Week 7: Multiple Regression

Brandon Stewart[1]

Princeton

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[1]Many of these slides adapted from Matt Blackwell, Adam Glynn, Justin Grimmer, Jens Hainmueller and Erin Hartman.

Where We've Been and Where We're Going...

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  - regression with two variables
  - omitted variables, multicollinearity, interactions

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- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

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- Outcome is a linear combination of the the **x**, **z**, and **u** vectors

Grouping Things into Matrices

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- We can also write this at the individual level, where $\mathbf{x}_i'$ is the ith row of $\mathbf{X}$:

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i$$

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- Here we need the equivalent of a matrix 'division' operator.

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Disclaimer: the final equation is exactly true for all non-intercept coefficients if you remove the intercept from $\mathbf{X}$ such that $\hat{\boldsymbol{\beta}}_{-0} = \text{Var}(\mathbf{X}_{-0})^{-1}\text{Cov}(\mathbf{X}_{-0}, \mathbf{y})$. The numerator and denominator are the variances and covariances if $\mathbf{X}$ and $\mathbf{y}$ are demeaned and normalized by the sample size minus 1.

Classical OLS Assumptions in Matrix Form

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1. Linearity: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$
2. Random/iid sample: $(y_i, \mathbf{x}_i')$ are a iid sample from the population.
3. No perfect collinearity: $\mathbf{X}$ is an $n \times (k+1)$ matrix with rank $k+1$
4. Zero conditional mean: $E[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
5. Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
6. Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I}_n)$

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3. No perfect collinearity: $\mathbf{X}$ is an $n \times (k+1)$ matrix with rank $k+1$
4. Zero conditional mean: $E[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
5. Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
6. Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I}_n)$

Assumption 3: No Perfect Collinearity

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(see essentially any linear algebra book for proof)
- Just like variation in X led us to be able to divide by the variance in simple OLS

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- Using the zero mean conditional error assumptions:

$$E[\mathbf{u}|\mathbf{X}] = \begin{bmatrix} E[u_1|\mathbf{X}] \\ E[u_2|\mathbf{X}] \\ \vdots \\ E[u_n|\mathbf{X}] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

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So, yes!

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- The variance of a vector is actually a matrix:

$$\text{var}[\mathbf{u}] = \Sigma_u = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_n) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_n) \\ \vdots & & \ddots & \\ \text{cov}(u_n, u_1) & \text{cov}(u_n, u_2) & \dots & \text{var}(u_n) \end{bmatrix}$$

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- This matrix is always symmetric since $\text{cov}(u_i, u_j) = \text{cov}(u_j, u_i)$ by definition.

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- In less matrix notation:
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Rule: Variance of Linear Function of Random Vector

Recall that for a linear transformation of a random variable X we have $V[aX + b] = a^2 V[X]$ with constants a and b.

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Definition (Variance of Linear Transformation of Random Vector)

Let $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{B}$ be a linear transformation of a random vector $\mathbf{u}$ with non-random vectors or matrices $\mathbf{A}$ and $\mathbf{B}$. Then the variance of the transformation is given by:

$$V[f(\mathbf{u})] = V[\mathbf{A}\mathbf{u} + \mathbf{B}] = \mathbf{A}V[\mathbf{u}]\mathbf{A}'$$

Conditional Variance of $\hat{\beta}$

$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \boldsymbol{\beta} + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \boldsymbol{\beta}$ so the OLS estimator is a linear function of the errors. Thus:

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&= \sigma^2\mathbf{I}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}
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&= \sigma^2\mathbf{I}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

Conditional Variance of $\hat{\beta}$

$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \boldsymbol{\beta} + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \boldsymbol{\beta}$ so the OLS estimator is a linear function of the errors. Thus:

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&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V[\mathbf{u}|\mathbf{X}]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \quad \text{(by homoskedasticity)} \\
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To estimate $V[\hat{\boldsymbol{\beta}}|\mathbf{X}]$, we replace σ^2 with its unbiased estimator $\hat{\sigma}^2$, which is now written using matrix notation as:

$$
\hat{\sigma}^2 = \frac{\sum_i \hat{u}_i^2}{n-(k+1)} = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}
$$

Sampling Variance for $\hat{\beta}$

Under assumptions 1-5, the variance-covariance matrix of the OLS estimators is given by:

$$V[\hat{\beta}|\mathbf{X}] = \sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1} =$$

$$\begin{array}{c|cccccc}
 & \widehat{\beta}_0 & \widehat{\beta}_1 & \widehat{\beta}_2 & \cdots & \widehat{\beta}_k \\
\hline
\widehat{\beta}_0 & V[\widehat{\beta}_0] & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_1] & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_2] & \cdots & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_k] \\
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\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
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Recall that standard errors are the square root of the diagonals of this matrix.

Overview of Inference in the General Setting

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$$\frac{\widehat{\beta}_j - \beta_j}{\widehat{SE}[\widehat{\beta}_j]} \sim N(0, 1)$$

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- Thus, confidence intervals and hypothesis tests proceed in essentially the same way.

Properties of the OLS Estimator: Summary

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Theorem

Under Assumptions 1–6, the $(k+1) \times 1$ vector of OLS estimators $\hat{\boldsymbol{\beta}}$, conditional on $\mathbf{X}$, follows a multivariate normal distribution *with mean $\boldsymbol{\beta}$ and variance-covariance matrix $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$:*

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- *With a large sample,* $\hat{\boldsymbol{\beta}}$ *approximately follows the same distribution under Assumptions 1–5 only, i.e., without assuming the normality of* $\mathbf{u}$.

Implications of the Variance-Covariance Matrix

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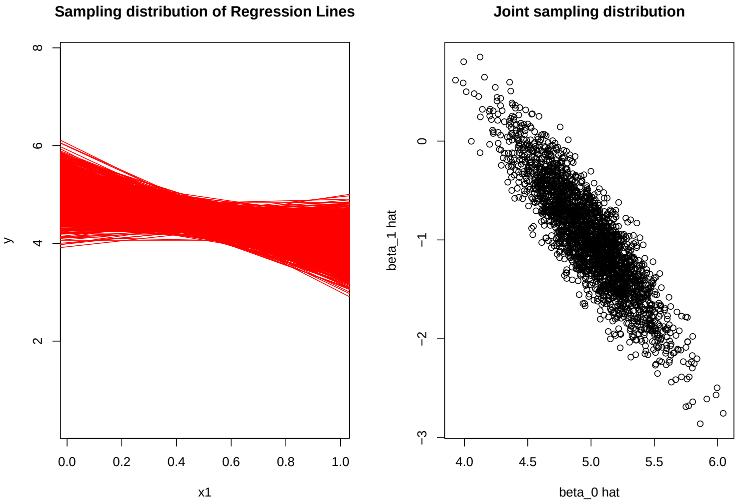
- This is because the sampling distribution of the terms in $\hat{\beta}$ are correlated.
- In a practical sense, this means that our uncertainty about coefficients is correlated across variables.

Multivariate Normal: Simulation

$Y = \beta_0 + \beta_1 X_1 + u$ with $u \sim N(0, \sigma_u^2 = 4)$ and $\beta_0 = 5$, $\beta_1 = -1$, and $n = 100$:



Marginals of Multivariate Normal RVs are Normal

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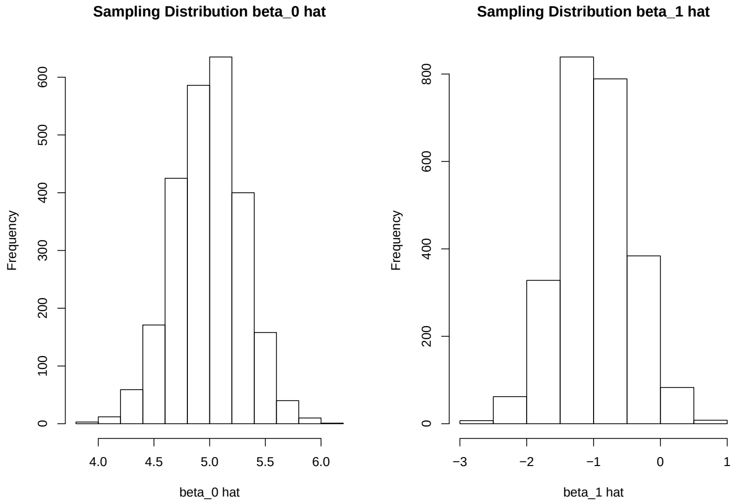

1. Matrix Notation of Regression

2. OLS Properties Under Gauss-Markov Assumptions
   - Unbiasedness
   - Classical Standard Errors

3. Asymptotics of OLS with Only IID Sampling

4. Standard Hypothesis Tests
   - t-Tests
   - Adjusted R^2
   - F Tests for Joint Significance

Asymptotics of the OLS Estimator with IID Sampling

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- Finite-sample guarantees came at the price of strong assumption (CEF is linear for unbiasedness, homoskedastic-normal errors for standard errors)
- Asymptotic results will hold more generally under essentially only IID sampling, but require more data.
- Define the Best Linear Projection (BLP) model as

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$$
$$E[\mathbf{X}'\mathbf{u}] = 0$$
$$E[\mathbf{X}'\mathbf{X}] > 0 \text{ (invertability)}$$

NB: ideally we would rewrite these all in terms of scalar random variable Y and random vectors for X. I've opted to keep them in matrices so the notation is parallel (you can imagine the vectors for Y are length 1 and matrices for X have one row), but the key idea remains that the expectations are over random draws.

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- Plug-in principle $\rightsquigarrow$ replace population expectation with sample versions:

$$\hat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{y}$$

Asymptotic OLS inference

- With this representation, we can write the OLS estimator as follows:

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- Note that we have only used IID sampling and no perfect collinearity so far.

Theorem (Asymptotic Normality of OLS Under Random Sampling)

Under the Best Linear Projection model,

$$\sqrt{N}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \mathbf{\Omega})$$

where

$$\mathbf{\Omega} = E[\mathbf{X}'\mathbf{X}]^{-1} E[\mathbf{X}' Diag(\mathbf{u^2})\mathbf{X}] E[\mathbf{X}'\mathbf{X}]^{-1}$$

and

- $\hat{\beta}$ *is approximately normal asymptotically with mean* β *and variance* $\mathbf{\Omega}/n$.
- $\mathbf{\Omega}/n$ *is the asymptotic covariance matrix of* $\hat{\beta}$ *and the square root of the diagonal are the standard errors.*
- *Asymptotically this provides us with all the necessary infrastructure for confidence intervals and tests.*

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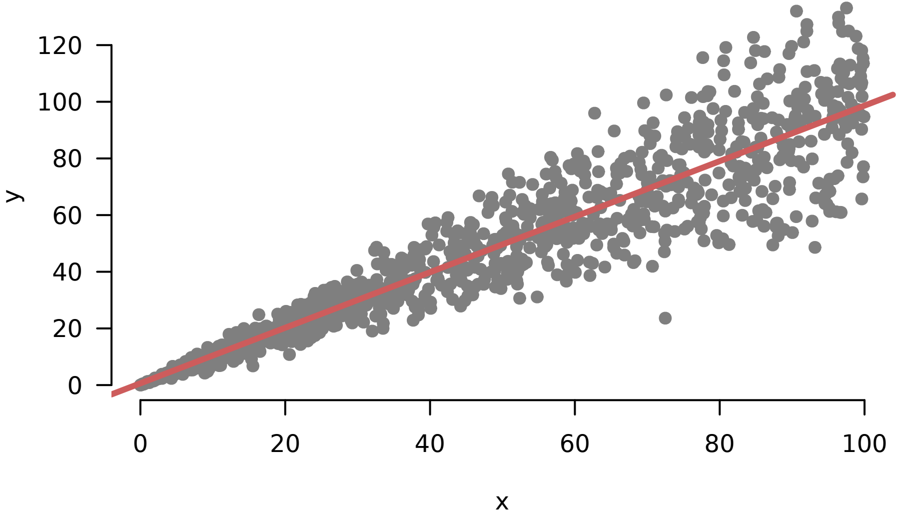
- Replace σ^2 with estimate $\widehat{\sigma}^2$ will give us our estimate of the covariance matrix

What Does This Rule Out?



Non-constant Error Variance

- Homoskedastic:

$$V[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I} = \left[\begin{array}{ccccc} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{array}\right]$$

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- Idea: If we can consistently estimate the components of $\mathbf{\Sigma}$, we could directly use this expression by replacing $\mathbf{\Sigma}$ with its estimate, $\hat{\mathbf{\Sigma}}$.

White's Heteroskedasticity Consistent Estimator

Suppose we have heteroskedasticity of unknown form (but zero covariance):

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The estimate based on the above is called the heteroskedasticity consistent (HC) or robust standard errors. This also coincides with the agnostic standard errors!

Intuition for Robust Standard Errors

Core intuition: while $\widehat{\mathbf{\Sigma}}$ is an $n \times n$ matrix, $\mathbf{X}'\widehat{\mathbf{\Sigma}}\mathbf{X}$ is a $(k+1) \times (k+1)$ matrix.

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- There are various small sample corrections to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

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- As we saw, we can arrive at White's heteroskedasticity consistent standard errors using the plug-in principle and thus in some ways, these are the natural way of getting standard errors in the agnostic regression framework.
- Robust SEs converge to same point as the bootstrap.

1. Matrix Notation of Regression

2. OLS Properties Under Gauss-Markov Assumptions
   - Unbiasedness
   - Classical Standard Errors

3. Asymptotics of OLS with Only IID Sampling

4. Standard Hypothesis Tests
   - t-Tests
   - Adjusted R^2
   - F Tests for Joint Significance

Running Example: Chilean Referendum on Pinochet

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- We model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

Hypothesis Testing in R

Model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,        Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

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where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.
That is, take the variance-covariance matrix of $\hat{\boldsymbol{\beta}}$ and square root the diagonal element corresponding to j.

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We can pull classical variance-covariance matrix in `R` from the `lm()` object:

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educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
```

We can pull classical variance-covariance matrix in R from the `lm()` object:

R Code

```
> V <- vcov(fit)
> V
              (Intercept)           fem          educ           age
(Intercept)  2.642311e-03 -3.455498e-04 -5.270913e-04 -3.357119e-05
fem         -3.455498e-04  5.623170e-04  2.249973e-05  8.285291e-07
educ        -5.270913e-04  2.249973e-05  1.922354e-04  3.411049e-06
age         -3.357119e-05  8.285291e-07  3.411049e-06  6.914098e-07

> sqrt(diag(V))
 (Intercept)          fem         educ          age
0.0514034097 0.0237132251 0.0138648980 0.0008315105
```

Using the t-Value as a Test Statistic

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$$P\left(-t_{\alpha/2} \leq T \leq t_{\alpha/2}\right) = 1 - \alpha$$

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$$P\left(-t_{\alpha/2} \leq T \leq t_{\alpha/2}\right) = 1 - \alpha$$
3. Decide whether the realized value of T in our data is unusual given the distribution of the test statistic under the null hypothesis.
4. Finally, either declare that we reject H_0 or not, or report the p-value.

Confidence Intervals

To construct confidence intervals, there is again no difference compared to the case of $k = 1$, except that we need to use t_{n-k-1} instead of t_{n-2}

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$$\left[\widehat{\beta}_j - t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\,,\, \widehat{\beta}_j + t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\right]$$

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We rearrange:

$$\left[\widehat{\beta}_j - t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\,,\; \widehat{\beta}_j + t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\right]$$

and thus can construct the confidence intervals as usual using:

$$\hat{\beta}_j \pm t_{\alpha/2} \cdot \hat{SE}[\hat{\beta}_j]$$

Confidence Intervals in R

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
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age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
```

R Code

```
> confint(fit)
                   2.5 %      97.5 %
(Intercept)  0.303407780  0.50504909
fem          0.089493169  0.18251357
educ        -0.087954435 -0.03356629
age          0.002147755  0.00540954
```

Testing Hypothesis About a Linear Combination of β_j

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R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> summary(fit)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       4452.7      783.4   5.684 2.07e-07 ***
RegionAfrica     -2552.8     1204.5  -2.119   0.0372 *
RegionAsia         148.9     1149.8   0.129   0.8973
RegionLatAmerica  -271.3     1007.0  -0.269   0.7883
RegionOecd        9671.3     1007.0   9.604 5.74e-15 ***
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```

- $\hat{\beta}_{Asia}$ and $\hat{\beta}_{LAm}$ are close. So we may want to test the null hypothesis:

$$H_0: \ \beta_{LAm} = \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} = 0$$

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against the alternative of

$$H_1 : \ \beta_{LAm} \neq \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} \neq 0$$

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- What would be an appropriate test statistic for this hypothesis?

Testing Hypothesis About a Linear Combination of β_j

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```

- Let's consider a t-value:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})}$$

We will reject H_0 if T is sufficiently different from zero.

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- Let's consider a t-value:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})}$$

  We will reject H_0 if T is sufficiently different from zero.

- Note that unlike the test of a single hypothesis, both $\hat{\beta}_{LAm}$ and $\hat{\beta}_{Asia}$ are random variables, hence the denominator.

Testing Hypothesis About a Linear Combination of β_j

- Our test statistic:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \ \sim \ t_{n-k-1}$$

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$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \sim t_{n-k-1}$$

- How do you find $\hat{SE}(\hat{\beta}_{LAm} - \hat{\beta}_{Asia})$?

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- Recall the following property of the variance:

$$V(X \pm Y) = V(X) + V(Y) \pm 2\text{Cov}(X, Y)$$

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Therefore, the standard error for a linear combination of coefficients is:

$$\hat{SE}(\widehat{\beta}_1 \pm \widehat{\beta}_2) = \sqrt{\widehat{V}(\widehat{\beta}_1) + \widehat{V}(\widehat{\beta}_2) \pm 2\widehat{\text{Cov}}[\widehat{\beta}_1, \widehat{\beta}_2]}$$

which we can calculate from the estimated covariance matrix of $\hat{\boldsymbol{\beta}}$.

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  which we can calculate from the estimated covariance matrix of $\hat{\boldsymbol{\beta}}$.
- Since the estimates of the coefficients are correlated, we need the covariance term.

Example: GDP per capita on Regions

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R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> V <- vcov(fit)
> V
                (Intercept) RegionAfrica RegionAsia RegionLatAmerica
(Intercept)        613769.9    -613769.9  -613769.9        -613769.9
RegionAfrica      -613769.9    1450728.8   613769.9         613769.9
RegionAsia        -613769.9     613769.9  1321965.9         613769.9
RegionLatAmerica  -613769.9     613769.9   613769.9        1014054.6
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                 RegionOecd
(Intercept)       -613769.9
RegionAfrica       613769.9
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RegionLatAmerica   613769.9
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```

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[1] 1052.844
>
> tstat <- (coef(fit)[4] - coef(fit)[3])/se
> tstat
RegionLatAmerica
      -0.3990977
```

$$t = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \quad \text{where}$$

$$\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}) = \sqrt{\widehat{V}(\widehat{\beta}_{LAm}) + \widehat{V}(\widehat{\beta}_{Asia}) - 2\widehat{\text{Cov}}[\widehat{\beta}_{LAm}, \widehat{\beta}_{Asia}]}$$

Plugging in we get $t \approx -0.40$. So what do we conclude?

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Plugging in we get $t \approx -0.40$. So what do we conclude?

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NB: this same strategy can be used to get standard errors for interactions at any value of the other variables.

Adjusted R^2

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R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
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age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,       Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

Adjusted R^2

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$$R^2 = 1 - \frac{\text{SS}_{\text{res}}}{\text{SS}_{\text{tot}}}$$

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- This makes R^2 more 'comparable' across models with different numbers of variables, but the next section will show you an even better way to approach that problem in a testing framework.

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- So, how do we penalize more complex models? Adjusted R^2
- This makes R^2 more 'comparable' across models with different numbers of variables, but the next section will show you an even better way to approach that problem in a testing framework.
- Still since people report it, the next slide derives adjusted R^2.

Adjusted R^2

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- Key idea: rewrite R^2 in terms of variances

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$$R^2 = 1 - \frac{\text{SS}_{\text{res}}/n}{\text{SS}_{\text{tot}}/n}$$

Adjusted R^2

- Key idea: rewrite R^2 in terms of variances

$$R^2 = 1 - \frac{\text{SS}_{\text{res}}/n}{\text{SS}_{\text{tot}}/n} = 1 - \frac{\tilde{V}(\text{SS}_{\text{res}})}{\tilde{V}(\text{SS}_{\text{tot}})}$$

where $\tilde{V}$ is a biased estimator of the population variance.

Adjusted R^2

- Key idea: rewrite R^2 in terms of variances

$$
\begin{aligned}
R^2 &= 1 - \frac{\text{SS}_{\text{res}}/n}{\text{SS}_{\text{tot}}/n} \\
&= 1 - \frac{\tilde{V}(\text{SS}_{\text{res}})}{\tilde{V}(\text{SS}_{\text{tot}})}
\end{aligned}
$$

  where $\tilde{V}$ is a biased estimator of the population variance.

- What if we replace the biased estimator with the unbiased estimators

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- Key idea: rewrite R^2 in terms of variances

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- Adjusted R^2 will always be smaller than R^2 and can sometimes be negative!

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- Regardless the intuition here is useful: as we add more variables R^2 must stay constant or go up mechanically and that creates complications.

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- F tests allows us to to test joint hypothesis

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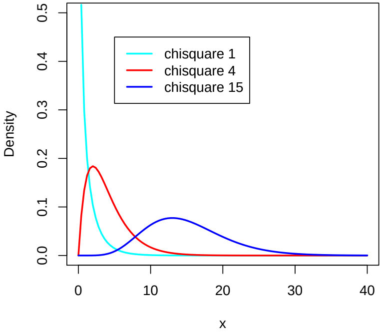

Properties: $X > 0$, $E[X] = n$ and $V[X] = 2n$. In R: `dchisq()`, `pchisq()`, `rchisq()`

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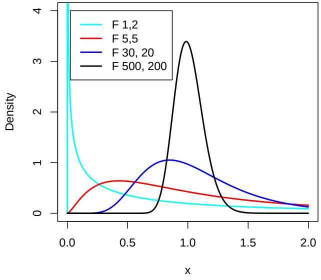

In R: `df()`, `pf()`, `rf()`

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- Asymptotically under IID sampling, $F_0 \overset{d}{\sim} \mathcal{F}_{q,n-k-1}$

Unrestricted Model (UR)

R Code

```
> fit.UR <- lm(vote1 ~ fem + educ + age + fem:age + fem:educ, data = Chile)
> summary(fit.UR)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
educ        -0.038571   0.019578  -1.970 0.048988 *
age          0.005482   0.001114   4.921 9.44e-07 ***
fem:age     -0.003779   0.001673  -2.259 0.024010 *
fem:educ    -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,	Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Restricted Model (R)

R Code

```
> fit.R <- lm(vote1 ~ educ + age, data = Chile)
> summary(fit.R)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4878039  0.0497550   9.804  < 2e-16 ***
educ        -0.0662022  0.0139615  -4.742 2.30e-06 ***
age          0.0035783  0.0008385   4.267 2.09e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4921 on 1700 degrees of freedom
Multiple R-squared:  0.03275,       Adjusted R-squared:  0.03161
F-statistic: 28.78 on 2 and 1700 DF,  p-value: 5.097e-13
```

F Test in R

R Code

```
> SSR.UR <- sum(resid(fit.UR)^2)  # = 402
> SSR.R <- sum(resid(fit.R)^2)    # = 411

> DFdenom <- df.residual(fit.UR)  # = 1703
> DFnum <- 3

> F <- ((SSR.R - SSR.UR)/DFnum) / (SSR.UR/DFdenom)
> F
[1] 13.01581

> qf(0.99, DFnum, DFdenom)
[1] 3.793171
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Given above, what do we conclude?

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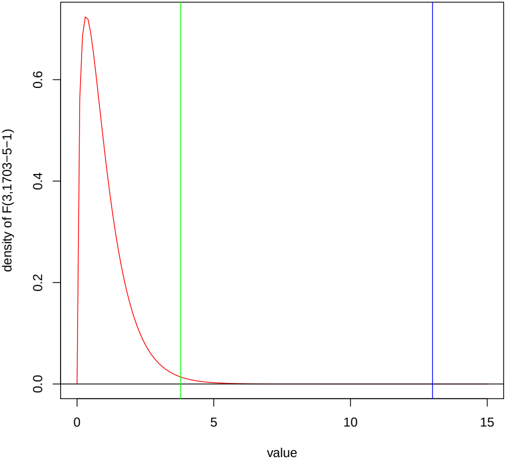
Given above, what do we conclude?

$F_0 = 13$ is greater than the critical value for a .01 level test. So we *reject* the null hypothesis.

Null Distribution, Critical Value, and Test Statistic

Note that the F statistic is always positive, so we only look at the right tail of the reference F (or χ^2 in a large sample) distribution.



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- Have any of you used an F-test like this in your research?
- This is called the omnibus test and is routinely reported by statistical software.

Omnibus Test in R

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Coefficients:
             Estimate Std. Error t value Pr(>|t|)
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Omnibus Test in R with Random Noise

R Code

```
> set.seed(08540)
> p <- 10; x <- matrix(rnorm(p*1000), nrow=1000)
> y <- rnorm(1000); summary(lm(y~x))
~~~~~

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0115475  0.0320874  -0.360   0.7190
x1          -0.0019803  0.0333524  -0.059   0.9527
x2           0.0666275  0.0314087   2.121   0.0341 *
x3          -0.0008594  0.0321270  -0.027   0.9787
x4           0.0051185  0.0333678   0.153   0.8781
x5           0.0136656  0.0322592   0.424   0.6719
x6           0.0102115  0.0332045   0.308   0.7585
x7          -0.0103903  0.0307639  -0.338   0.7356
x8          -0.0401722  0.0318317  -1.262   0.2072
x9           0.0553019  0.0315548   1.753   0.0800 .
x10          0.0410906  0.0319742   1.285   0.1991
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.011 on 989 degrees of freedom
Multiple R-squared:  0.01129,       Adjusted R-squared:  0.001294
F-statistic: 1.129 on 10 and 989 DF,  p-value: 0.3364
```

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Next, let's consider:

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- What question are we asking?
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- How many restrictions?

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$$H_0 : \beta_1 = \beta_2 = \beta_3$$

- What question are we asking?
  $\rightarrow$ Are the coefficients X_1, X_2 and X_3 different from each other?
- How many restrictions?
  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

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  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)
- How do we fit the restricted model?
  $\rightarrow$ The null hypothesis implies that the model can be written as:

  $$Y = \beta_0 + \beta_1 (X_1 + X_2 + X_3) + ... + \beta_k X_k + u$$

  So we create a new variable $X^* = X_1 + X_2 + X_3$ and fit:

  $$Y = \beta_0 + \beta_1 X^* + ... + \beta_k X_k + u$$

Testing Equality of Coefficients in R

R Code

```
> fit.UR2 <- lm(REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> summary(fit.UR2)
~~~~~
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1899.9      914.9   2.077   0.0410 *
Asia          2701.7     1243.0   2.173   0.0327 *
LatAmerica    2281.5     1112.3   2.051   0.0435 *
Transit       2552.8     1204.5   2.119   0.0372 *
Oecd         12224.2     1112.3  10.990   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 3034 on 80 degrees of freedom
Multiple R-squared: 0.7096,        Adjusted R-squared: 0.6951
F-statistic: 48.88 on 4 and 80 DF,  p-value: < 2.2e-16
```

Are the coefficients on *Asia*, *LatAmerica* and *Transit* statistically significantly different?

Testing Equality of Coefficients in R

R Code

```
> D$Xstar <- D$Asia + D$LatAmerica + D$Transit
> fit.R2 <- lm(REALGDPCAP ~ Xstar + Oecd, data = D)

> SSR.UR2 <- sum(resid(fit.UR2)^2)
> SSR.R2 <- sum(resid(fit.R2)^2)

> DFdenom <- df.residual(fit.UR2)

> F <- ((SSR.R2 - SSR.UR2)/2) / (SSR.UR2/DFdenom)
> F
[1] 0.08786129

> pf(F, 2, DFdenom, lower.tail = F)
[1] 0.9159762
```

So, what do we conclude?

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So, what do we conclude?

The three coefficients are statistically indistinguishable from each other, with the p-value of 0.916.

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

t Test vs. F Test

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- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2)/\hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the t test.

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- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR})\big/(SSR_R/(n-k-1))$ and do the F test.

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It turns out these two tests give identical results. This is because

$$X \sim t_{n-k-1} \iff X^2 \sim \mathcal{F}_{1,n-k-1}$$

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It turns out these two tests give identical results. This is because

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- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.
- Usually, the t test is used for single hypotheses and the F test is used for joint hypotheses.

Some More Notes on F Tests

- The F-value can also be calculated from R^2:

$$F = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n - k - 1)}$$

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- F tests only work for testing nested models, i.e. the restricted model must be a special case of the unrestricted model.

  For example F tests cannot be used to test

$$Y = \beta_0 + \beta_1 X_1 \qquad\qquad + \beta_3 X_3 + u$$

against

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \qquad\quad + u$$

Some More Notes on F Tests

Joint significance does not necessarily imply the significance of individual coefficients, or vice versa:

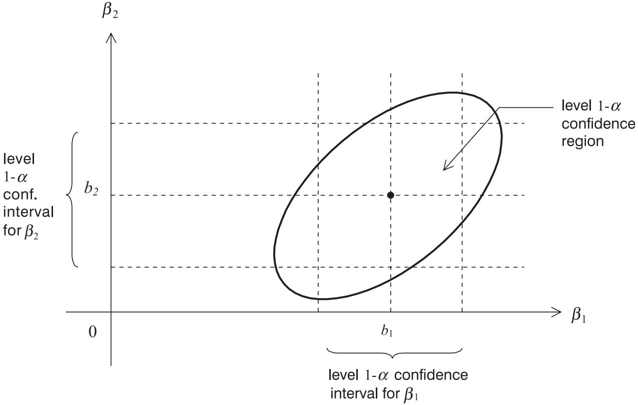

Figure 1.5: t- versus F-Tests

Image Credit: Hayashi (2011) *Econometrics*

Goal Check: Understand `lm()` Output

```
Call:
lm(formula = sr ~ pop15, data = LifeCycleSavings)

Residuals:
   Min     1Q Median     3Q    Max
-8.637 -2.374  0.349  2.022 11.155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.49660    2.27972   7.675 6.85e-10 ***
pop15       -0.22302    0.06291  -3.545 0.000887 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.03 on 48 degrees of freedom
Multiple R-squared:  0.2075,Adjusted R-squared:  0.191
F-statistic: 12.57 on 1 and 48 DF,  p-value: 0.0008866
```

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Next week: Diagnostics!