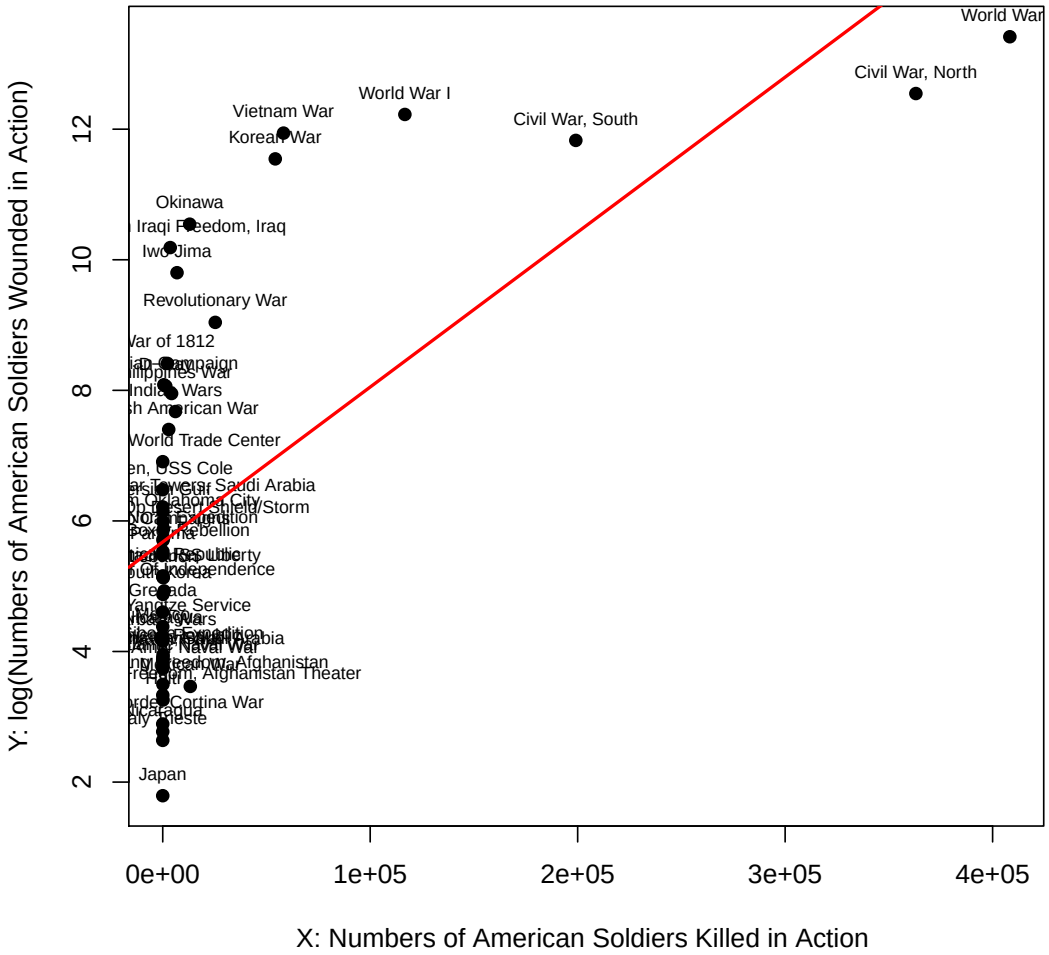
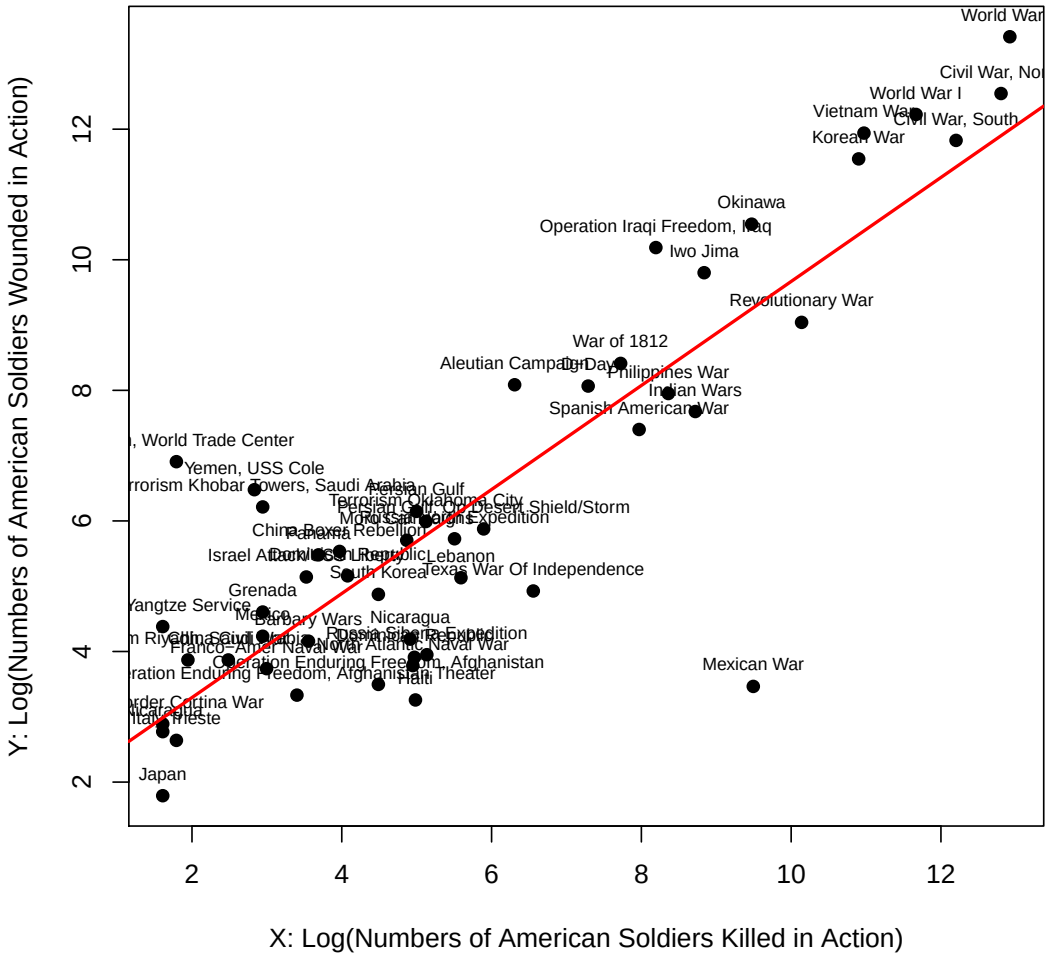


$\hat{\beta}_1 = 0.0000237 \longrightarrow$ One additional soldier killed predicts 0.0023 percent increase in the number of soldiers wounded

Regression: Log-Log



$\hat{\beta}_1 = 0.797 \longrightarrow$

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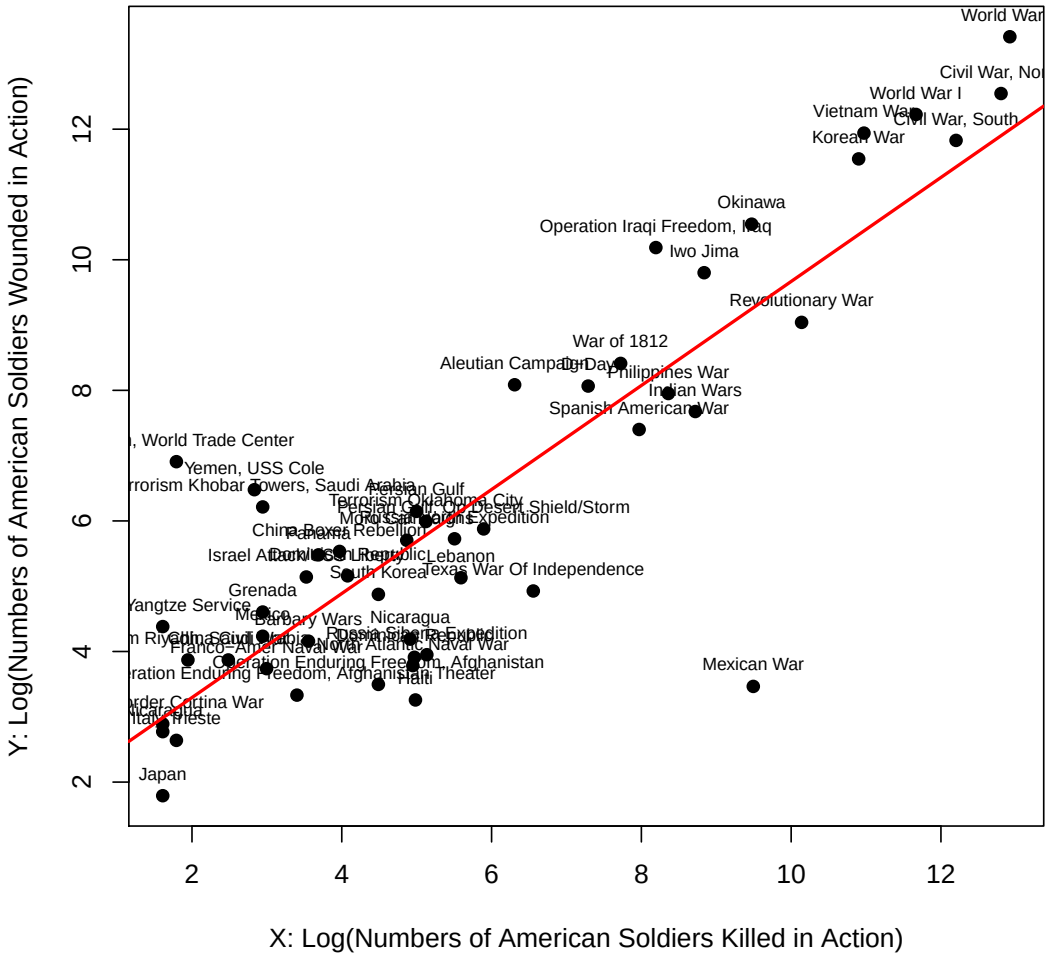

$\hat{\beta}_1 = 0.797 \longrightarrow$ A percent increase in deaths predicts 0.797 percent increase in the wounded

Four Most Commonly Used Models

Model	Equation	β_1 Interpretation
Level-Level	$Y = \beta_0 + \beta_1 X$	$\Delta Y = \beta_1 \Delta X$
Log-Level	$log(Y) = \beta_0 + \beta_1 X$	$\%\Delta Y = 100\beta_1 \Delta X$
Level-Log	$Y = \beta_0 + \beta_1 log(X)$	$\Delta Y = (\beta_1/100)\%\Delta X$
Log-Log	$log(Y) = \beta_0 + \beta_1 log(X)$	$\%\Delta Y = \beta_1 \%\Delta X$

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This can be derived from a series expansion of the log function. Numerically, when $|x| \leq .1$, the approximation is within 0.001.

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Applying our approximation and multiplying by 100 we find,

$$p \approx 100\left(\log(a) - \log(b)\right)$$

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Assume we have: $log(Y) = \beta_0 + \beta_1 X$ where X is binary with values 1 or 0. Assume $\beta_1 > .2$. What is the problem with saying that a one unit increase in X is associated with a $\beta_1 \cdot 100$ percent change in Y?

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A one unit change in X (ie. going from 0 to 1) creates $100(exp(\hat{\beta}_1) - 1)$ percent increase in our prediction $\hat{Y}$.

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- What are we characterizing? The geometric mean.

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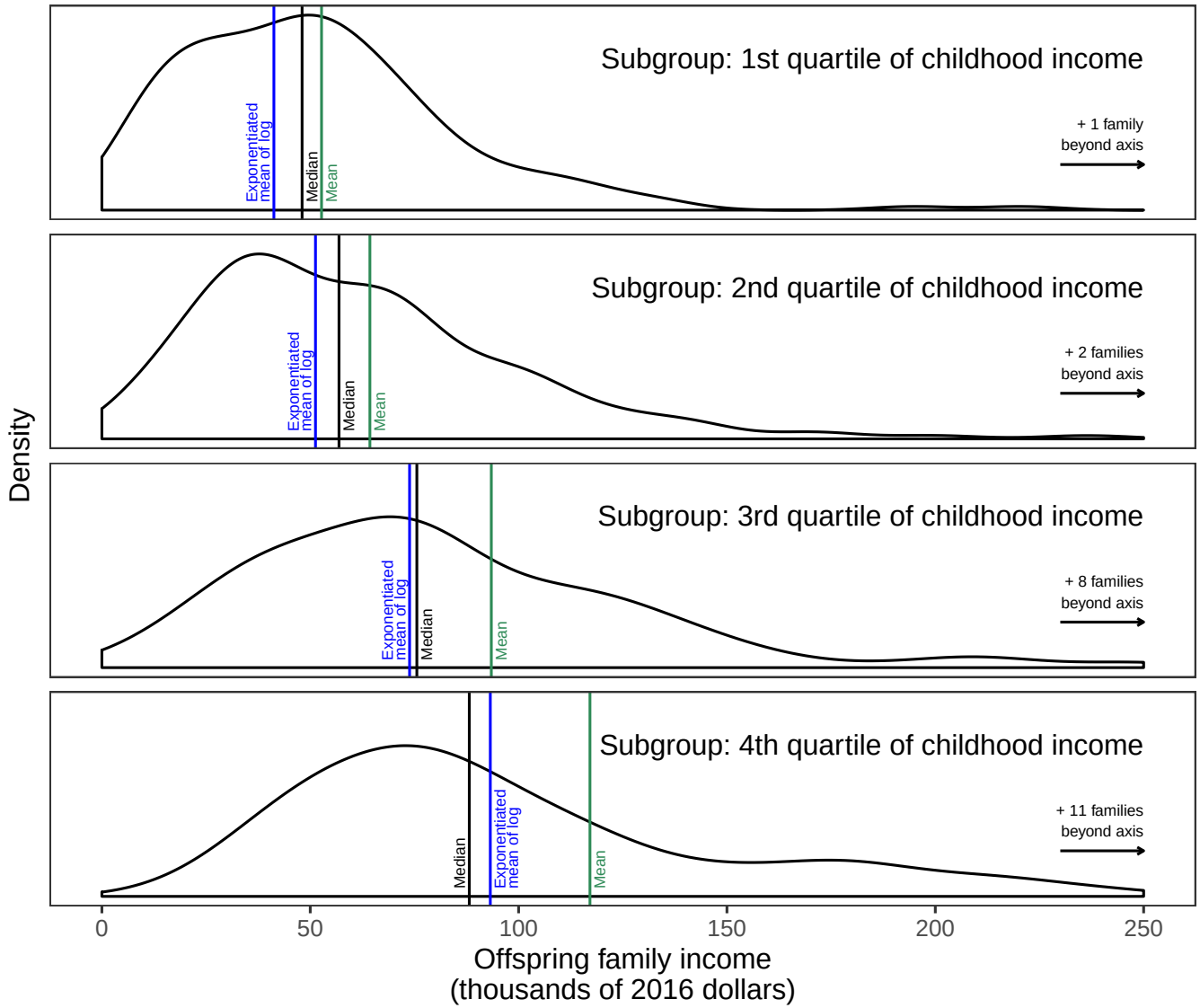
The geometric mean is a robust measure of central tendency.

Geometric Mean is Closer to the Median Than the Mean



(Lundberg and Stewart, 2020)

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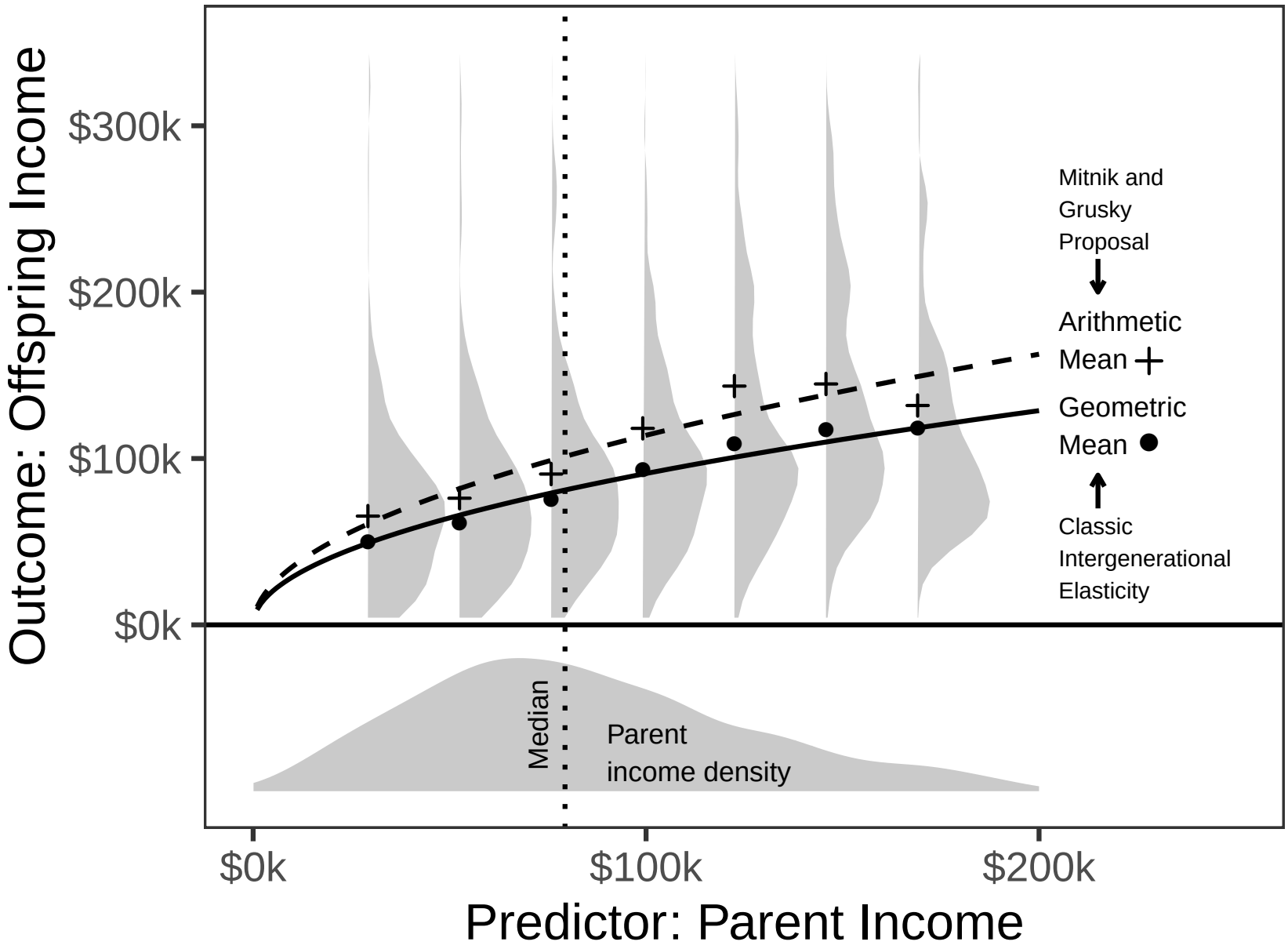

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   - Polynomials
   - LOESS
   - Log Transformations
2. Non-Normality
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- Fix $\mathbf{x}_i'$ and the distribution of errors are Normal.

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- The sample size (n) needed for approximation to hold depends on how far the errors are from Normal.
- Reasonable question: if we have enough data are non-normal errors even a problem?

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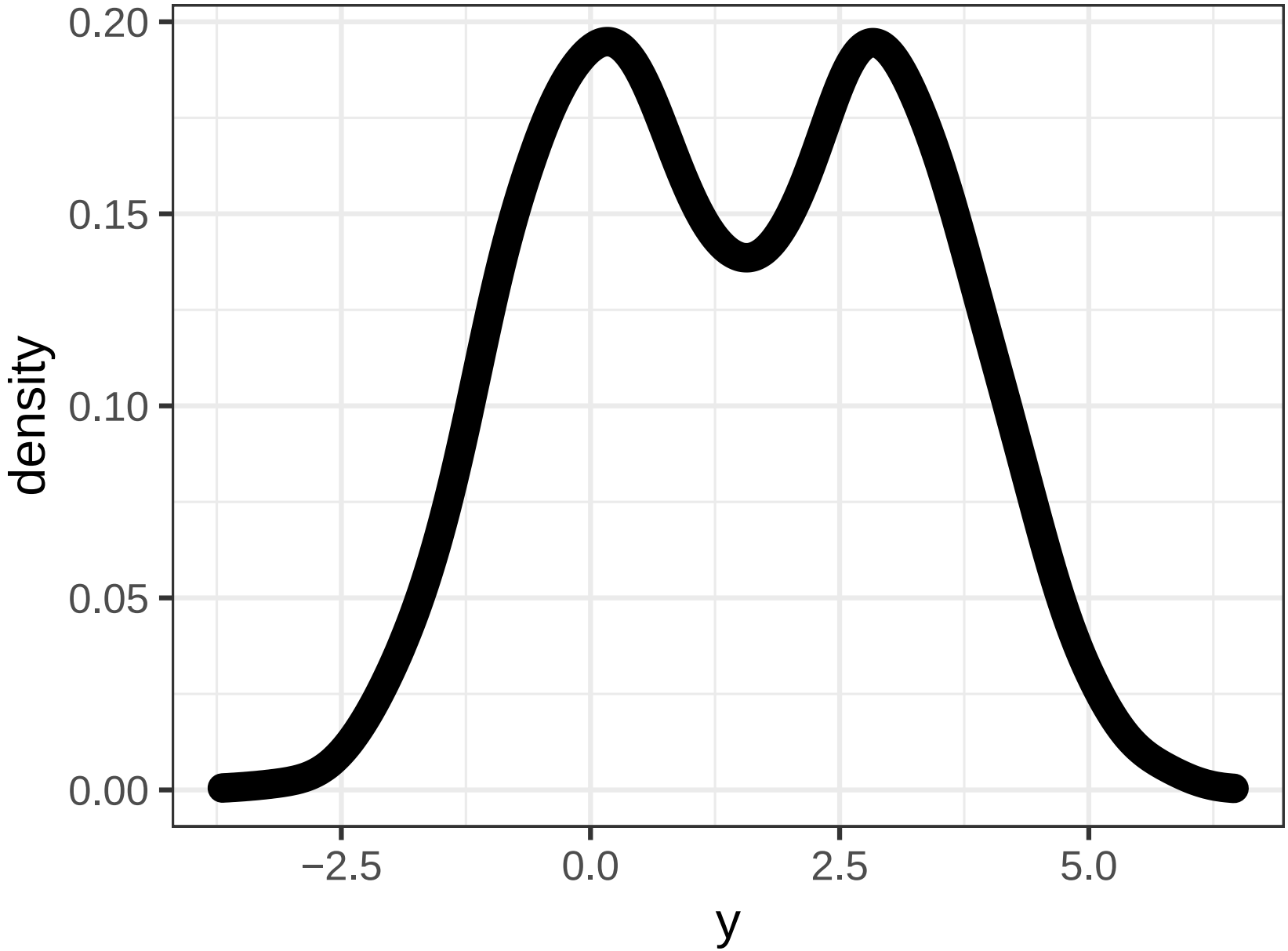
Example: Is this a Violation?



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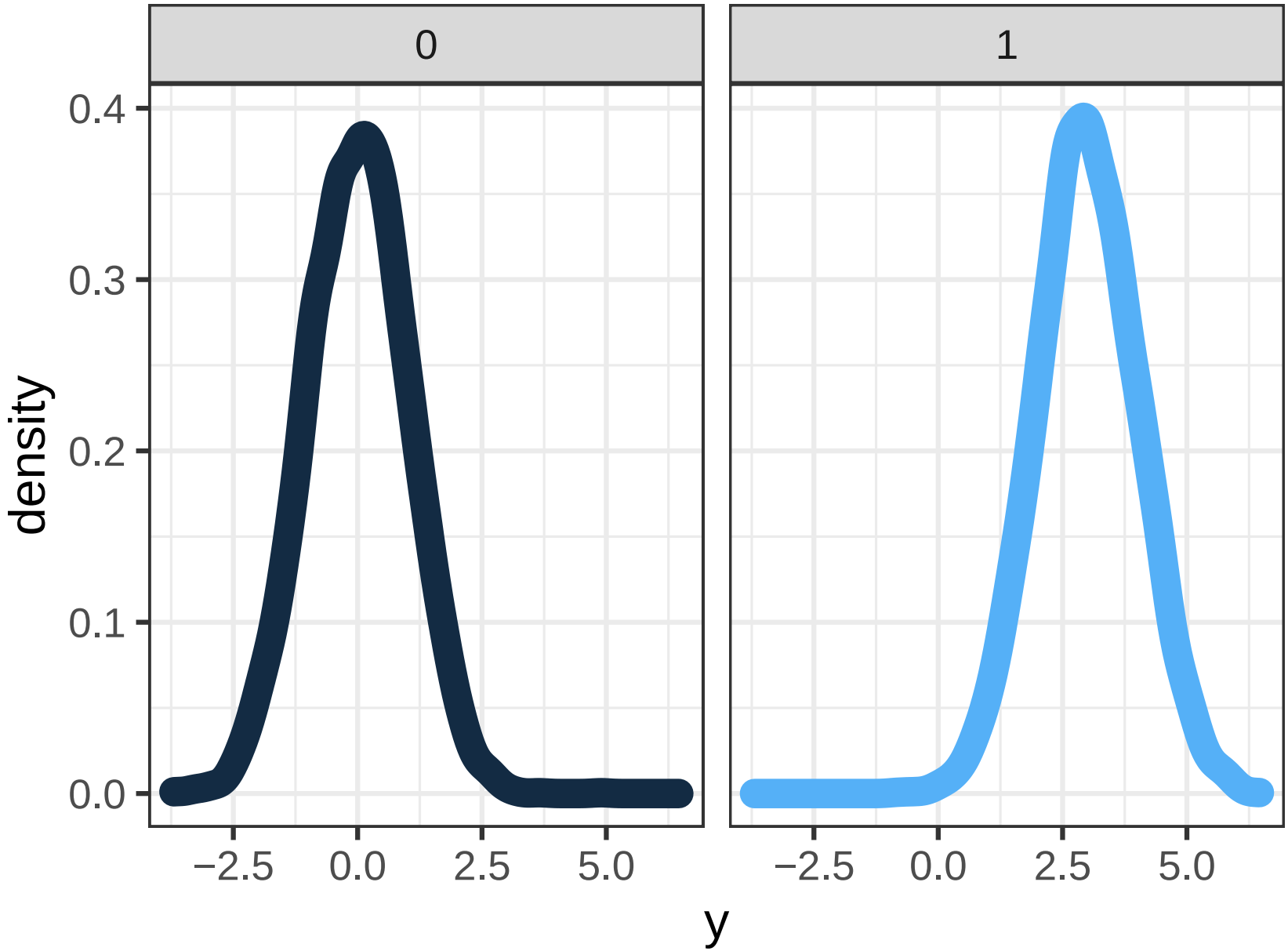

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Solution: Carefully investigate the residuals numerically and graphically.

To understand the relationship between residuals and errors, we need to derive the distribution of the residuals (which we will do over the next few slides).

Defining the Hat Matrix

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- Note that each residual is a function of all of the errors.

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The variance of the ith residual $\hat{u}_i$ is $V[\hat{u}_i] = \sigma^2(1 - h_{ii})$, where h_{ii} is the ith diagonal element of the matrix $\mathbf{H}$ (called the hat value).

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What if we could transform the residuals to address the two issues above?

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The standardized residuals are still not ideal, since the numerator and denominator of $\hat{u}_i'$ are not independent. This makes the distribution of $\hat{u}_i'$ nonstandard. If the distribution is non-standard, we can't easily check for violations.

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The Butterfly Did It: The Aberrant Vote for Buchanan in Palm Beach County, Florida

JONATHAN N. WAND *Cornell University*
KENNETH W. SHOTTS *Northwestern University*
JASJEET S. SEKHON *Harvard University*
WALTER R. MEBANE, JR. *Cornell University*
MICHAEL C. HERRON *Northwestern University*
HENRY E. BRADY *University of California, Berkeley*

We show that the butterfly ballot used in Palm Beach County, Florida, in the 2000 presidential election caused more than 2,000 Democratic voters to vote by mistake for Reform candidate Pat Buchanan, a number larger than George W. Bush's certified margin of victory in Florida. We use multiple methods and several kinds of data to rule out alternative explanations for the votes Buchanan received in Palm Beach County. Among 3,053 U.S. counties where Buchanan was on the ballot, Palm Beach County has the most anomalous excess of votes for him. In Palm Beach County, Buchanan's proportion of the vote on election-day ballots is four times larger than his proportion on absentee (nonbutterfly) ballots, but Buchanan's proportion does not differ significantly between election-day and absentee ballots in any other Florida county. Unlike other Reform candidates in Palm Beach County, Buchanan tended to receive election-day votes in Democratic precincts and from individuals who voted for the Democratic U.S. Senate candidate. Robust estimation of overdispersed binomial regression models underpins much of the analysis.

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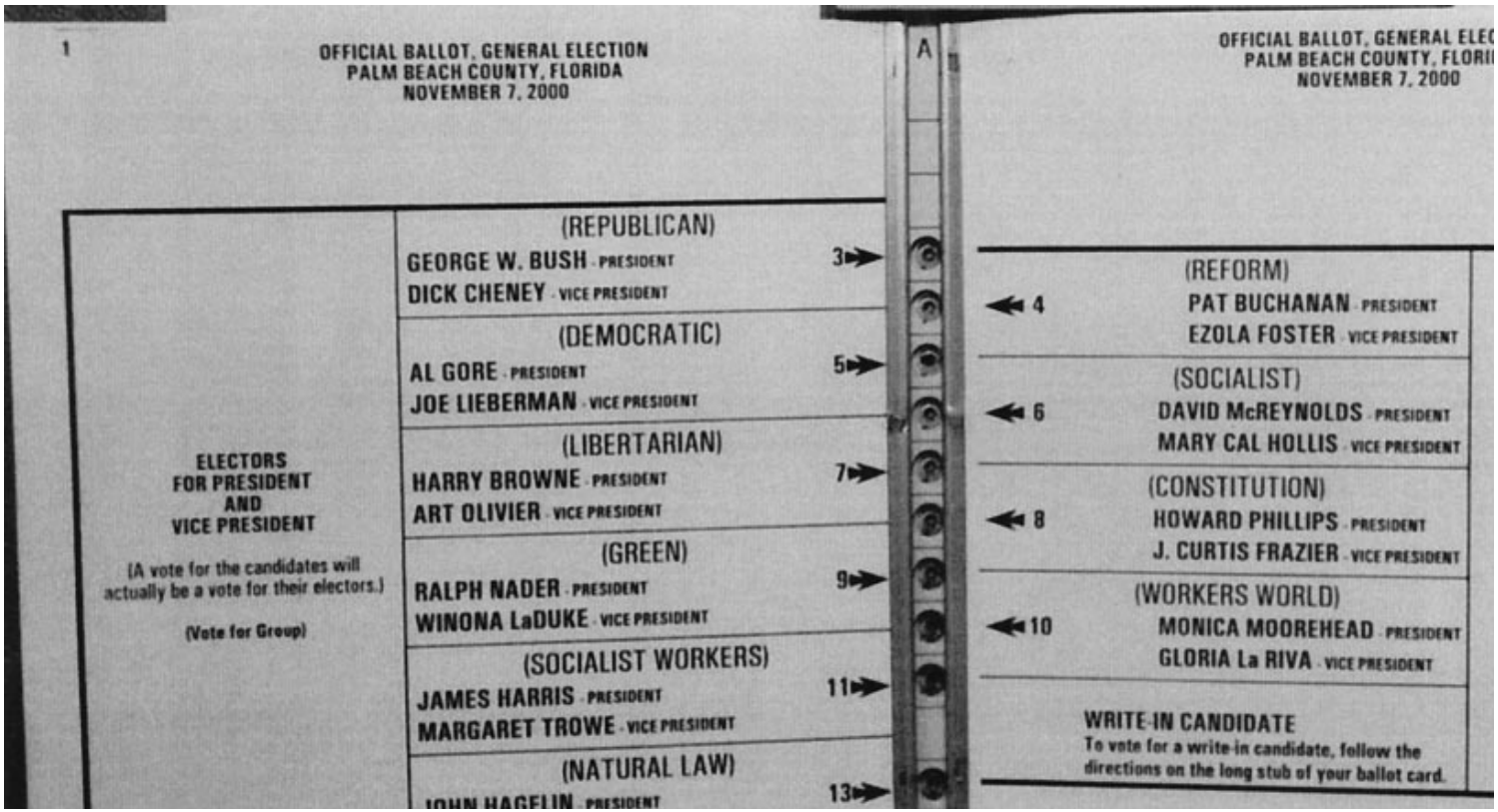
FIGURE 1. The Palm Beach County Bufferfly Ballot



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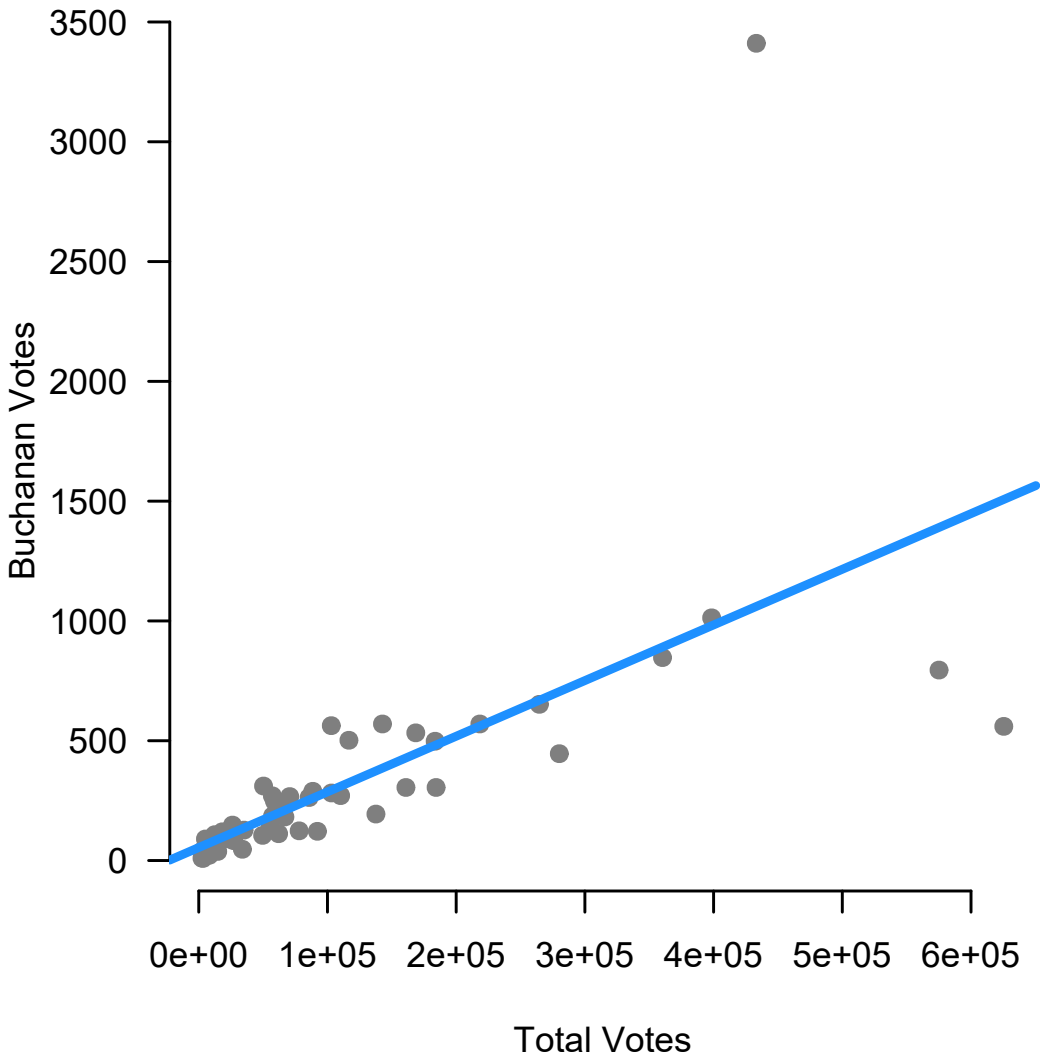

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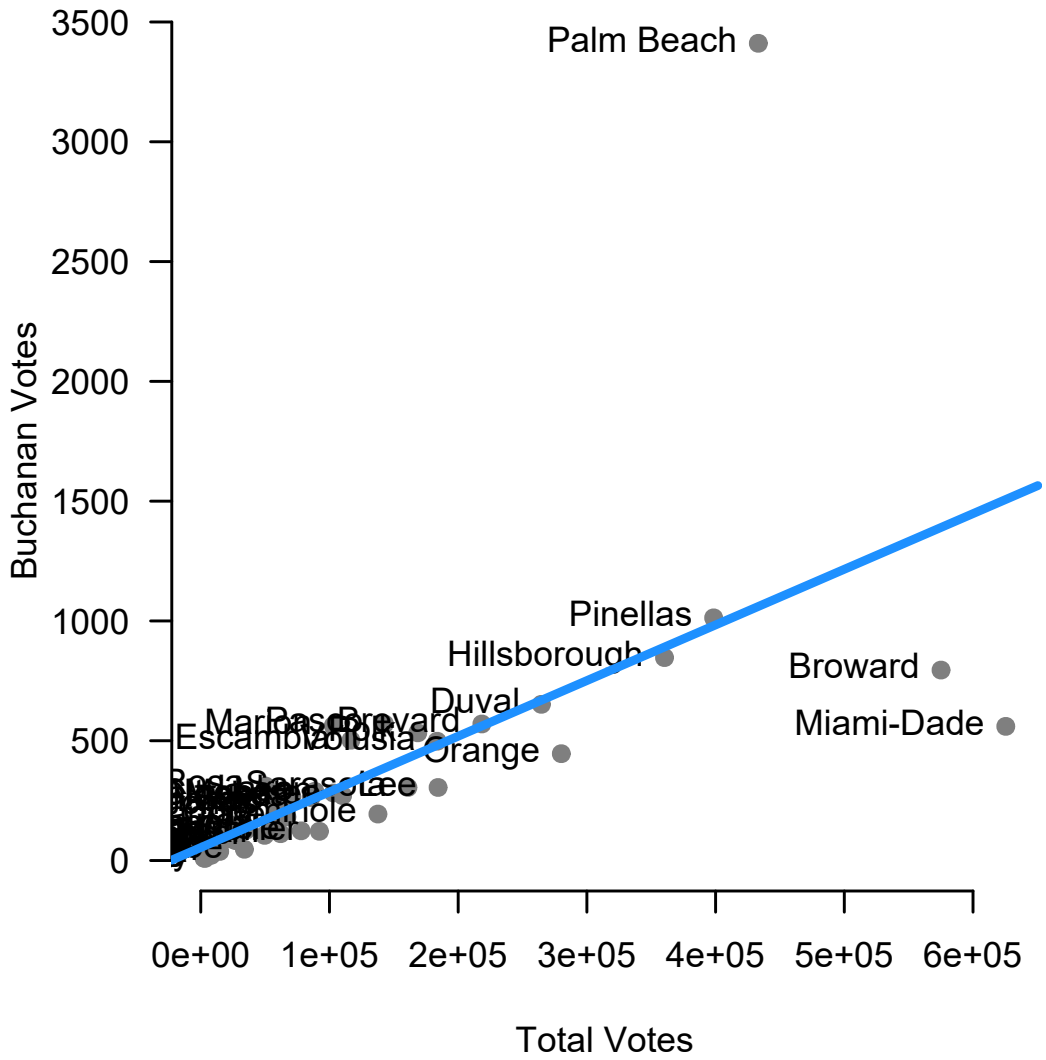

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- We examine data from the 2000 presidential election in Florida used in Wand et al. (2001).
- Our analysis takes place at the county level and we will regress the number of Buchanan votes in each county on the total number of votes in each county.

Buchanan Votes and Total Votes

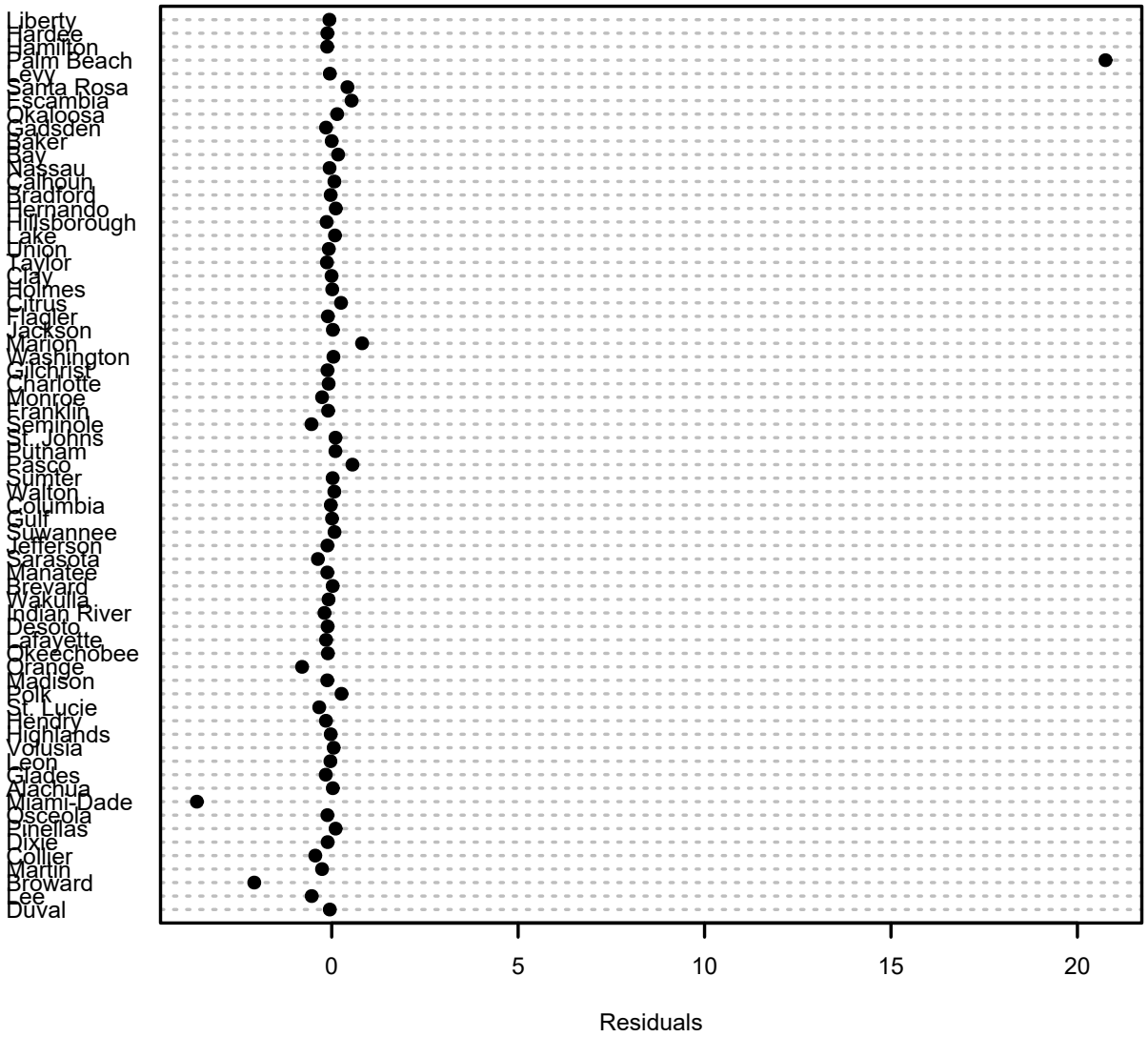
R Code

```
> mod1 <- lm(buchanan00~TotalVotes00,data=dta)
> summary(mod1)
Residuals:
    Min      1Q  Median      3Q     Max
-947.05  -41.74  -19.47   20.20 2350.54

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.423e+01  4.914e+01   1.104    0.274
TotalVotes00 2.323e-03  3.104e-04   7.483 2.42e-10 ***
---
Residual standard error: 332.7 on 65 degrees of freedom
Multiple R-squared: 0.4628,        Adjusted R-squared: 0.4545
F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10

> residuals              <- resid(mod1)
> standardized_residuals <- rstandard(mod1)
> studentized_residuals  <- rstudent(mod1)
> dotchart(residuals,dta$name,cex=.7,xlab="Residuals")
```

Plotting the residuals



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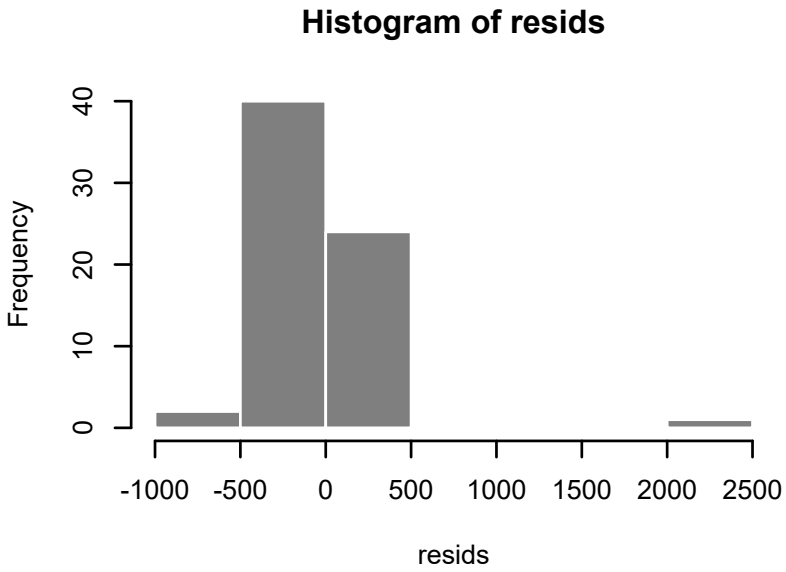

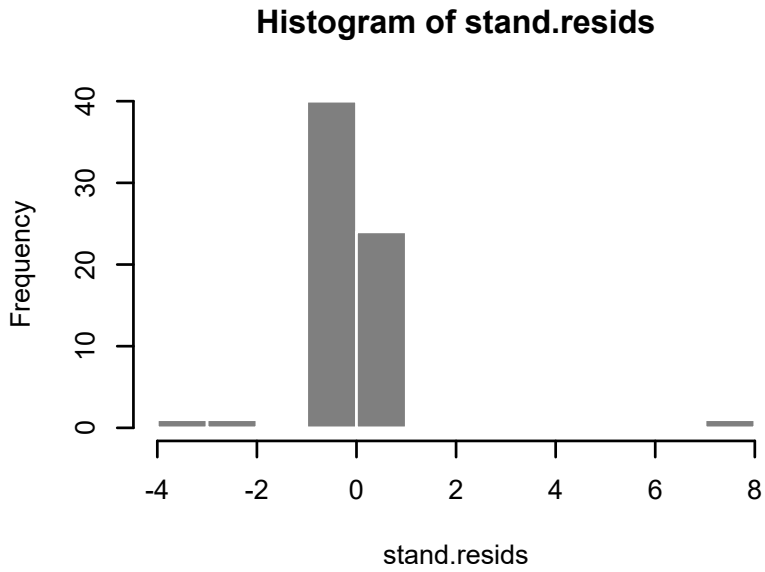

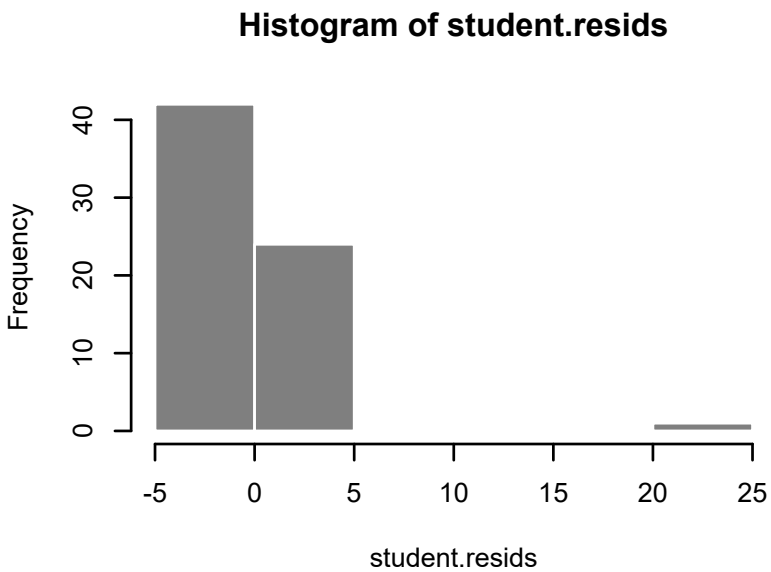

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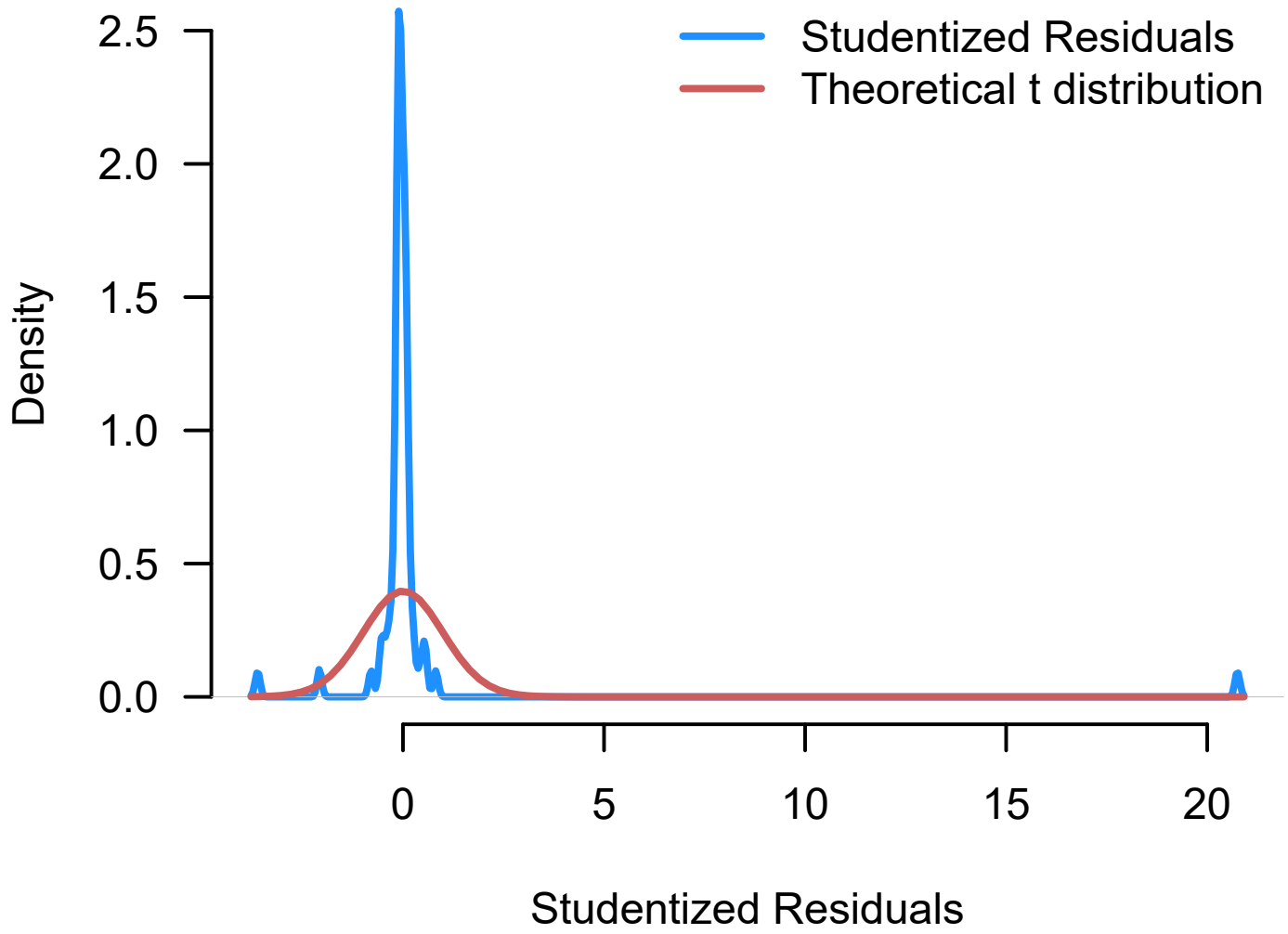

Quantile-Quantile plots

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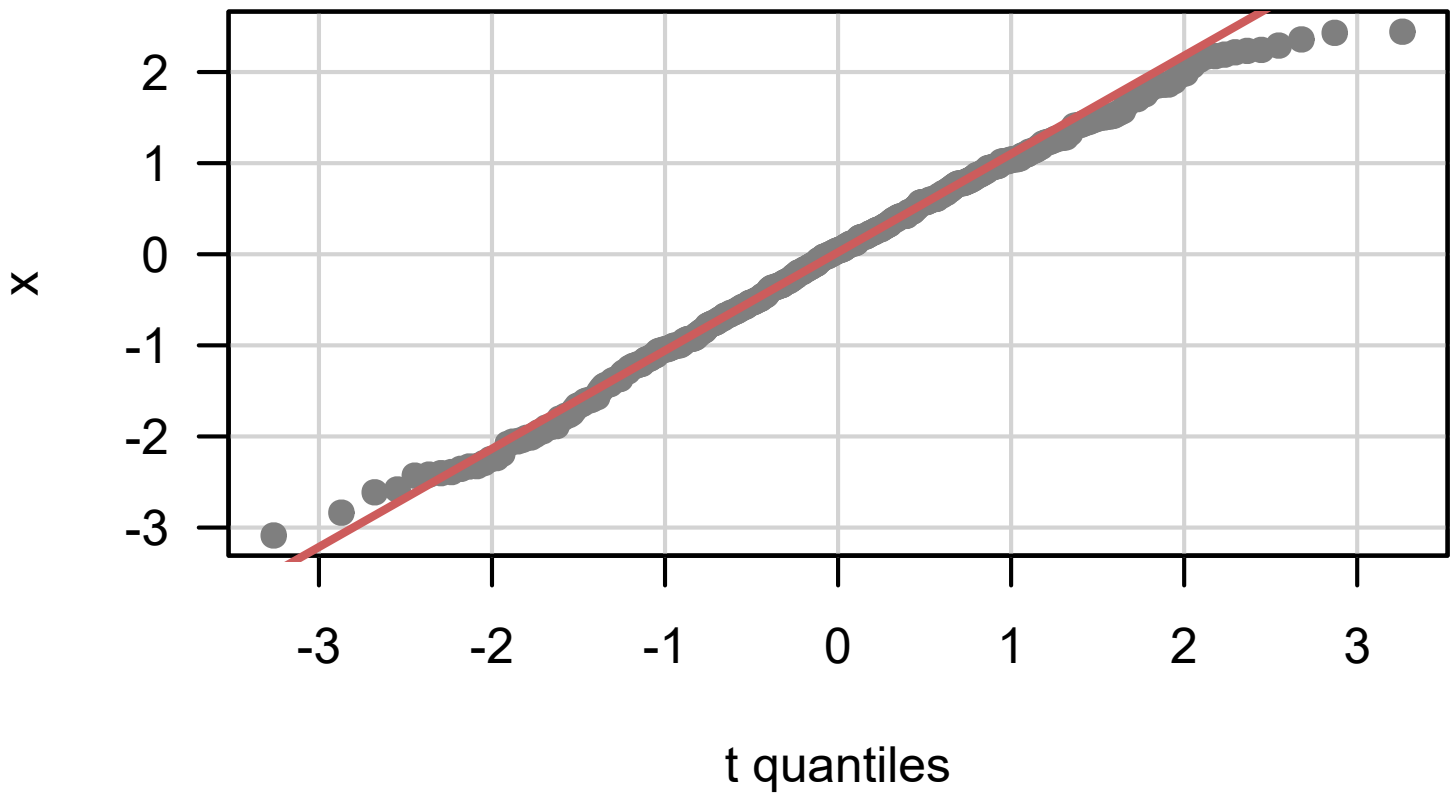
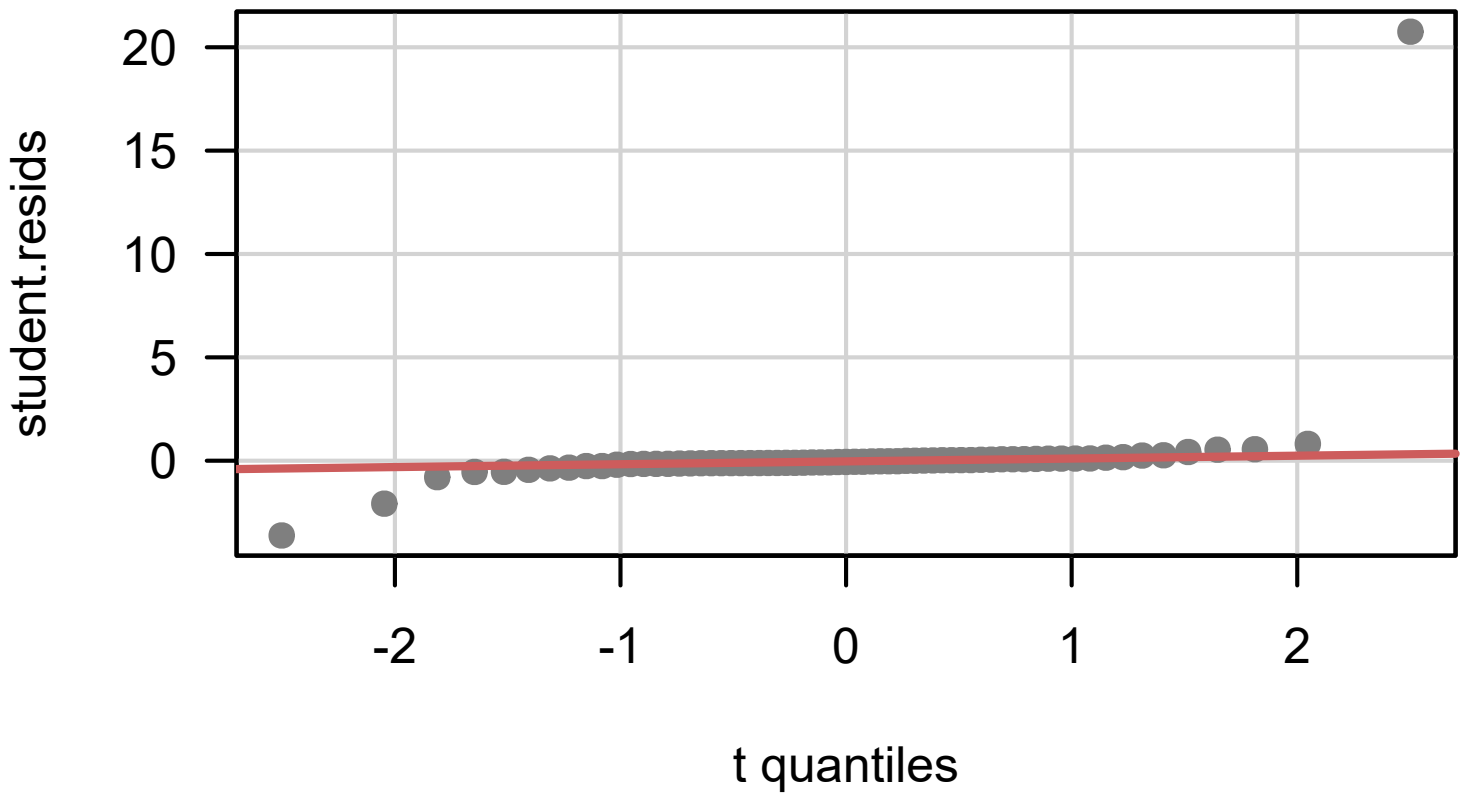
- If distributions are equal $\implies$ 45 degree line

Good QQ-plot



Buchanan QQ-plot



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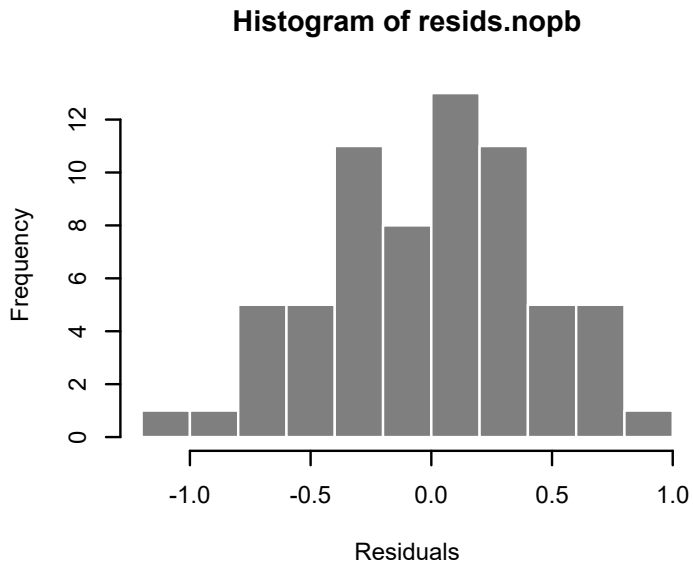
- Use estimators other than OLS that are robust to nonnormality (skipped this year- but see materials from prior years)

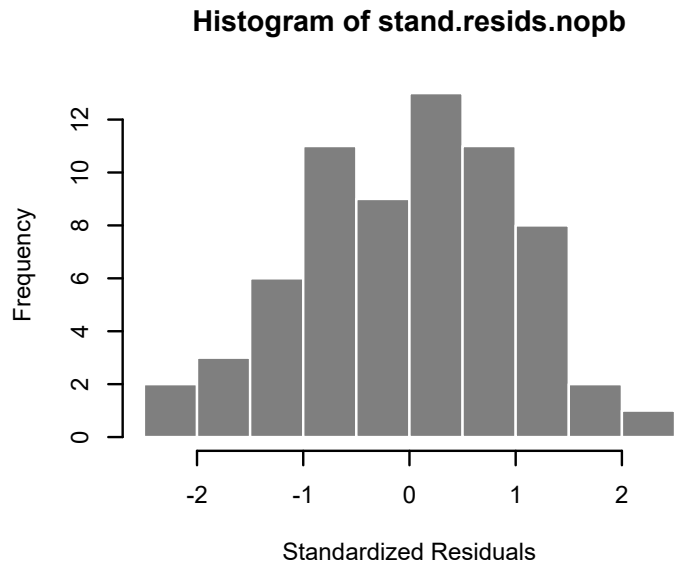
Buchanan Revisited

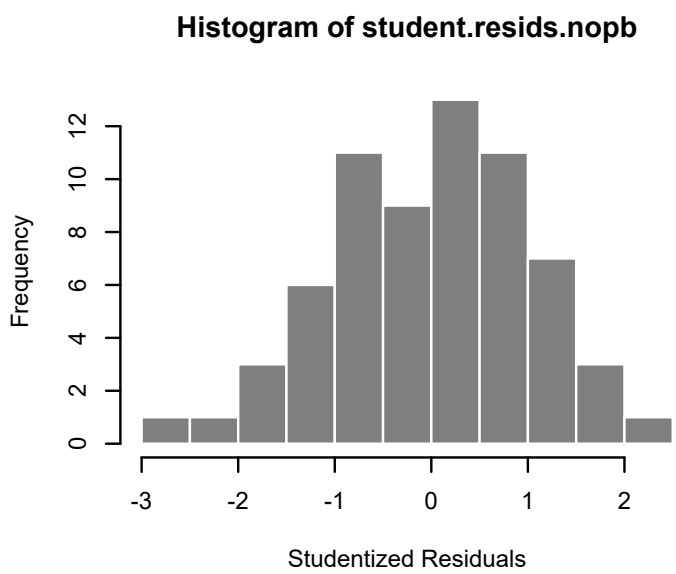
Let's delete Palm Beach and also use log transformations for both variables

```
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -2.48597    0.37889  -6.561 1.09e-08 ***
## log(edaytotal)  0.70311    0.03621  19.417  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4362 on 64 degrees of freedom
## Multiple R-squared:  0.8549, Adjusted R-squared:  0.8526
## F-statistic:   377 on 1 and 64 DF,  p-value: < 2.2e-16
```

Buchanan Revisited







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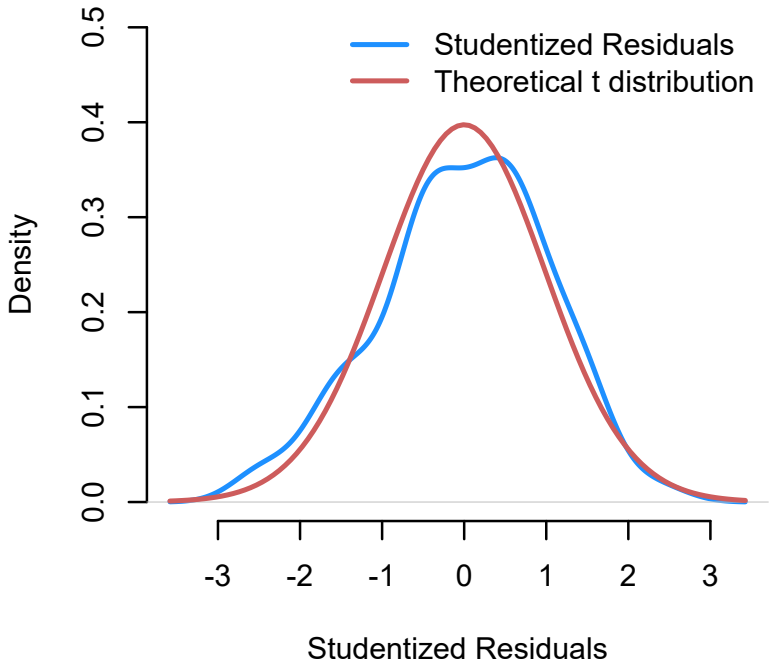

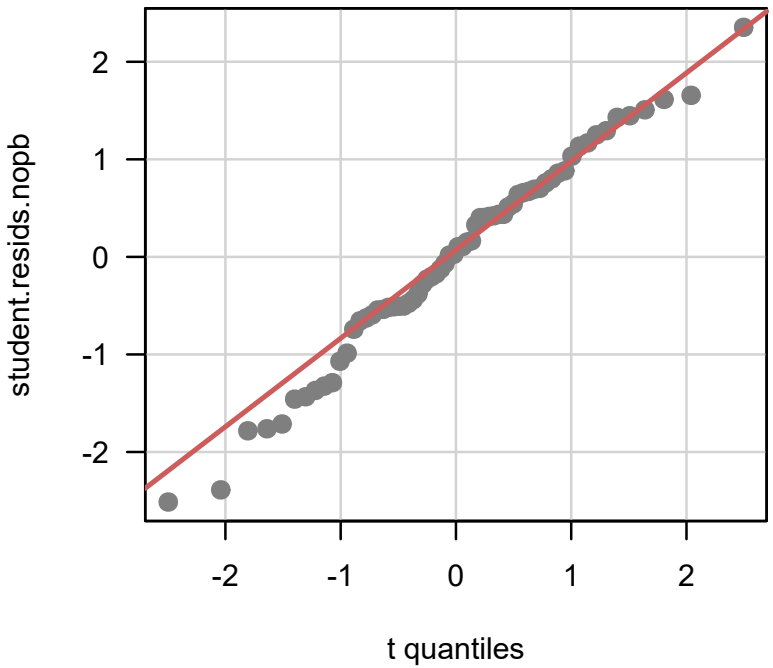

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   - LOESS
   - Log Transformations
2. Non-Normality
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   - Standardized and Studentized Residuals
   - QQ Plots
3. Extreme Values
   - Outliers
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	Constant	x_1	x_2	$x_1 \cdot x_2$
Norway Obs Included	.814	-.192	-.278	.137
	(4.7)	(2.0)	(2.4)	(2.9)
Norway Obs Excluded	.641	-.068	-.138	.054
	(4.8)	(0.9)	(1.5)	(1.3)

Creative Curve Fitting with Norway

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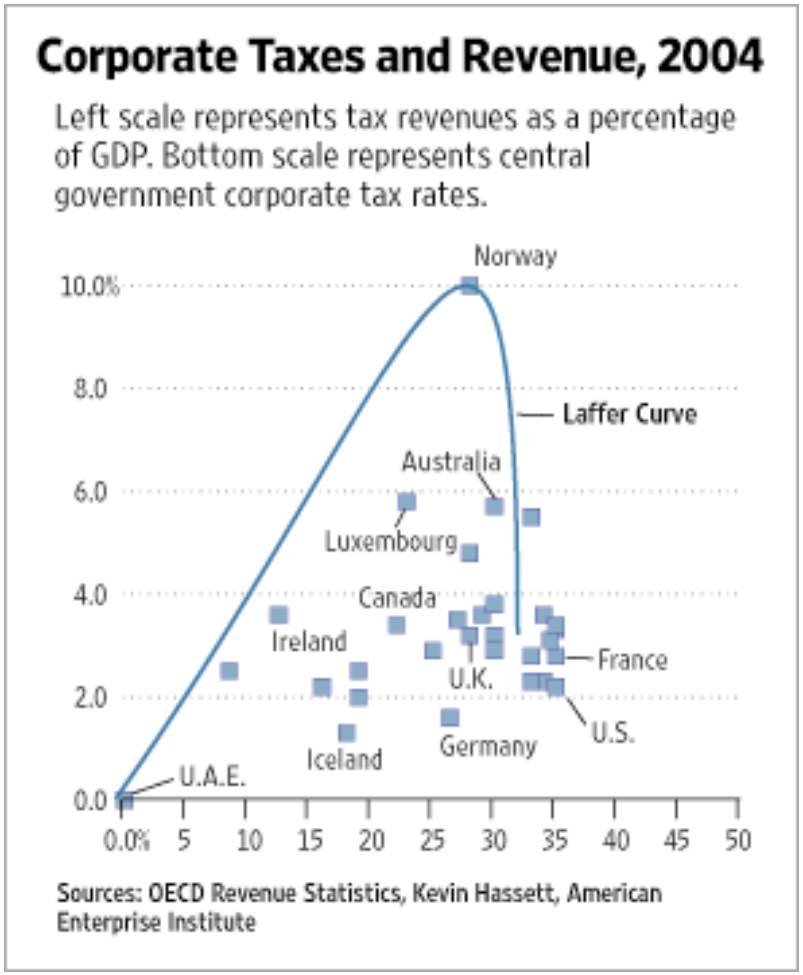

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Carefully Examine the Data First!!

1. Examine summary statistics: `summary(data)`
2. Scatterplot matrix for densities and bivariate relationships:
   E.g. `scatterplotMatrix(data)` from `car` library.
3. Further conditional plots for multivariate data:
   E.g. `ggplot2`

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- Can be a violation of iid (not identically distributed)

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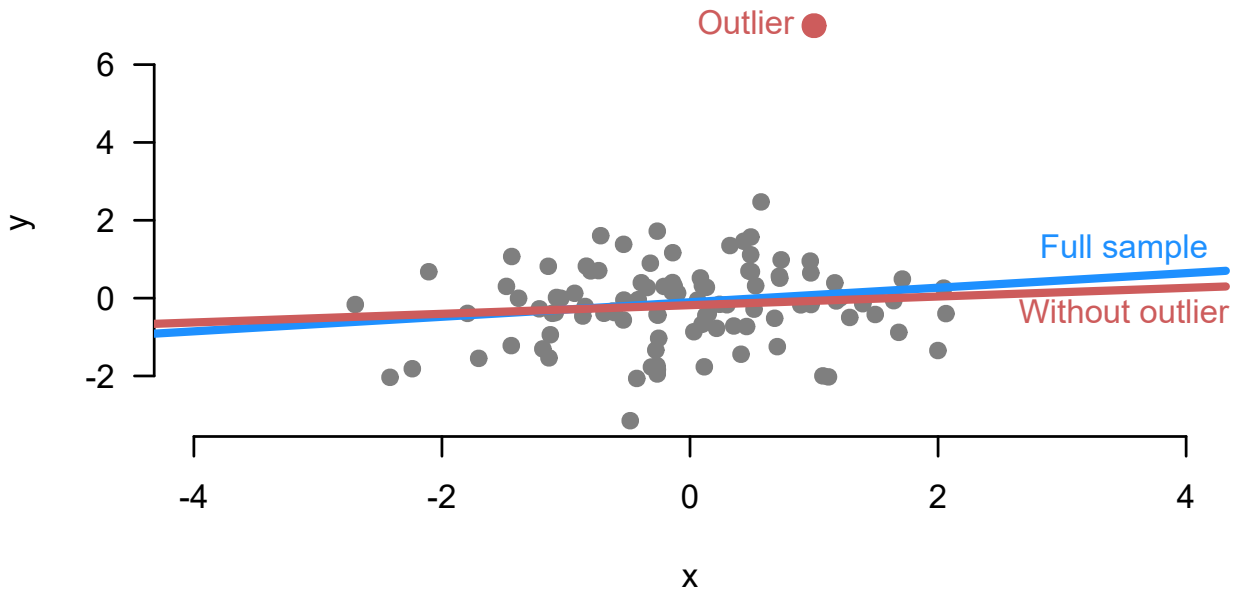

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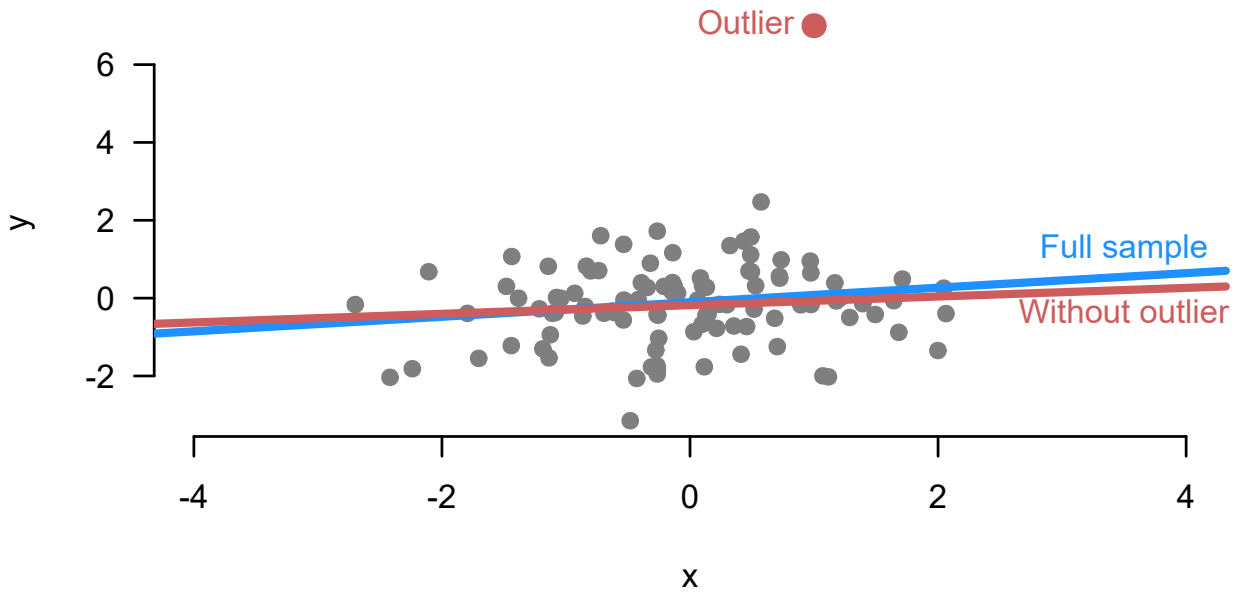

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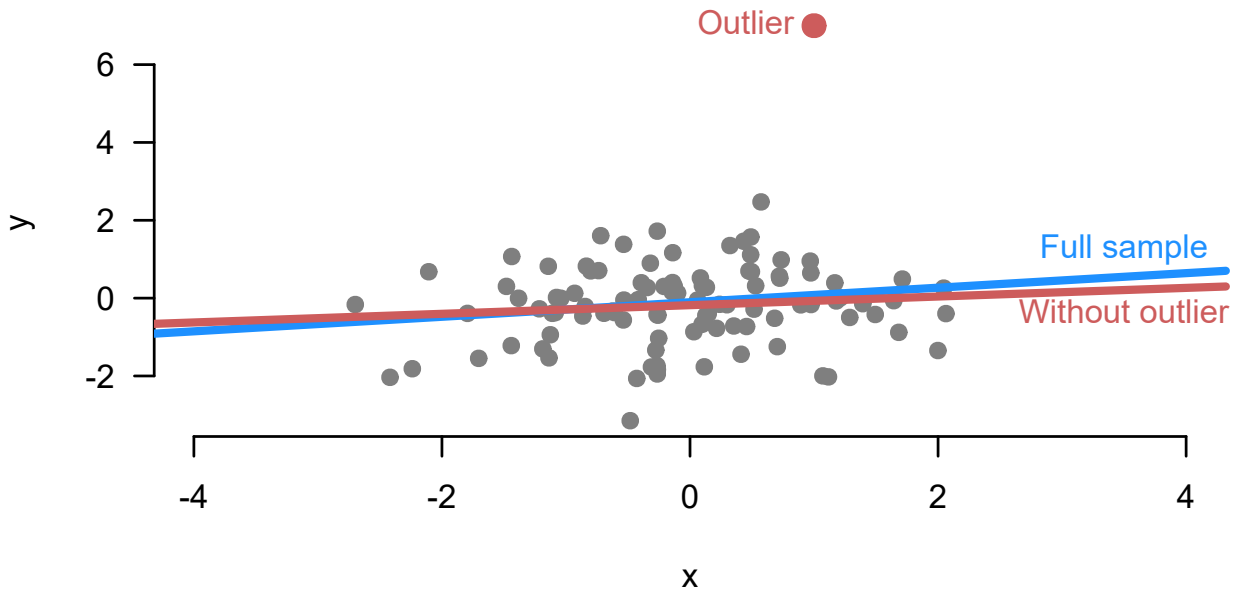

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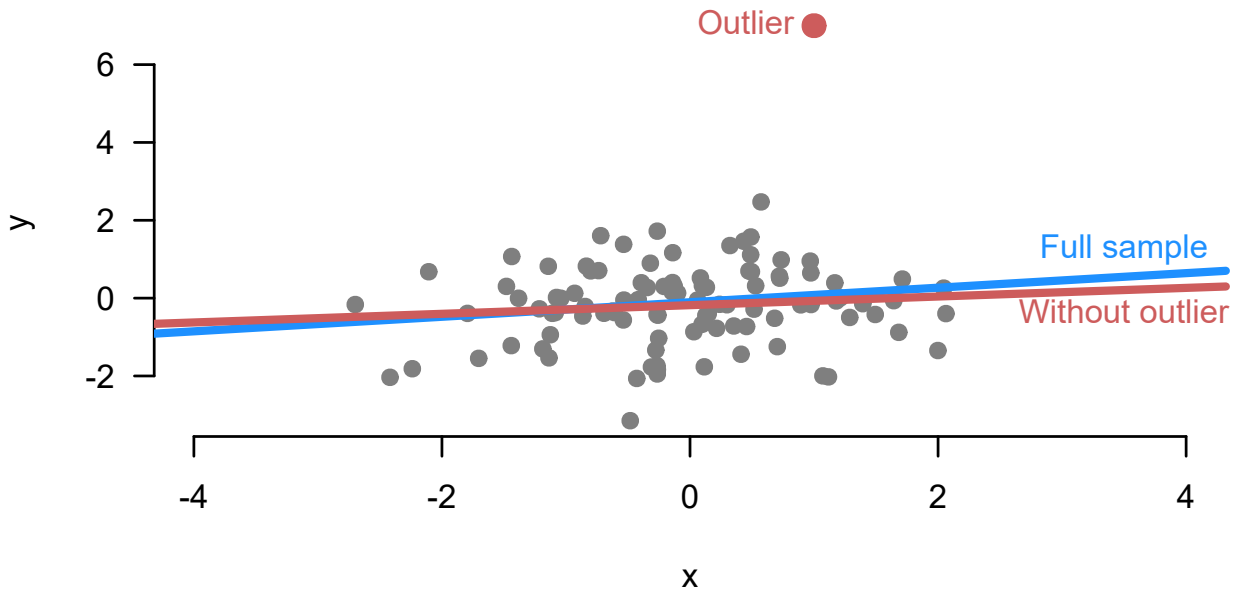

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- $\widehat{\sigma} > \widehat{\sigma}_{-i}$ because we drop the large residual from the outlier, and so $\widehat{u}_i' < \widehat{u}_i^*$

Cutoff Rules for Outliers

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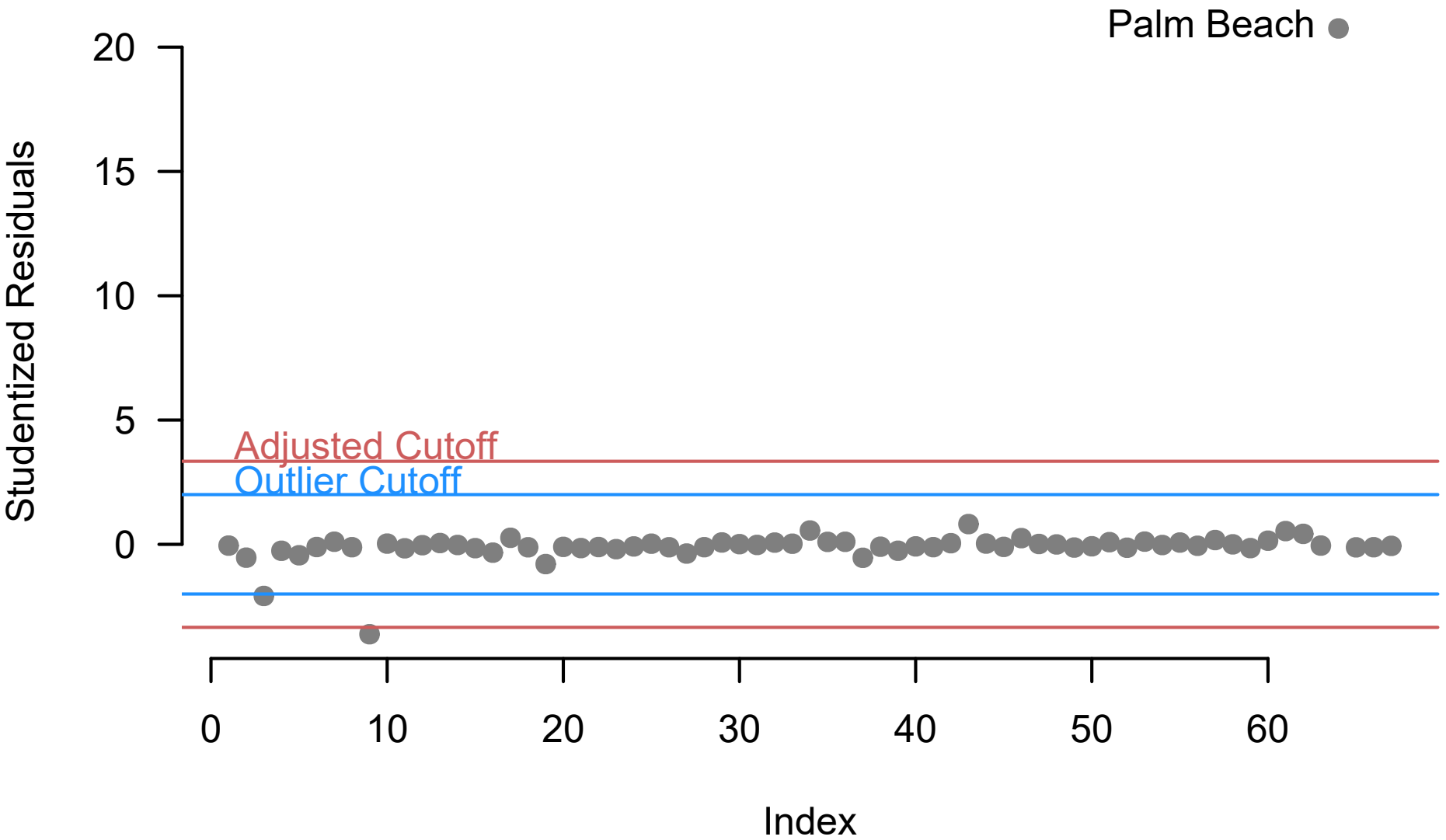
- People usually adjust cutoff for multiple testing

Buchanan outliers



What to do about outliers

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- Key question is what is the goal? If you want to estimate the expectation of a distribution and a property of that distribution is extreme observations, that's just part of the story.

A Cautionary Tale: The "Discovery" of the Ozone Hole

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- The ozone hole was detected in satellite data only when the raw data was reprocessed. When the software was rerun without the pre-processing flags, the ozone hole was seen as far back as 1976.

A Sociological Cautionary Tale

Comment on Herring, ASR, April 2009

AMERICAN SOCIOLOGICAL ASSOCIATION

American Sociological Review
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Does Diversity Pay? A Replication of Herring (2009)

Dragana Stojmenovska,[a] Thijs Bol,[a] and Thomas Leopold[a]

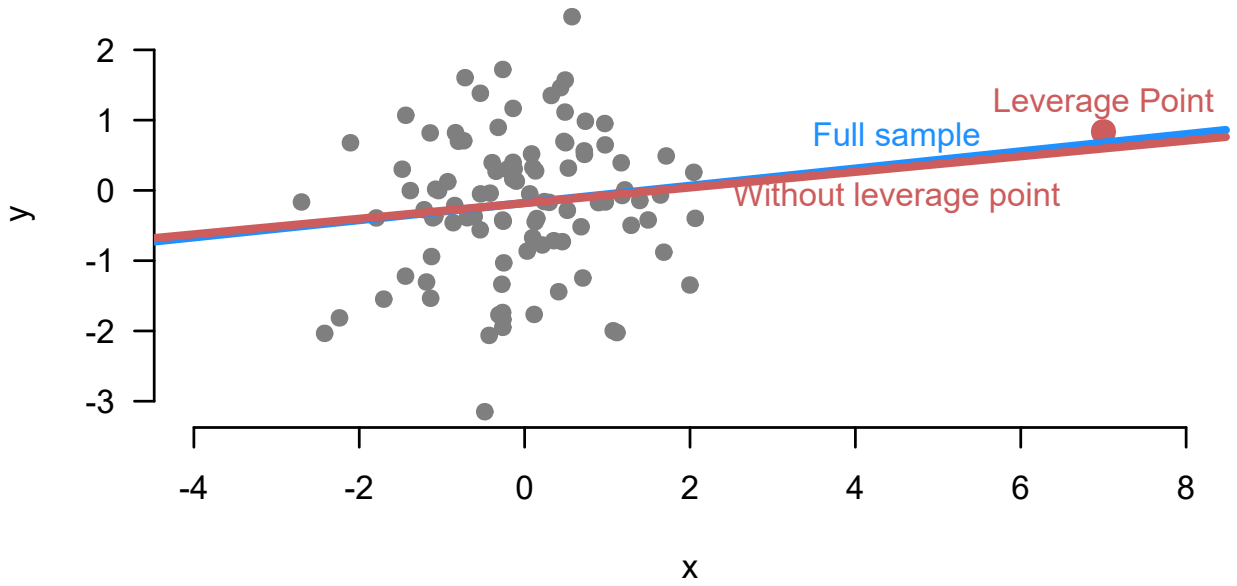
Abstract

In an influential article published in the *American Sociological Review* in 2009, Herring finds that diverse workforces are beneficial for business. His analysis supports seven out of eight hypotheses on the positive effects of gender and racial diversity on sales revenue, number of customers, perceived relative market share, and perceived relative profitability. This comment points out that Herring's analysis contains two errors. First, missing codes on the outcome variables are treated as substantive codes. Second, two control variables—company size and establishment size—are highly skewed, and this skew obscures their positive associations with the predictor and outcome variables. We replicate Herring's analysis correcting for both errors. The findings support only one of the original eight hypotheses, suggesting that diversity is nonconsequential, rather than beneficial, to business success.

A Sociological Cautionary Tale

In our correspondence with Herring, he did not offer a definitive explanation for these discrepancies, but indicated that he may have treated all codes other than "not applicable" (–999) as substantive codes. Given (1) the large difference between his sample size and the number of valid observations in the NOS, and (2) the large number of missing values due to reasons other than "not applicable"—in particular for sales revenue and number of customers—this coding error appears likely to account for much of the discrepancies. This means, for example, that 206 business organizations in which the sales revenue was unknown were treated as if they had sales of 88,888,888,888 US Dollars. Yet, even when we replicated this error (i.e., keeping all organizations with missing values other than –999 in our sample), we were unable to recover Herring's sample sizes, although the differences were smaller.

Leverage Point Definition



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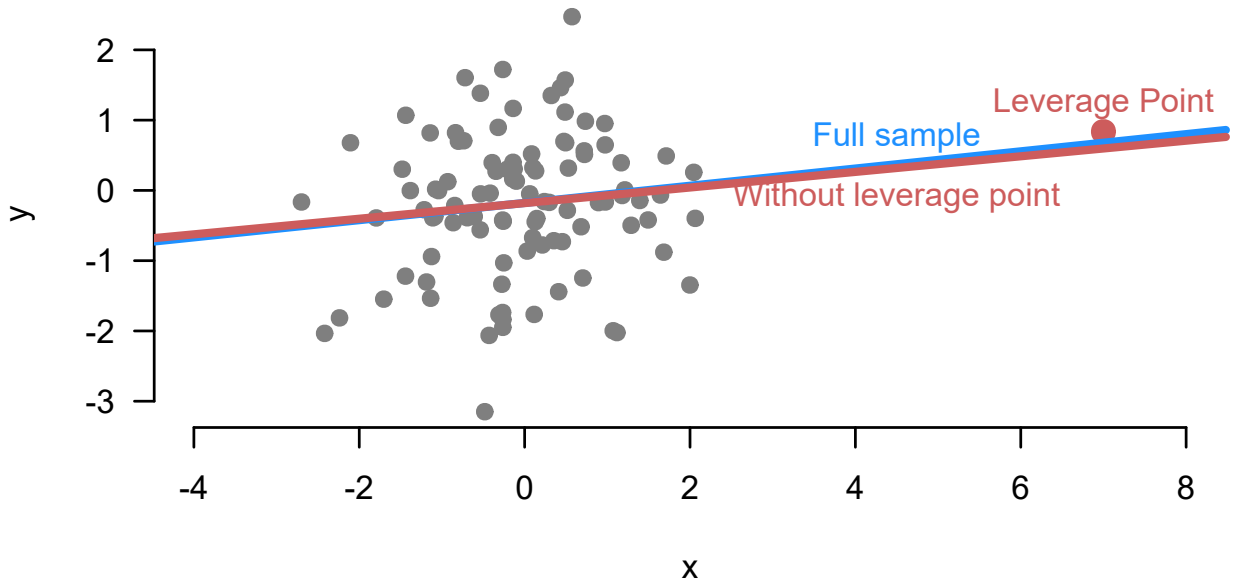

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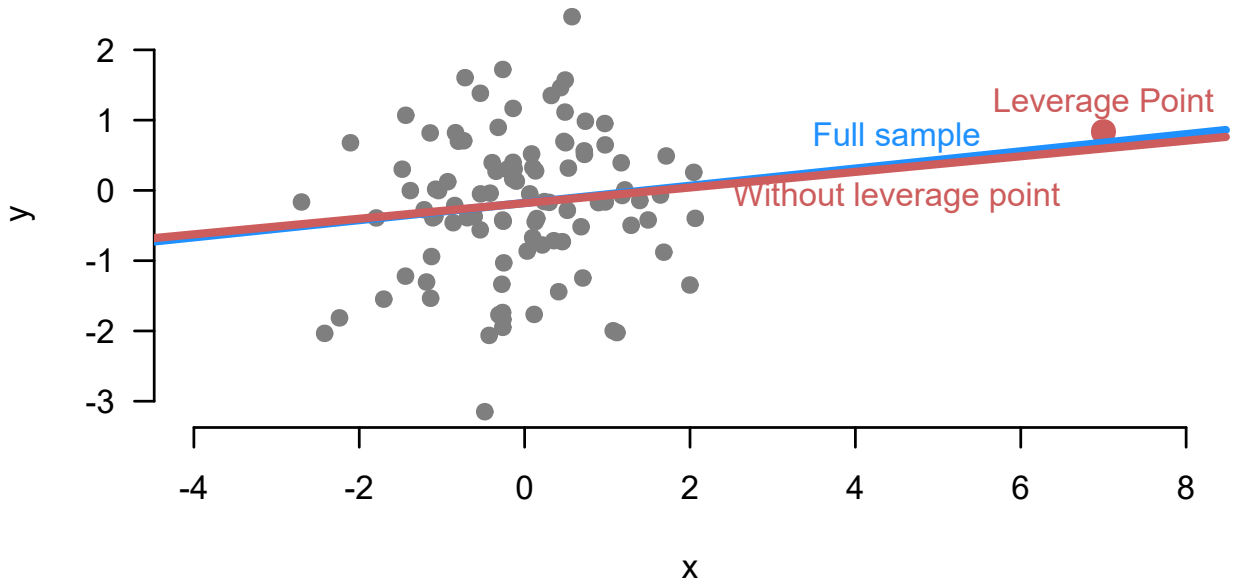

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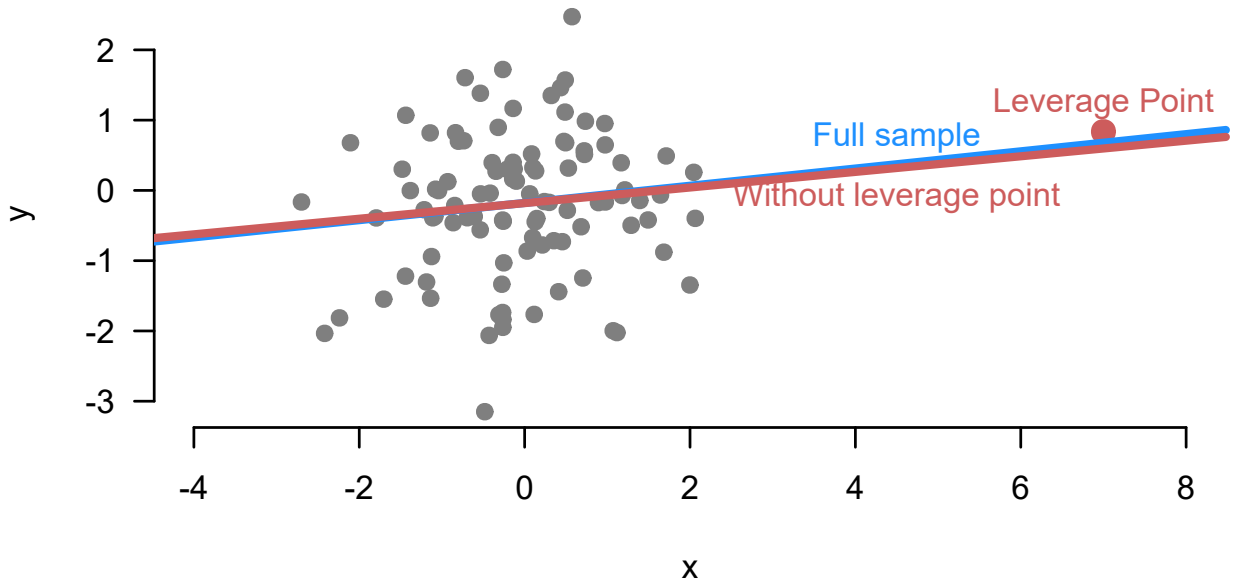

- Values that are extreme in the x direction
- That is, values far from the center of the covariate distribution
- Decrease SEs (more X variation)
- No bias if typical in y dimension

Leverage Points: Hat values

To measure leverage in multivariate data we will go back to the hat matrix $\mathbf{H}$:

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} = \mathbf{H}\mathbf{y}$$

$\mathbf{H}$ is $n \times n$, symmetric, and idempotent. It generates fitted values as follows:

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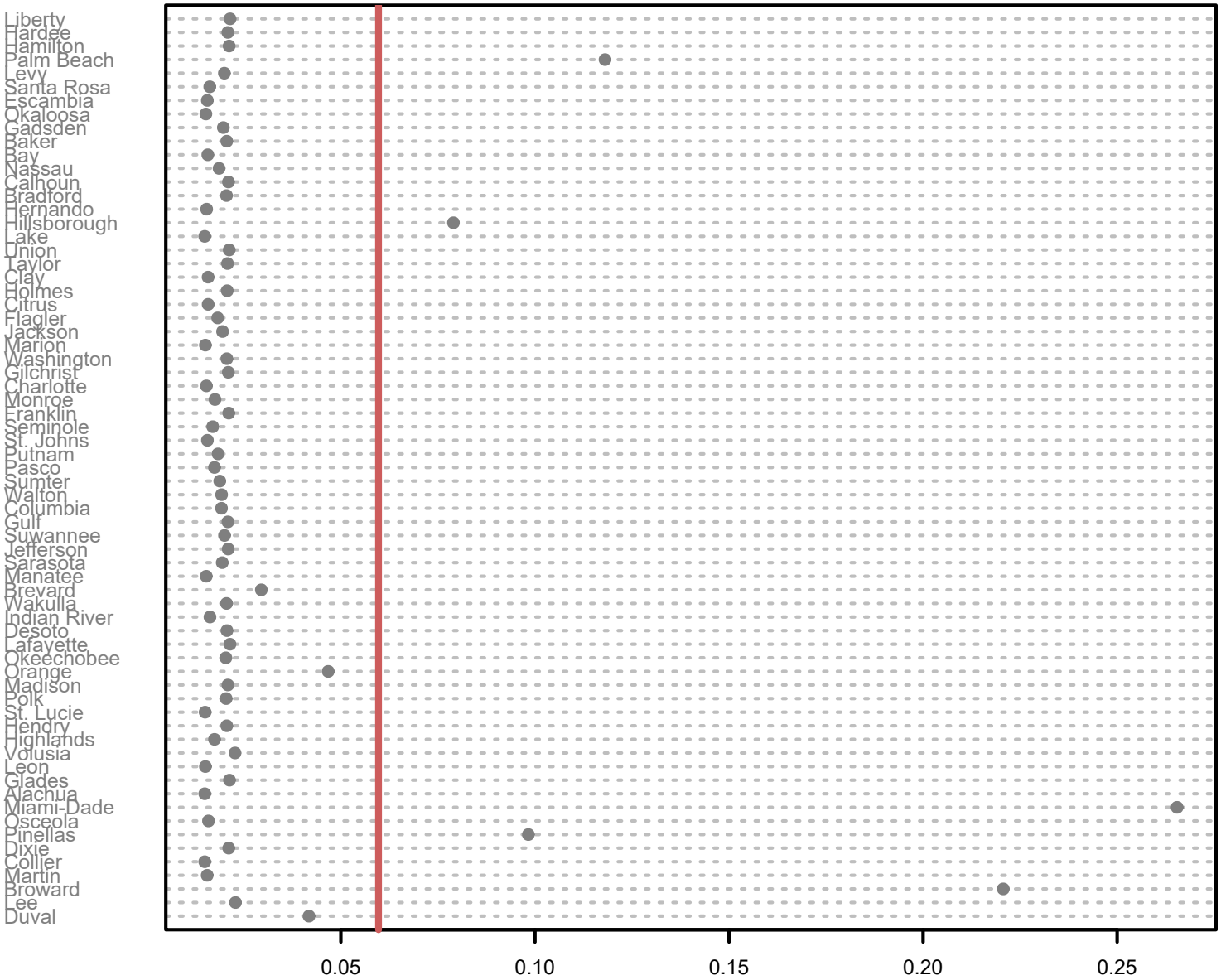
- Intuitively, the hat values measure how far a unit's vector of characteristics $\mathbf{x}_i$ is from the vector of means of **X**
- Rule of thumb: examine hat values greater than $2(k+1)/n$

Appendix: Facts about Hat Values

- $\sum_{i=1}^{n} h_i = k + 1$
- $1/n \geq h_i \geq 1$ for all i
- $\text{Var}[\widehat{u}_i] = (1 - h_i)\sigma^2$
- With a simple linear regression, we have

$$h_i = \frac{1}{n} + \frac{(X_i - \overline{X})^2}{\sum_{j=1}^{n}(X_j - \overline{X})^2}$$

Buchanan hats



Influence points

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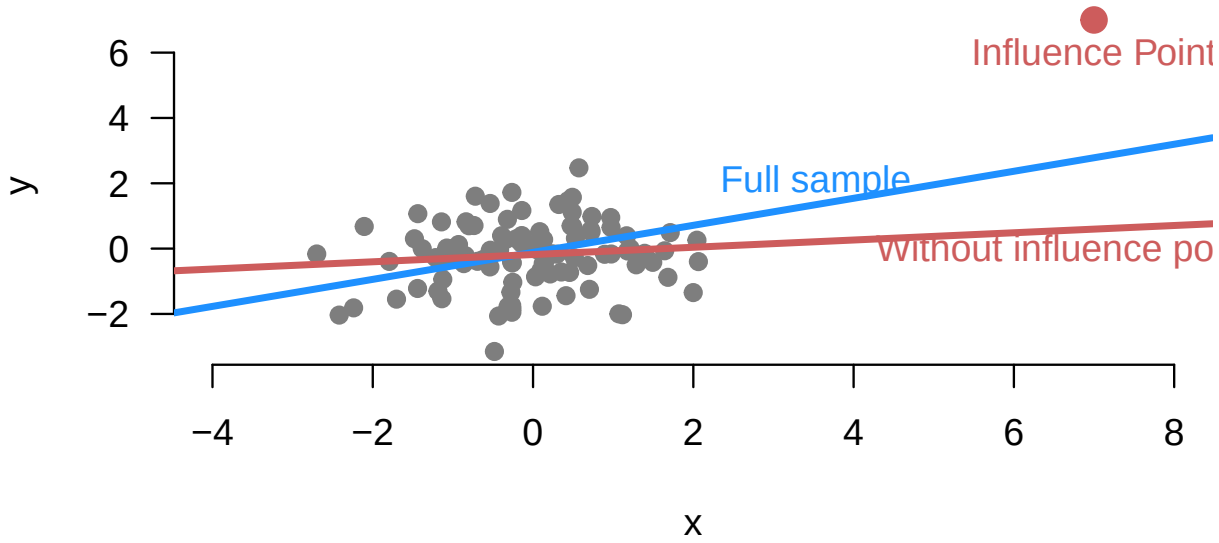

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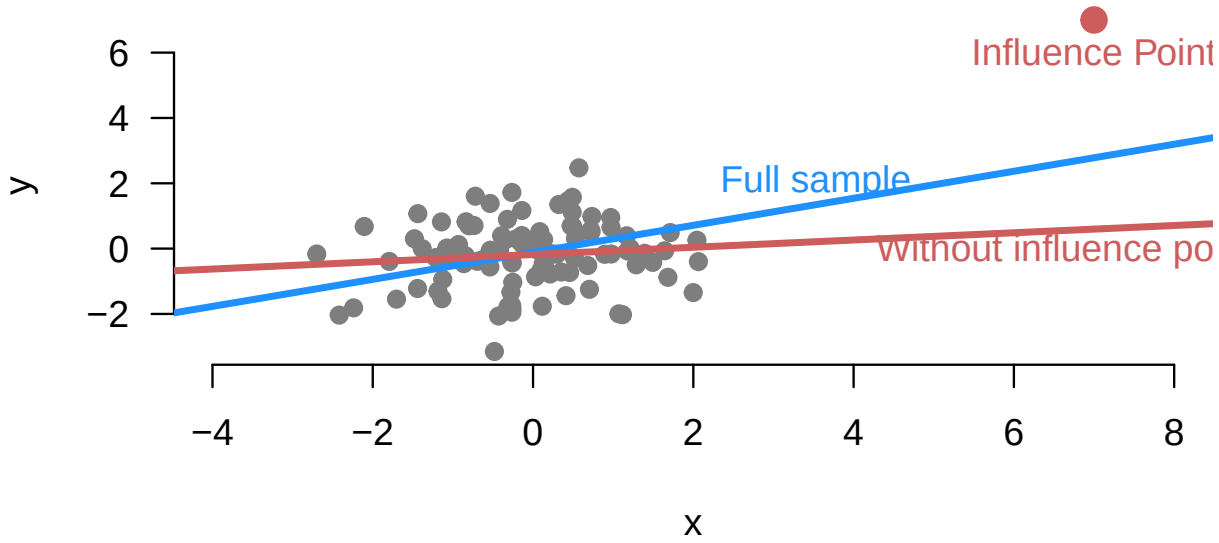

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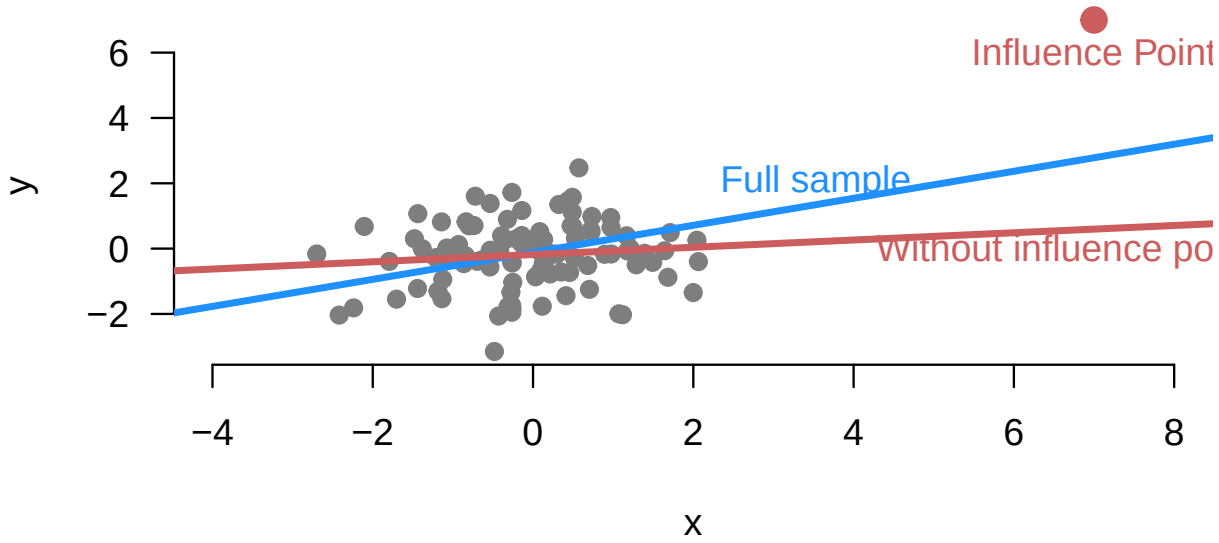

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- Causes the regression line to move toward it.

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- More formally: Measure the change that occurs in the slope estimates when an observation is removed from the data set. Let

$$D_{ij} = \hat{\beta}_j - \hat{\beta}_{j(-i)}, \quad i = 1, \ldots, n, \quad j = 0, \ldots, k$$

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- D_{ij} is called the DFbeta, which measures the influence of observation i on the estimated coefficient for the jth explanatory variable.

Standardized Influence

To make comparisons across coefficients, it is helpful to scale D_{ij} by the estimated standard error of the coefficients:

$$D_{ij}^{*} = \frac{\hat{\beta}_j - \hat{\beta}_{j(-i)}}{\hat{SE}_{-i}(\hat{\beta}_j)}$$

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- In R: `dfbetas(model)`

Buchanan influence

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -2.935e+01  5.520e+01  -0.532  0.59686
## edaytotal     1.100e-03  4.797e-04   2.293  0.02529 *
## absnbuchanan  6.895e+00  2.129e+00   3.238  0.00195 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 317.2 on 61 degrees of freedom
##   (3 observations deleted due to missingness)
## Multiple R-squared:  0.5361, Adjusted R-squared:  0.5209
## F-statistic: 35.24 on 2 and 61 DF,  p-value: 6.711e-11
```

Buchanan influence

```
##      (Intercept)     edaytotal  absnbuchanan
## 1  0.3454475146  0.4050504921 -0.7505222758
## 2 -0.0234266617 -0.0241000045 -0.0131672181
## 3  0.0650795039 -0.7319311820  0.3401669862
## 4 -0.0333980968  0.0133802934 -0.0087505576
## 5 -0.0397626659 -0.0073746223  0.0096551713
## 6 -0.0009277798  0.0001505476  0.0002210247
```

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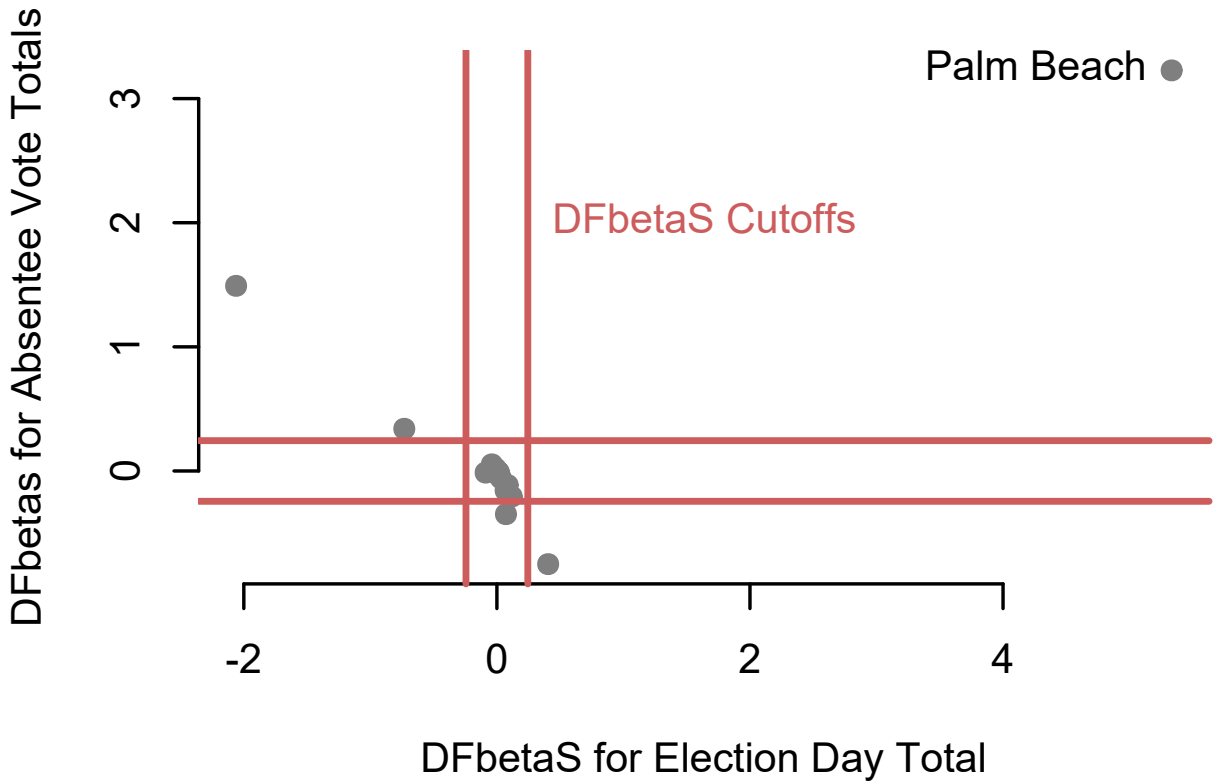

- Palm Beach county moves each of the coefficients by more than 3 standard errors!

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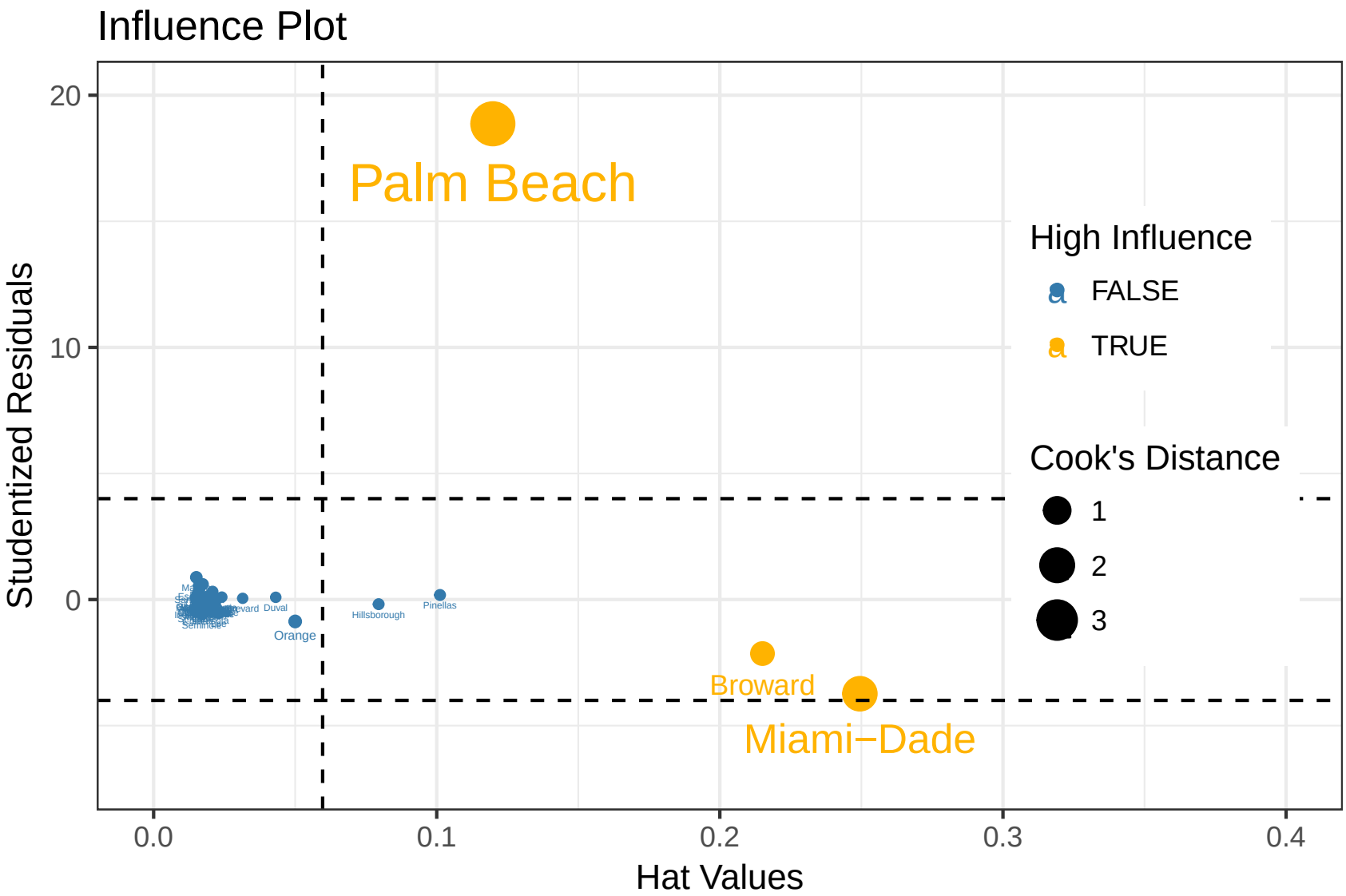
- The influence plot: the studentized residuals plotted against the hat values, size of points proportional to Cook's distance.

Influence Plot Buchanan



Courtesy of Erin Hartman

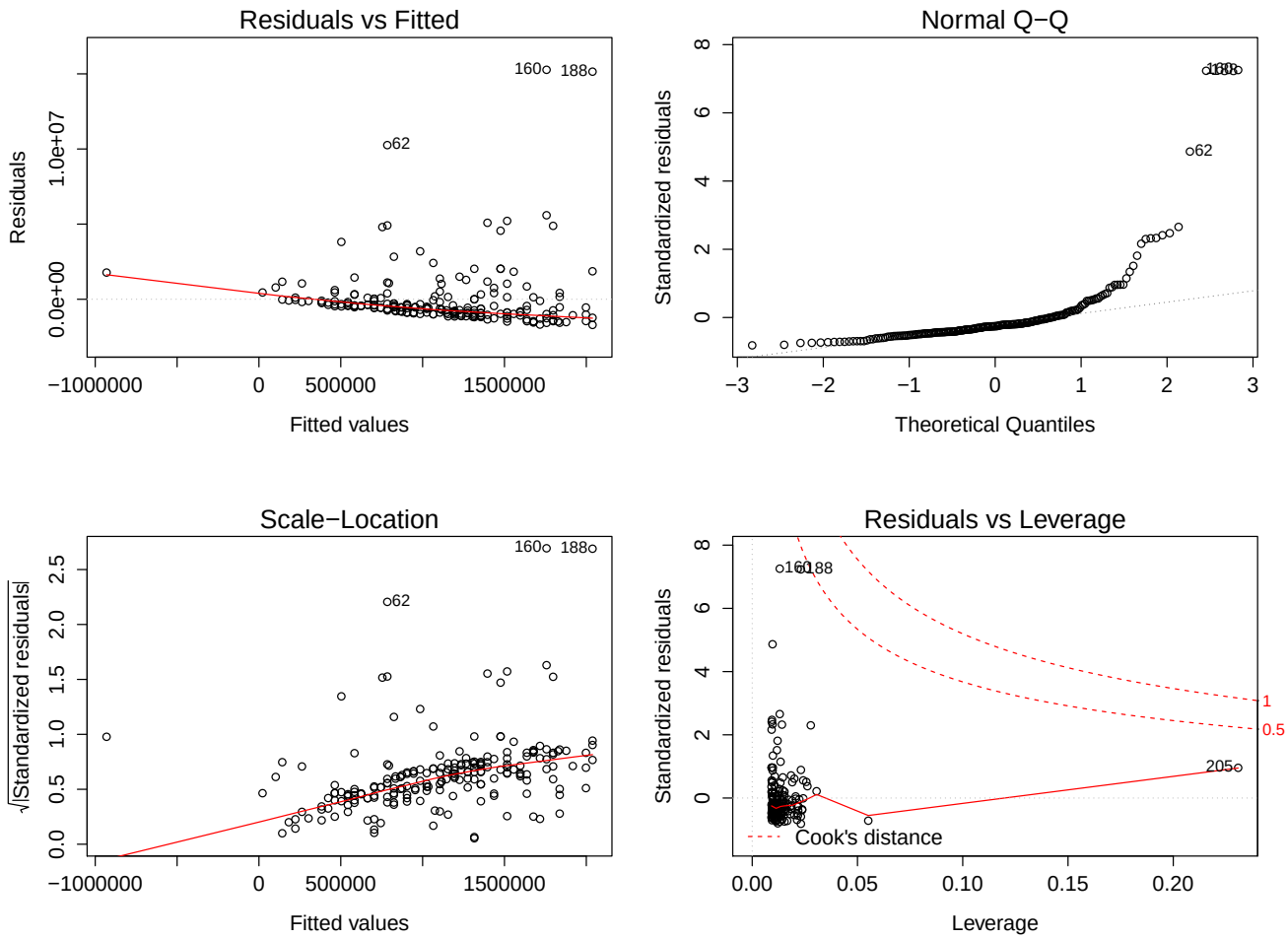
Code for Influence Plot

```
ggplot(fl_lm, aes(x = .hat, y = rstudent(fl_lm),
size = .cooksd,
col = .cooksd > 4/(nrow(fl_data) - 1 - 1),
label = fl_data$county)) +
geom_point() + geom_text(vjust = 2) +
xlab("Hat Values") + ylab("Studentized Residuals") +
geom_vline(xintercept = 2 * (fl_lm$rank - 1 + 1)/nrow(fl_data)
, linetype = 2) +
geom_hline(yintercept = c(-4, 4), linetype = 2) +
scale_color_manual("High Influence",
values = c("TRUE" = ucla_gold,
"FALSE" = ucla_blue)) +
scale_size("Cook's Distance") + theme_bw() +
theme(legend.position = c(0.9, 0.5)) + ylim(c(-7, 20)) +
xlim(c(0, 0.4)) + ggtitle("Influence Plot")
```

A Quick Function for Standard Diagnostic Plots

R Code

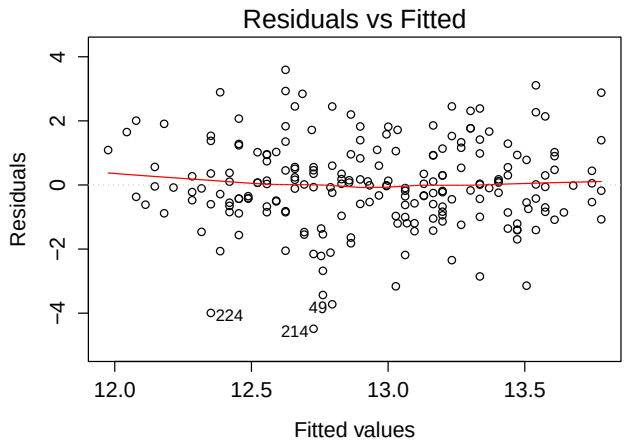
```
> par(mfrow=c(2,2))
> plot(mod1)
```

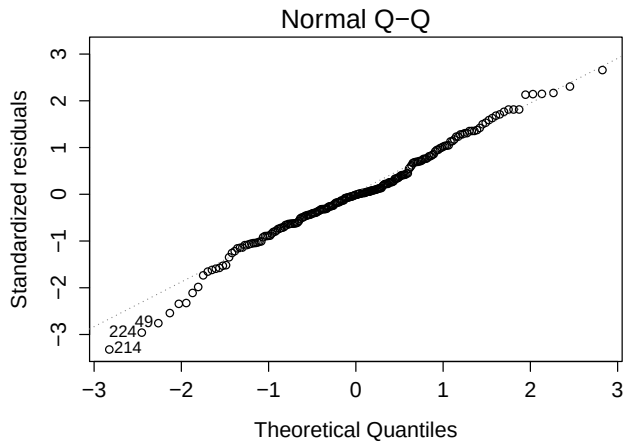


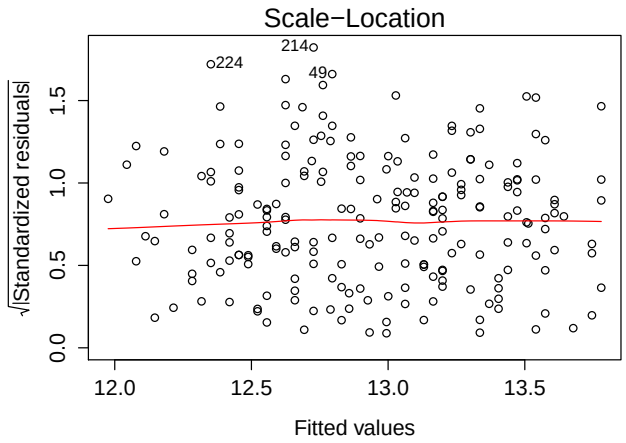
The Improved Model

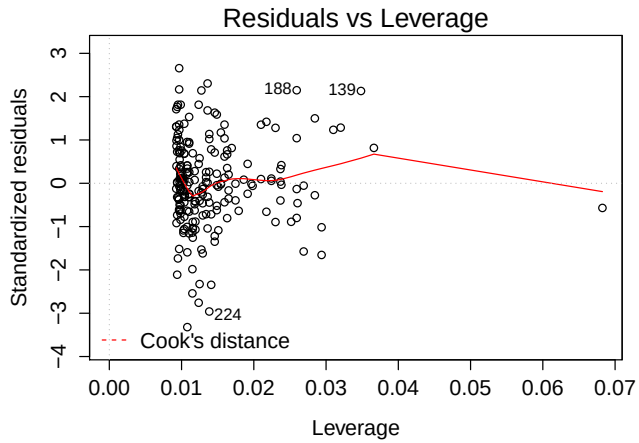
R Code

```
> par(mfrow=c(2,2))
> plot(mod2)
```









1. Non-Linearity
   - Polynomials
   - LOESS
   - Log Transformations
2. Non-Normality
   - Hat Matrix
   - Standardized and Studentized Residuals
   - QQ Plots
3. Extreme Values
   - Outliers
   - Leverage Points
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- Loosely speaking, you are tricking the regression into thinking it has more information than it does.

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- Under this clustered dependence, we can write this as:

$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1}\left(\sum_{g=1}^{m}\mathbf{X}_g'\Sigma_g\mathbf{X}_g\right)(\mathbf{X}'\mathbf{X})^{-1}$$

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- We can use a plugin estimator for the sampling variance on the previous slide with a small sample correction (many variants available, here we give the CR1 variant Stata uses):

$$\hat{V}_{CR1}[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \frac{m}{(m-1)}\frac{n-1}{n-k}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\sum_{g=1}^{m}\mathbf{X}_g'\hat{\boldsymbol{u}}_g\hat{\boldsymbol{u}}_g{}'\mathbf{X}_g\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- Another excellent strategy is to use the block bootstrap (resample clusters instead of units).

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- NB: this broad strategy of sandwich estimators works for other kinds of settings (time-series etc.). See `sandwich` package in `R`.

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Next week: Frameworks for Causal Inference!