Week 5: Simple Linear Regression

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Princeton

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[1] Many of these slides are adapted from Matt Blackwell, Adam Glynn, Erin Hartman and Jens Hainmueller. Illustrations by Shay O'Brien.

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- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

Narrow Goal for the Week: Understand `lm()` Output

```
Call:
lm(formula = sr ~ pop15, data = LifeCycleSavings)

Residuals:
   Min     1Q Median     3Q    Max
-8.637 -2.374  0.349  2.022 11.155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.49660    2.27972   7.675 6.85e-10 ***
pop15       -0.22302    0.06291  -3.545 0.000887 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.03 on 48 degrees of freedom
Multiple R-squared:  0.2075,Adjusted R-squared:  0.191
F-statistic: 12.57 on 1 and 48 DF,  p-value: 0.0008866
```

Macrostructure—This Semester

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The next few weeks,

- Linear Regression with Two Regressors
- Break Week
- Multiple Linear Regression
- What Can Go Wrong and How to Fix It
- Frameworks for Causal Inference
- Causality with Measured Confounding
- Sensitivity Analysis / Thanksgiving
- Unmeasured Confounding and Instrumental Variables
- Repeated Observations, Panel Data and Wrap-up

Reminder of Definitions

Definition (Conditional Expectation (Discrete))

Let Y and X be discrete random variables. The conditional expectation of Y given $X = x$ is defined as:

$$E[Y|X = x] = \sum_y y\, P(Y = y|X = x) = \sum_y y\, p_{Y|X}(y|x)$$

Definition (Conditional Expectation (Continuous))

Let Y and X be continuous random variables. The conditional expectation of Y given $X = x$ is given by:

$$E[Y|X = x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x) dy$$

Review of CEF Concepts

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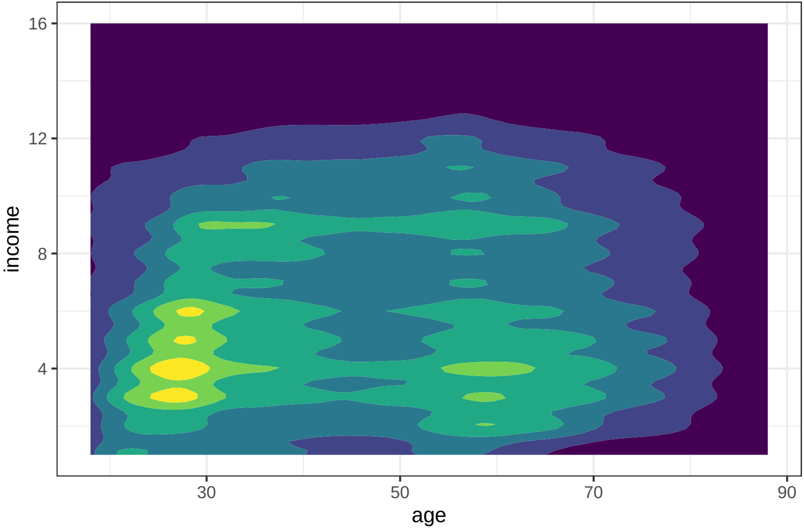

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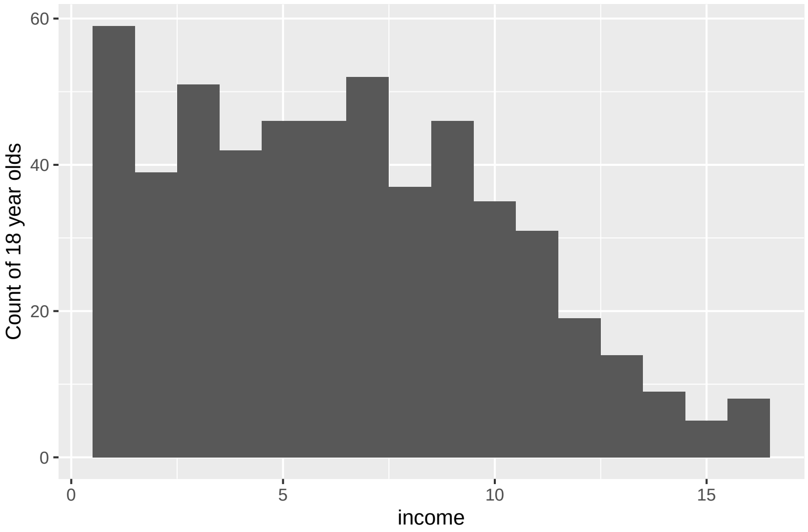

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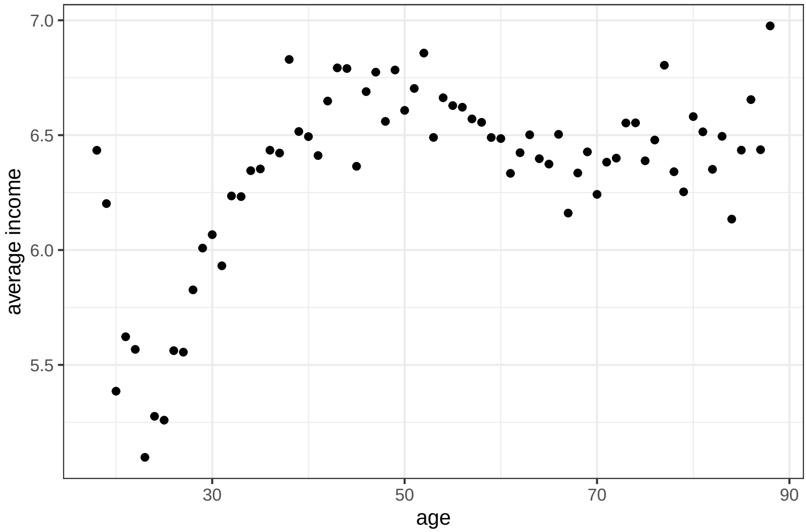

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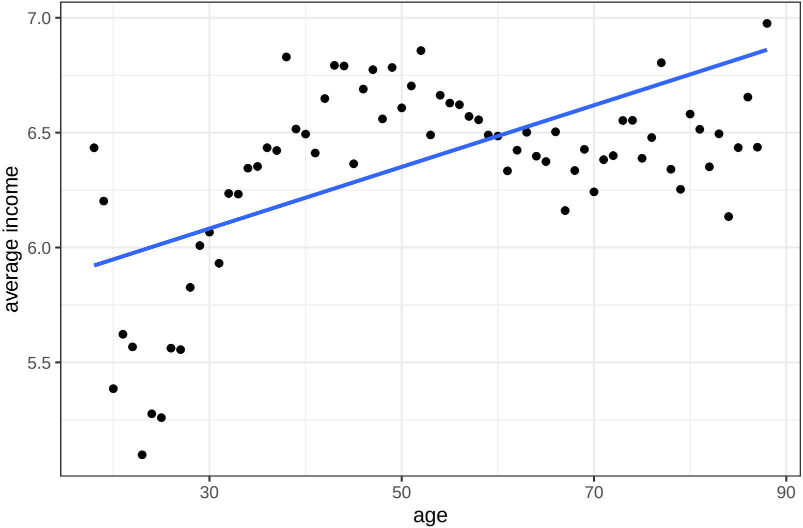

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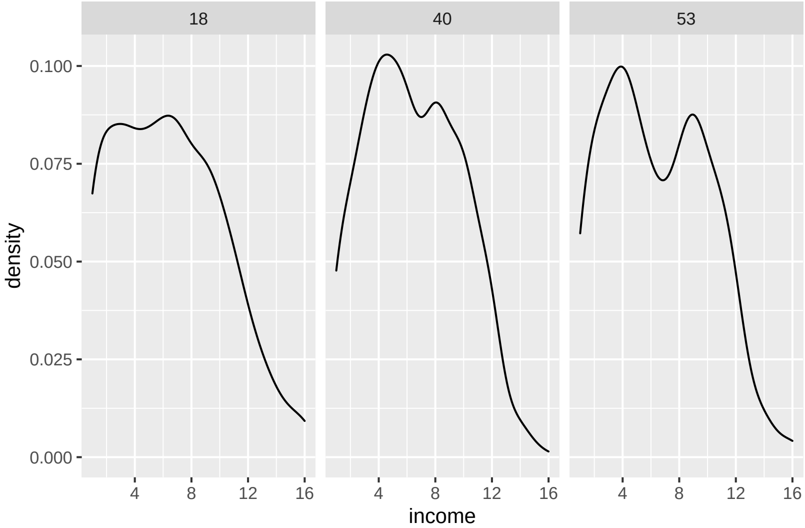

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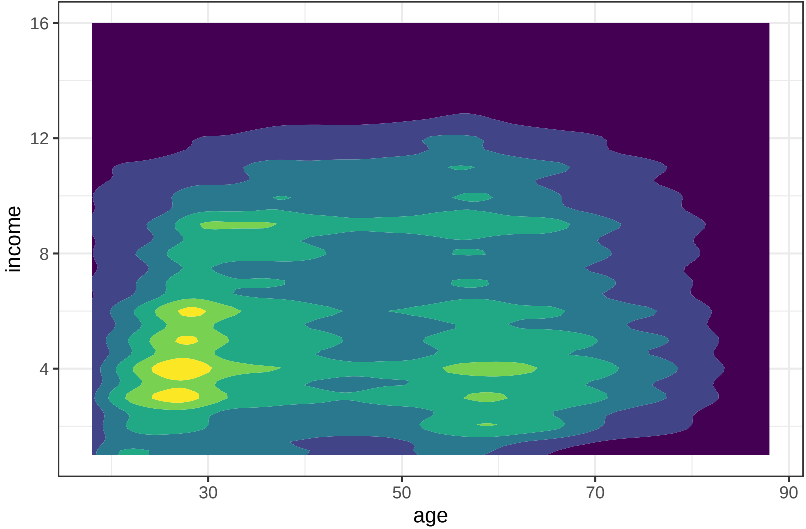

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- and the CEF is the lowest mean squared error predictor of Y given X.

For proofs see Aronow and Miller Thm 2.2.19, 2.2.20

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- Estimation just involves taking the means within the groups.

Non-binary CEFs

- If X is discrete with not too many categories, we can estimate the conditional expectation using the means within groups:
  - $\hat{\mu}(1) = \hat{E}[Y|X=1] = \frac{1}{n_{x=1}} \sum_{i:X_i=1} Y_i$
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- When X can take on many possible values (think income) or we have few observation for a given value of X, we have to write out a more general function.
- These functional forms are unknown which makes life hard.

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- We use two variables:
  - Y: income
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- Goal is to characterize the conditional expectation $E[Y|X = x]$, i.e. how average income varies with education level

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`educ`: Respondent's education:

- 1. 8 grades or less and no diploma or
- 2. 9-11 grades
- 3. High school diploma or equivalency test
- 4. More than 12 years of schooling, no higher degree
- 5. Junior or community college level degree (AA degrees)
- 6. BA level degrees; 17+ years, no postgraduate degree
- 7. Advanced degree

Nonparametric Regression with Discrete X

`income`: Respondent's family income:

- 1. None or less than \$2,999
- 2. \$3,000-\$4,999
- 3. \$5,000-\$6,999
- 4. \$7,000-\$8,999
- 5. \$9,000-\$9,999
- 6. \$10,000-\$10,999

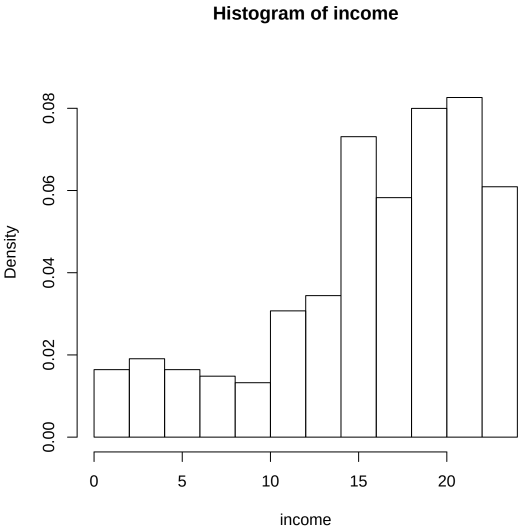
  ⋮

- 17. \$35,000-\$39,999
- 18. \$40,000-\$44,999

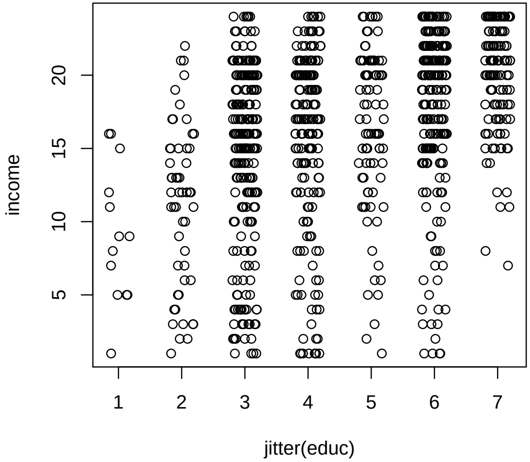
  ⋮

- 23. \$90,000-\$104,999
- 24. \$105,000 and over

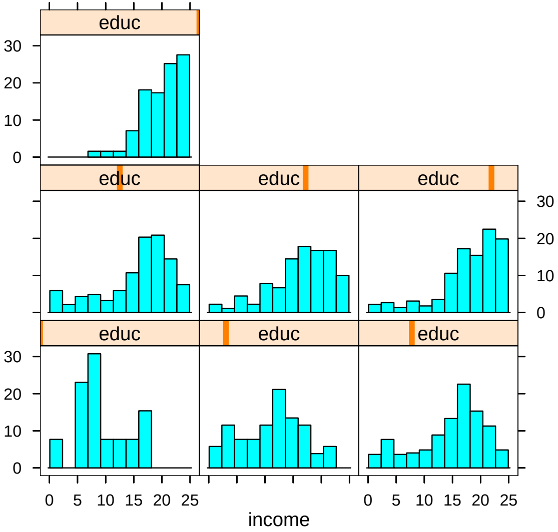
Marginal Distribution of Y (income)



Joint Distribution of X and Y (Income and Education)



Distribution of income given education $p(y|x)$



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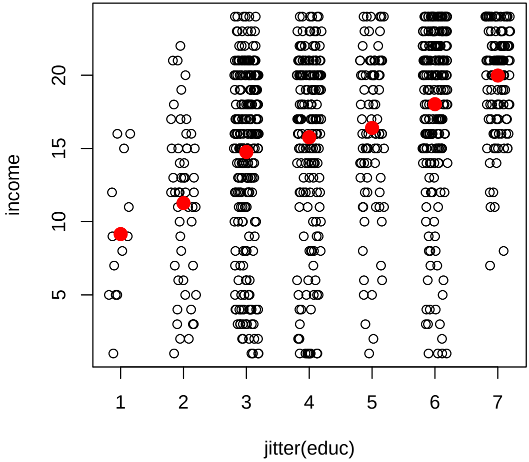

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  - X is discrete
  - there are a small number values of X
  - a small number of X variables
  - a lot of observations at each X value
- But what do we do when X is continuous and has many values?
- Let's talk through a few options.

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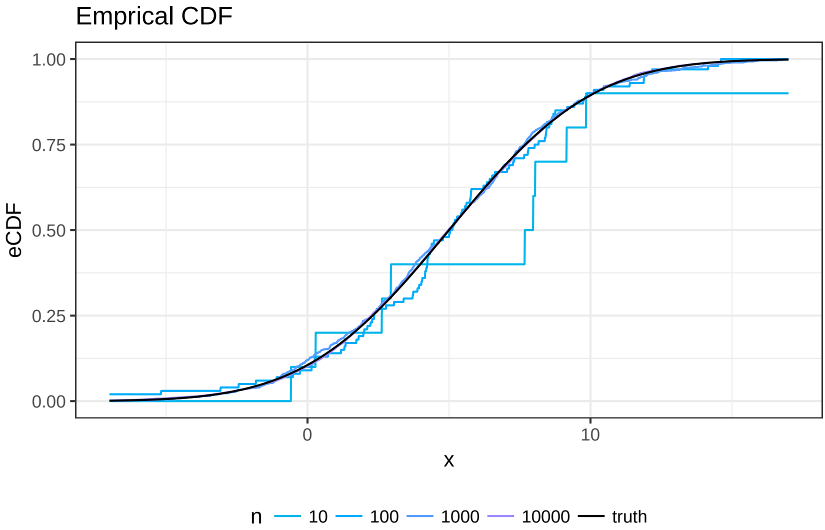
- The simplest way is to use the sample analog.
- Ex: If we're interested in the population mean, we use the sample mean
- This is justified because of the plug-in principle.
- The Weak Law of Large Numbers tells us that the empirical CDF is a good sample analog of the true CDF (which fully describes a distribution).

The Plug-in Principle in Action

Say we have a $\mathcal{N}(5, 4)$ distribution



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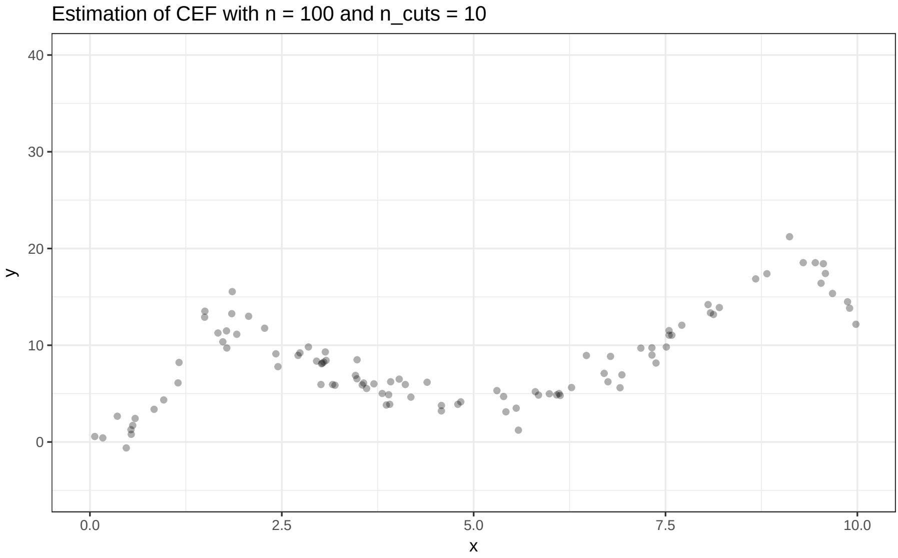
if T is well-behaved, then $\hat{\theta}$ is also asymptotically normal.

A Binning Approach to CEF for continuous random variables



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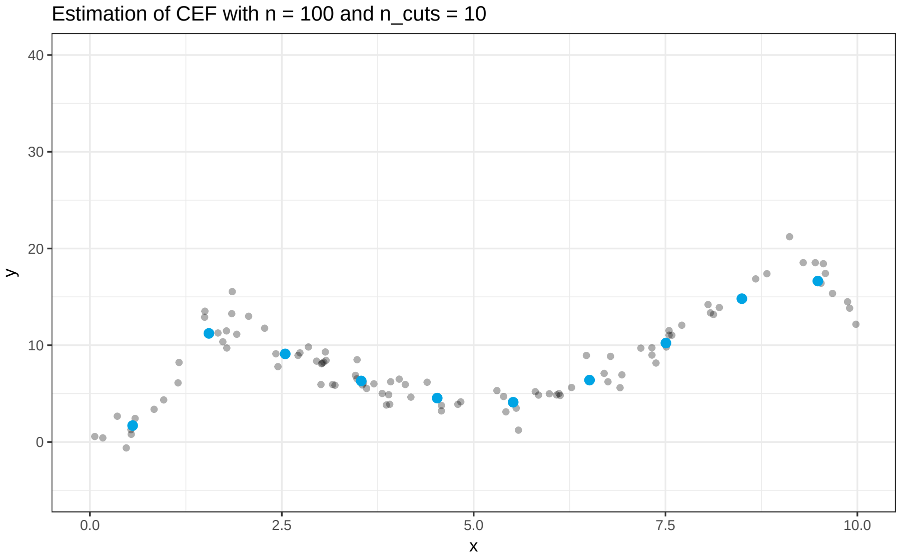

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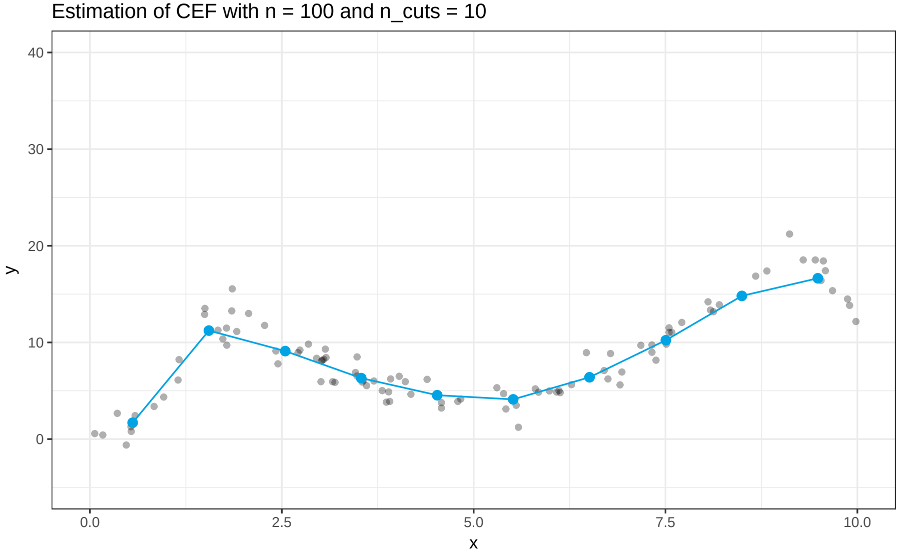

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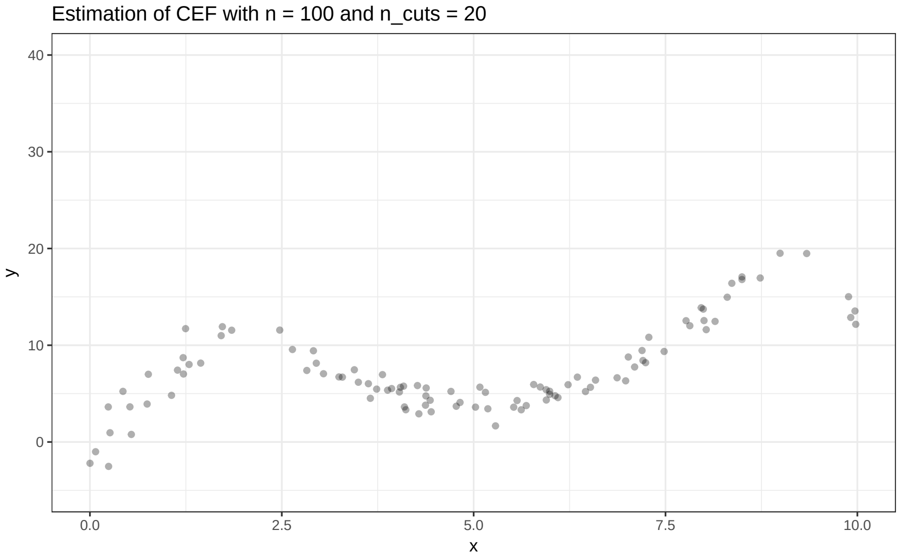

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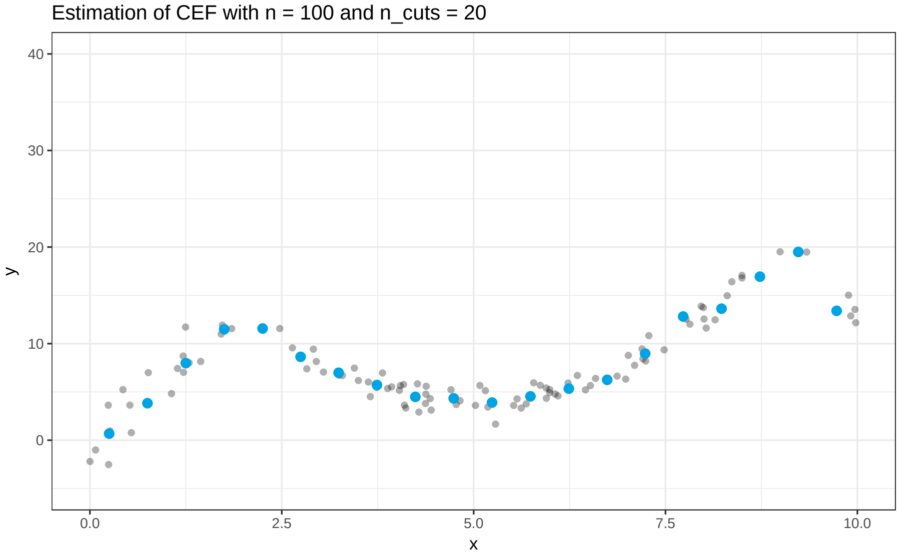

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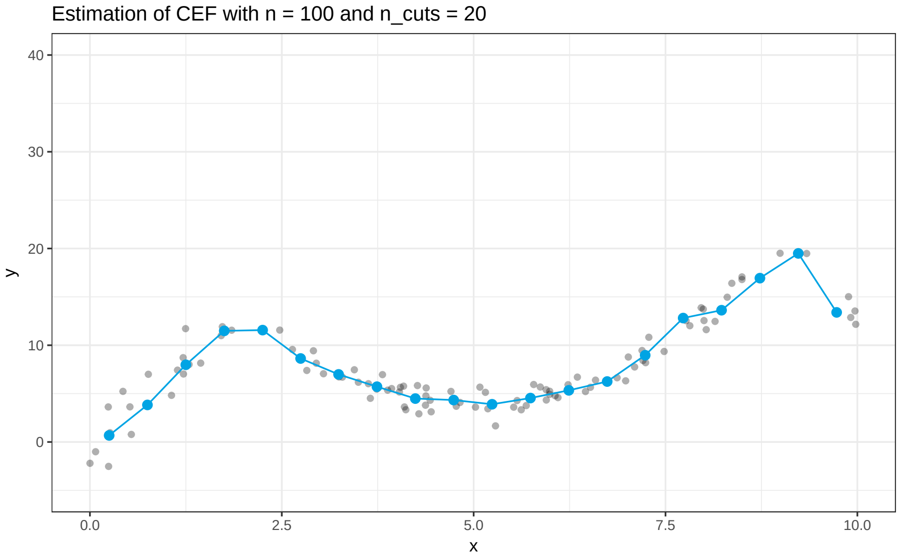

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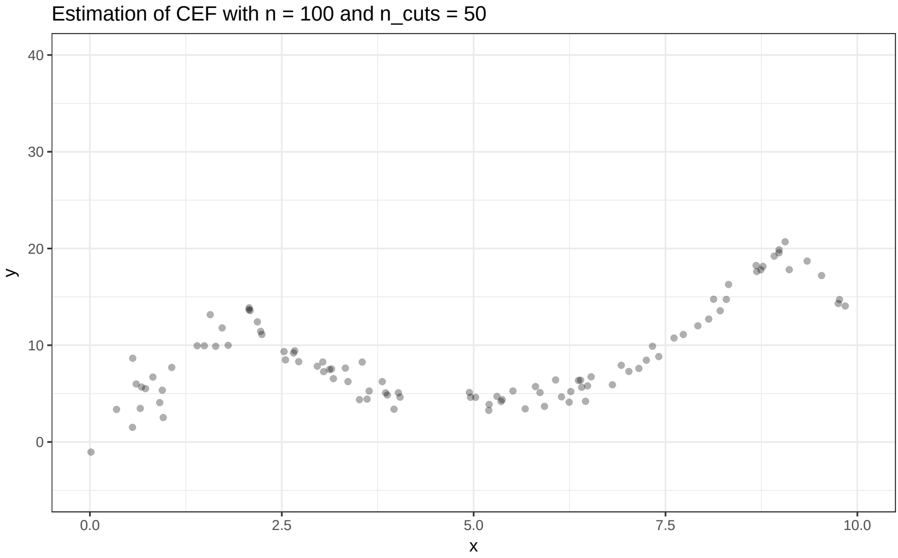

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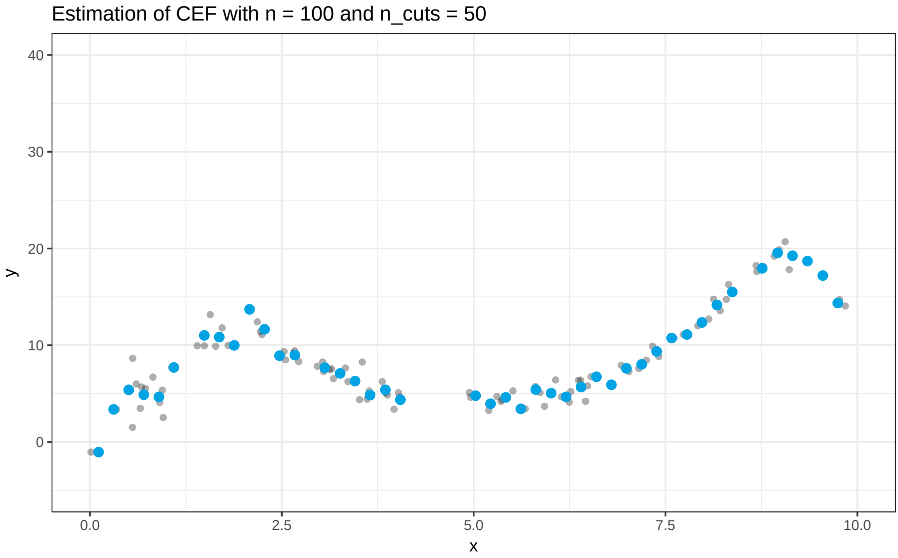

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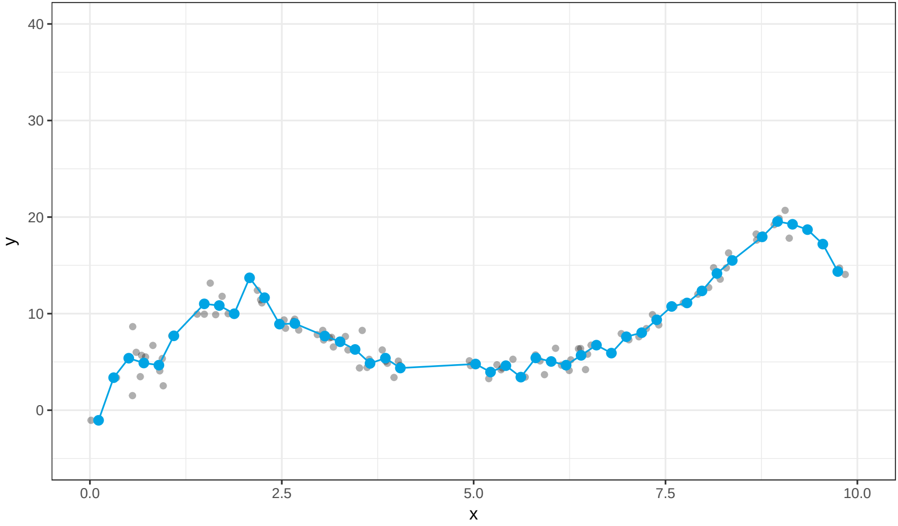
Estimation of CEF with n = 100 and n_cuts = 50



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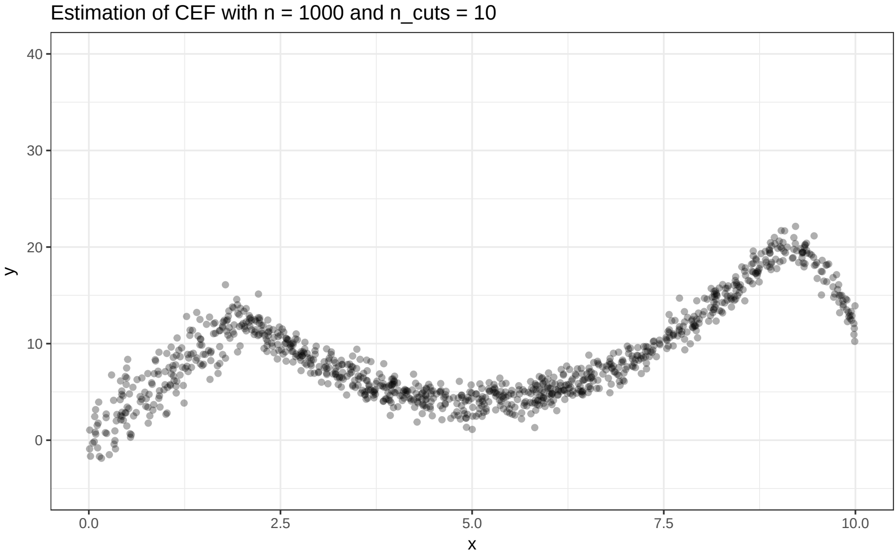

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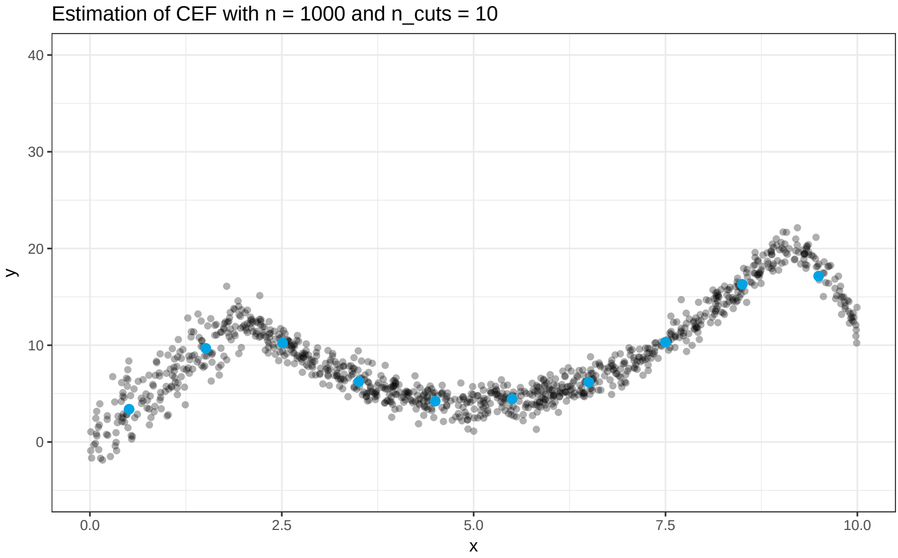

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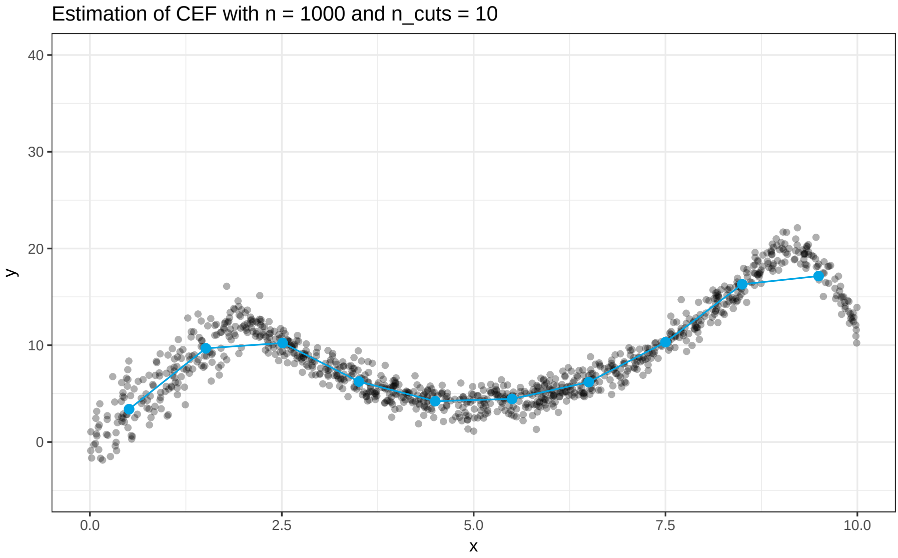

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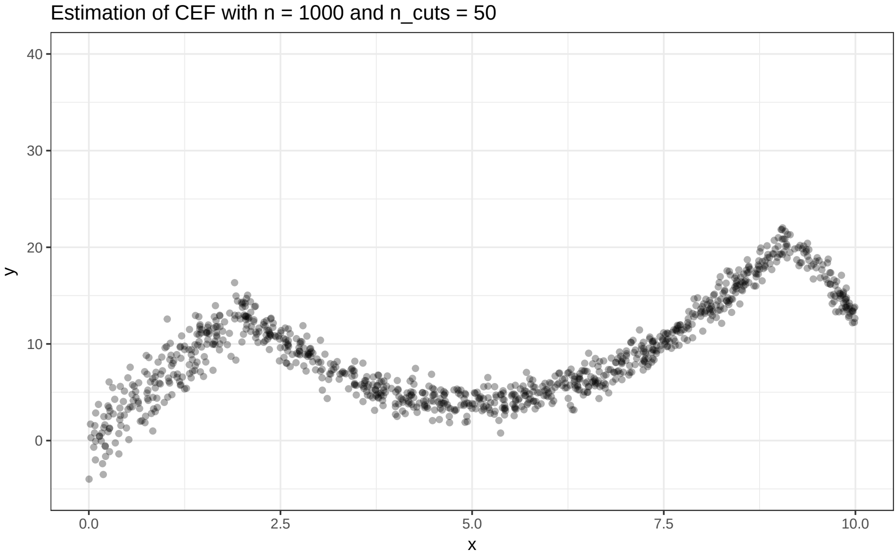

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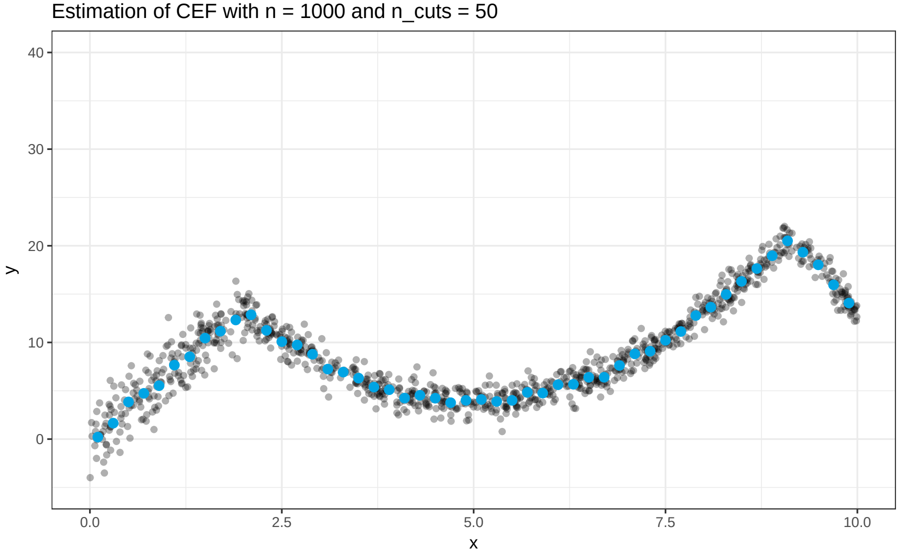

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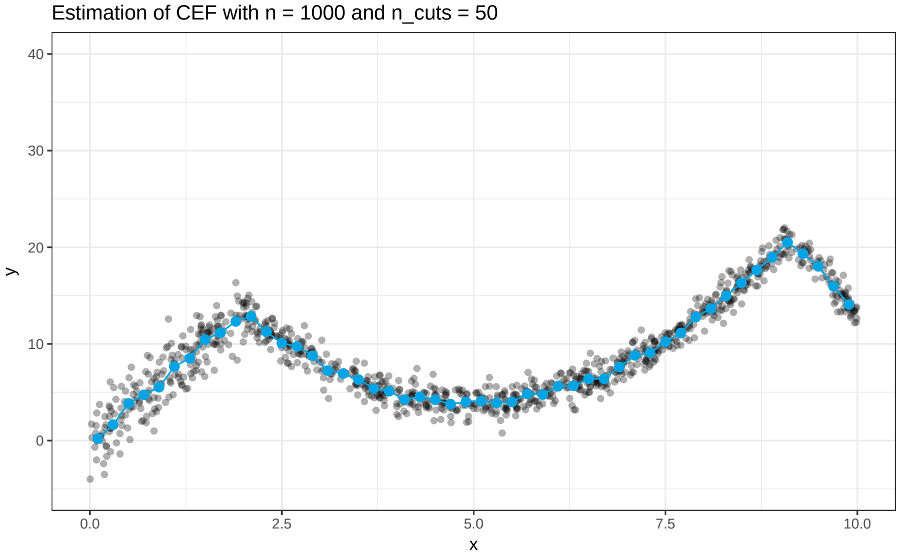

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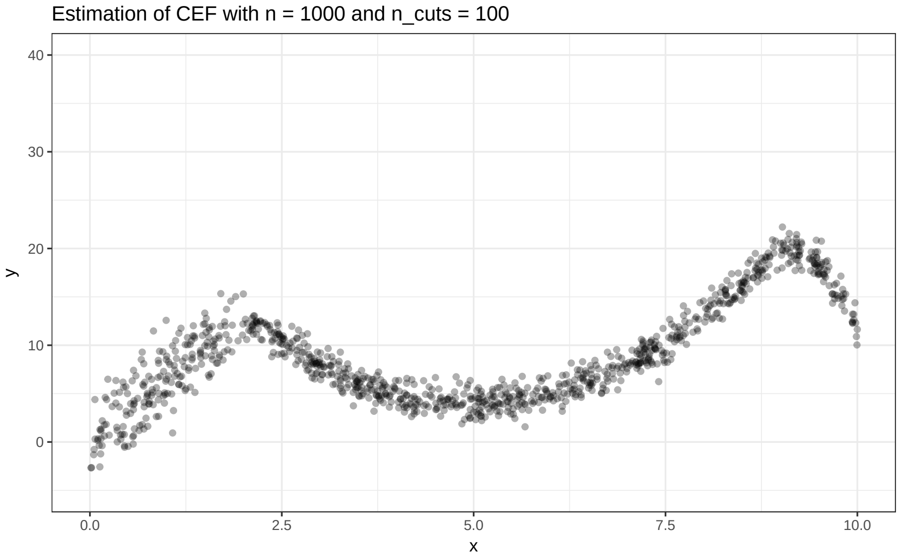

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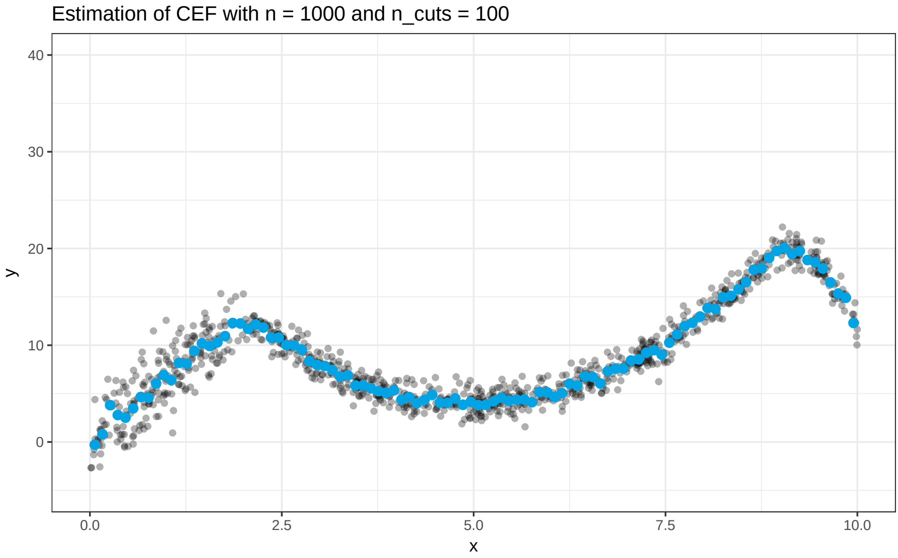

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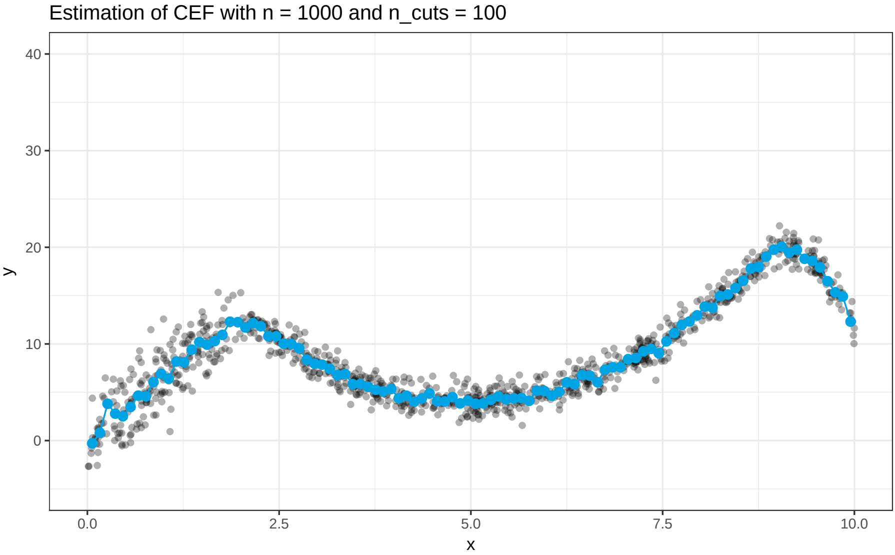

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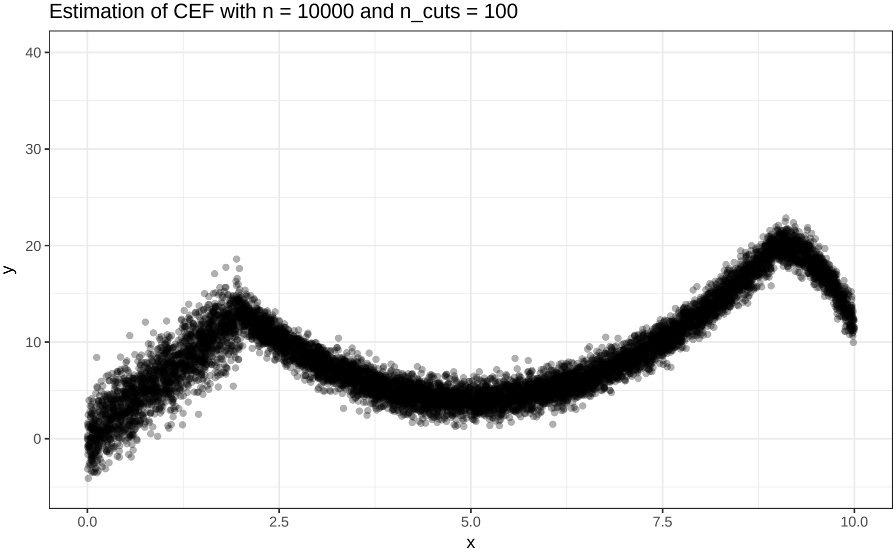

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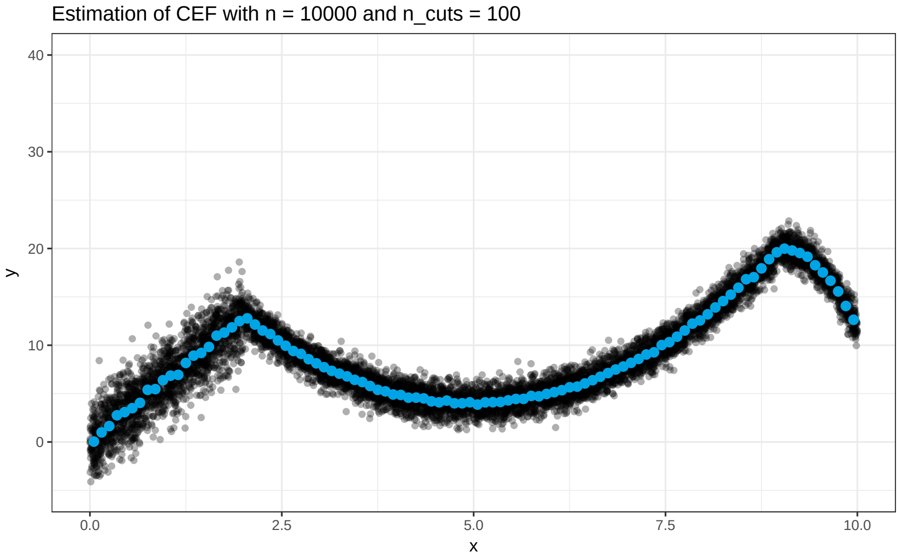

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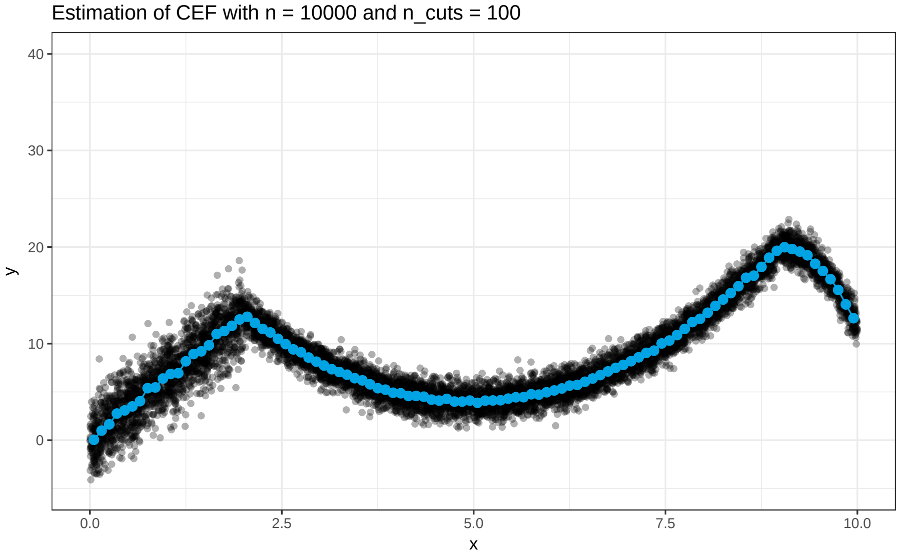

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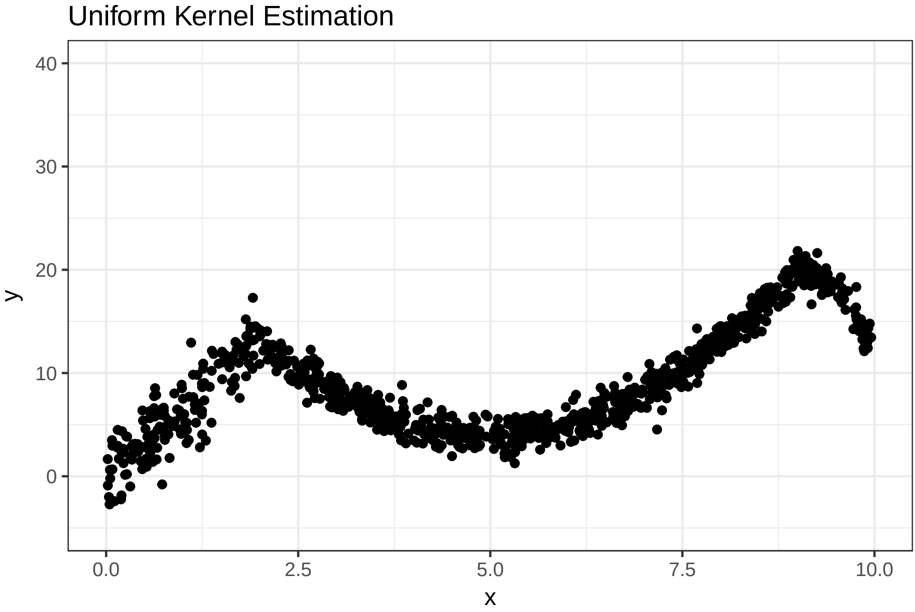
- Another approach is to use a moving local average to estimate $E[Y|X]$.
- We will call this approach uniform kernel regression for a reason that will become clear shortly.

Example of Kernel CEF Estimation



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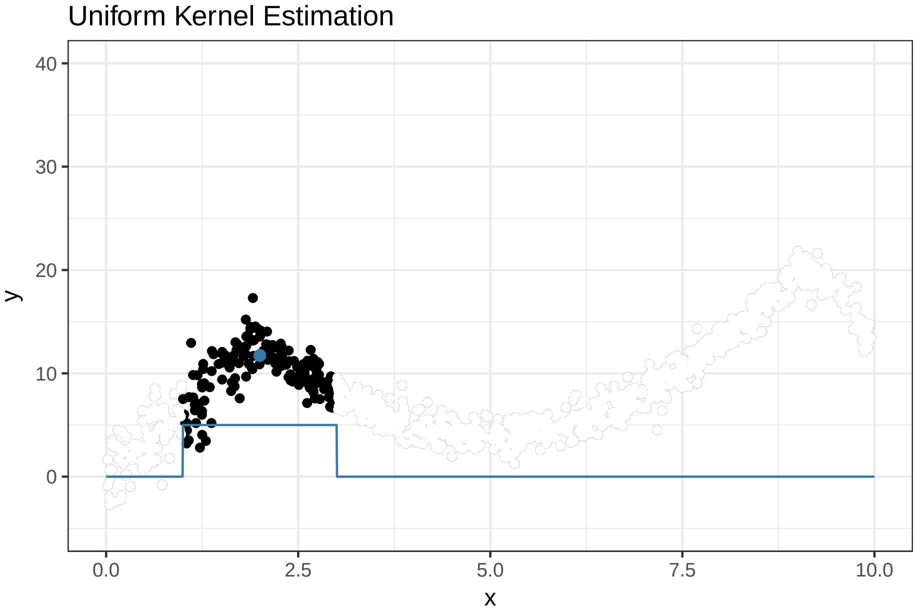

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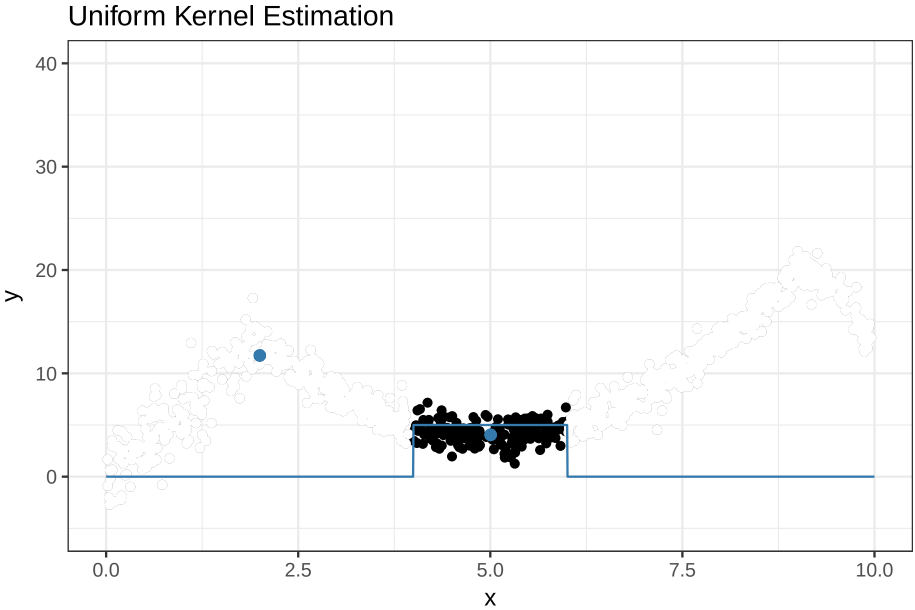

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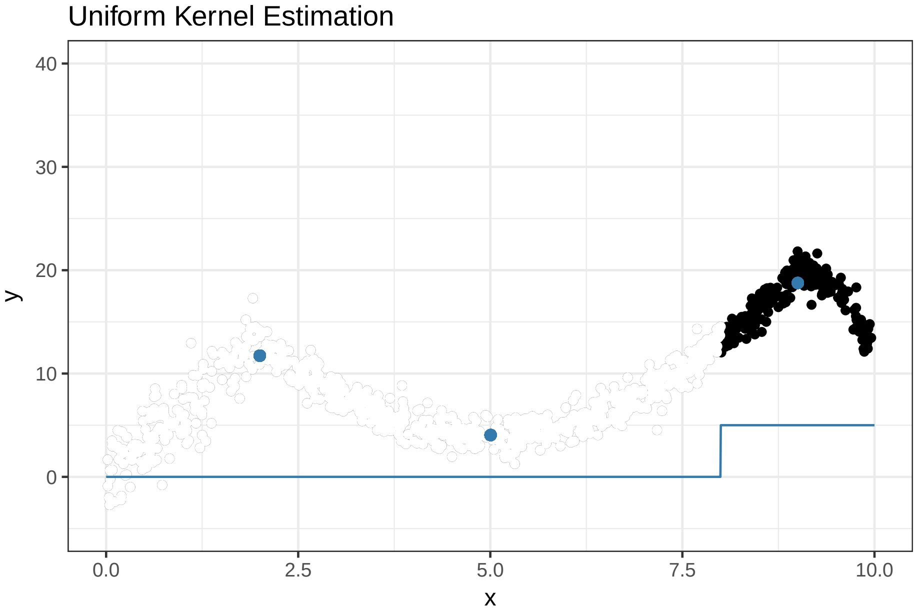

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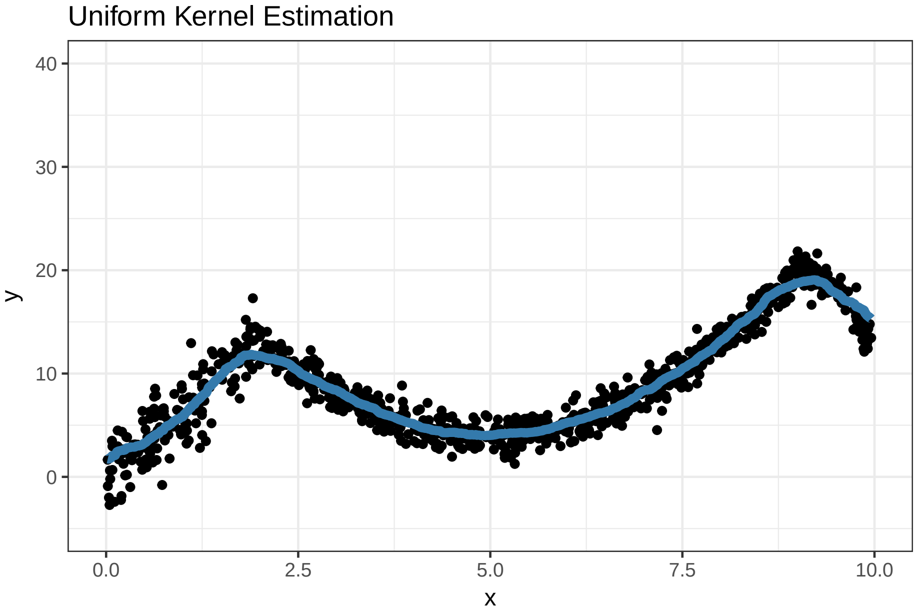

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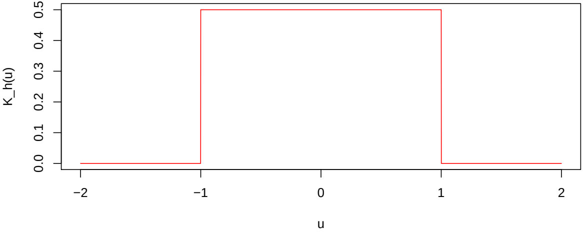

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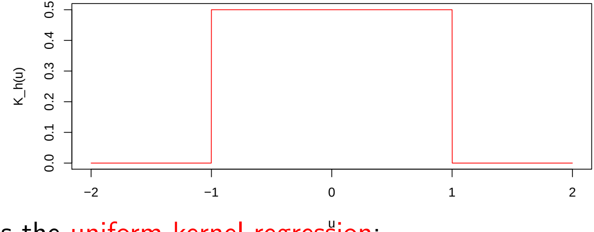
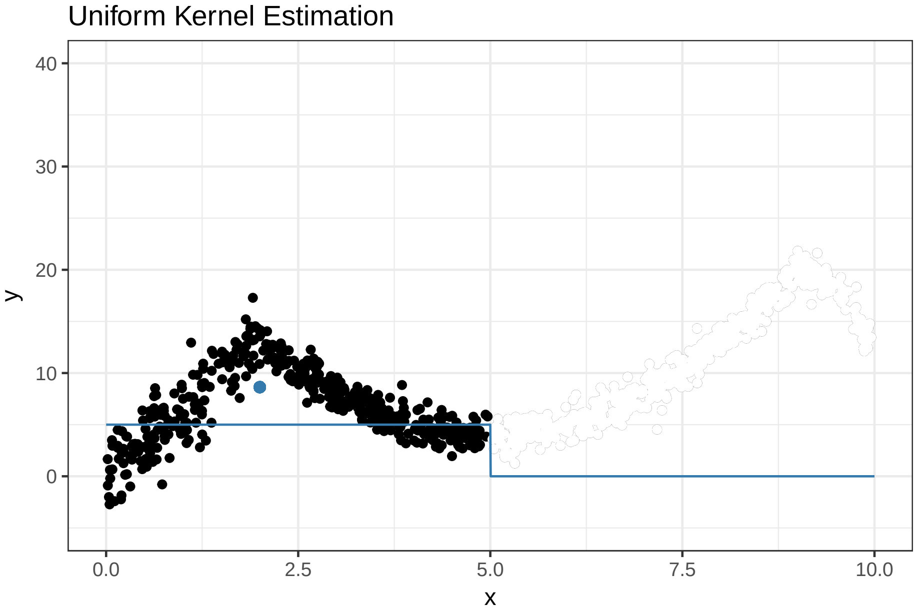


- This gives the uniform kernel regression:

$$\widehat{E}[Y|X = x_0] = \frac{\sum_{i=1}^{N} K_h((X_i - x_0)/h)\,Y_i}{\sum_{i=1}^{N} K_h((X_i - x_0)/h)} \text{ where } K_h(u) = \frac{1}{2}\,\mathbf{1}_{\{|u|\leq 1\}}$$

Changing the Bandwidth



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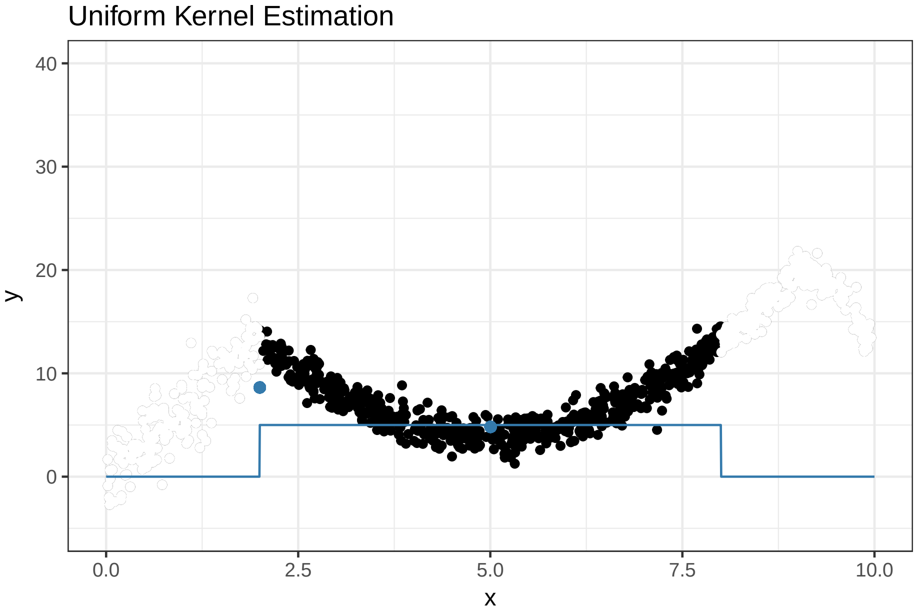

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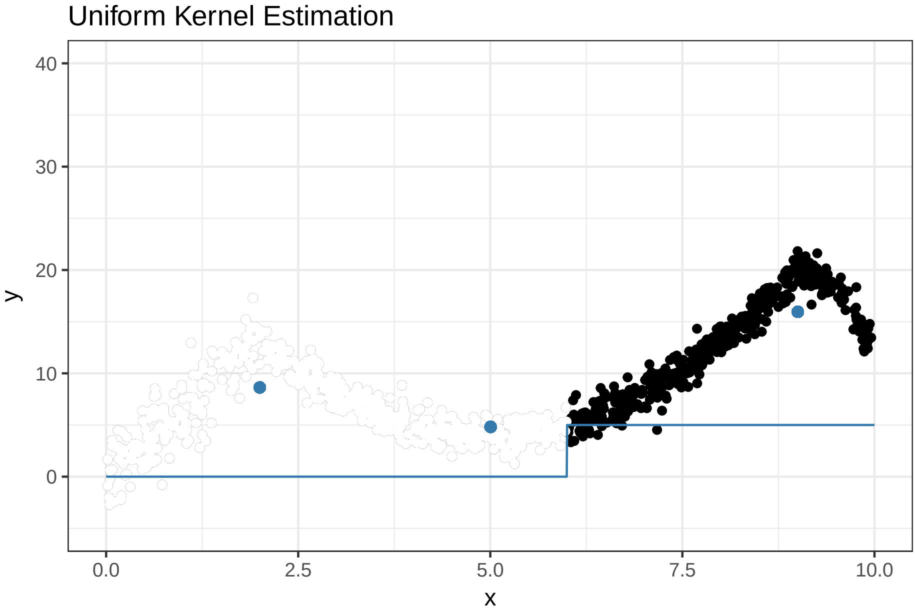

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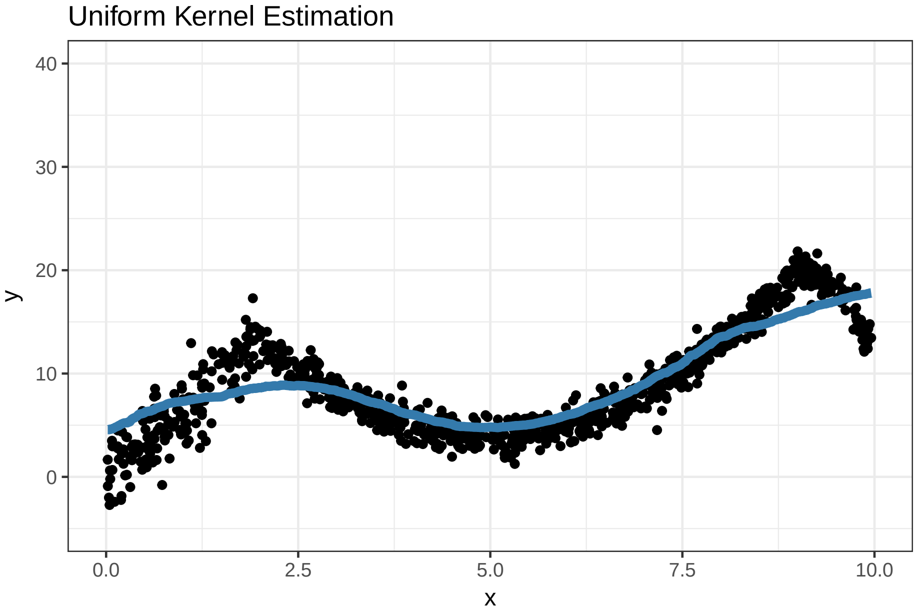

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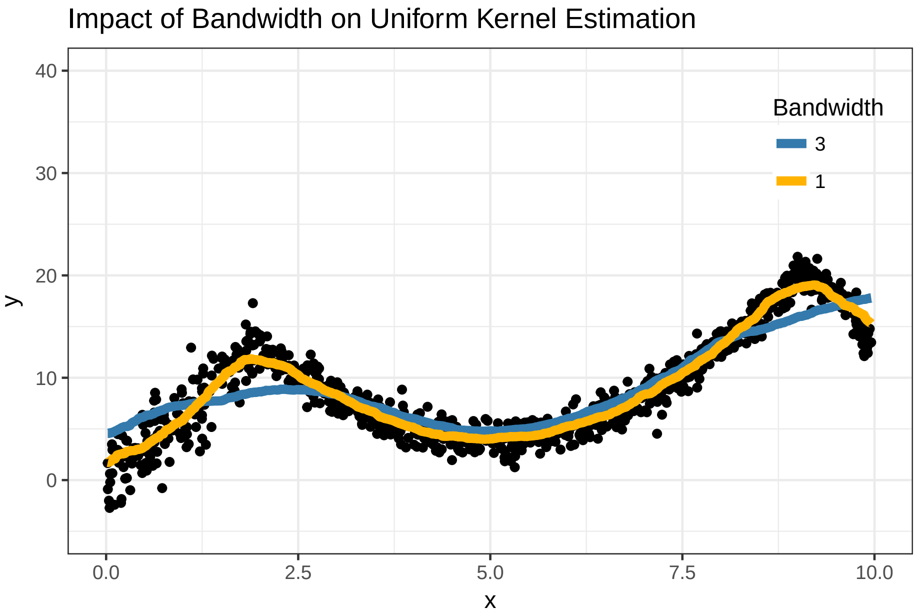

Uniform Kernel Regression: Properties

Theorem (Consistency of the Uniform Kernel Density Estimator)

For iid continuous random variables $X_1, X_2, \ldots X_n$, $\forall x \in \mathbb{R}$,

- *if the kernel is uniform, and*
- *if $h \rightarrow 0$ and*
- *$nh \rightarrow \infty$ as*
- *$n \rightarrow \infty$, then*

$$\hat{f}_K(x) \operatorname{plim} f(x).$$

Aronow and Miller Theorem 3.3.8. Proof by weak law of large numbers and the plug-in principle.

The More General Form of the Estimator

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Let $K : \mathbb{R} \to \mathbb{R}$ be a function which is symmetric about the y-axis satisfying $\int_{-\infty}^{\infty} K(x)dx = 1$, and let $K_h(x) = \frac{1}{h}K(\frac{x}{h}) \forall x \in \mathbb{R}$ and $h > 0$.

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Then a kernel density estimator of $f(x)$ is

$$\hat{f}_K(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i), \forall x \in \mathbb{R}$$

The function K is called the kernel and the scaling parameter h is called the bandwidth.
(Aronow and Miller Definition 3.3.7)

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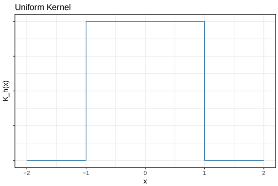
- Each observation within the interval is given equal weight, each observation outside the interval is given 0 weight



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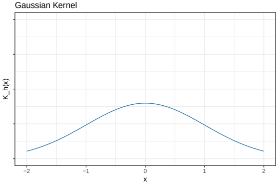
- Gaussian Kernel
- Distance weighted by how far from x_0 following the normal density

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$



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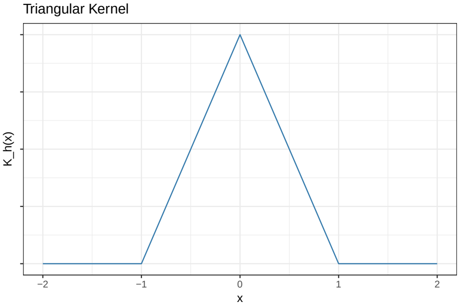
- Distance weighted by how far from x_0 using linear distance



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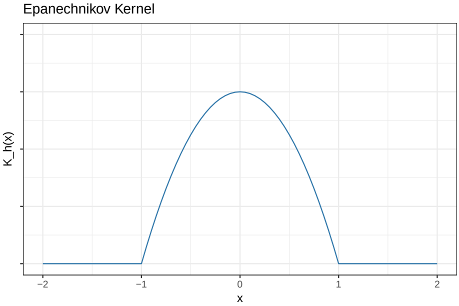
- Distance weighted by how far from x_0 using a parabolic function



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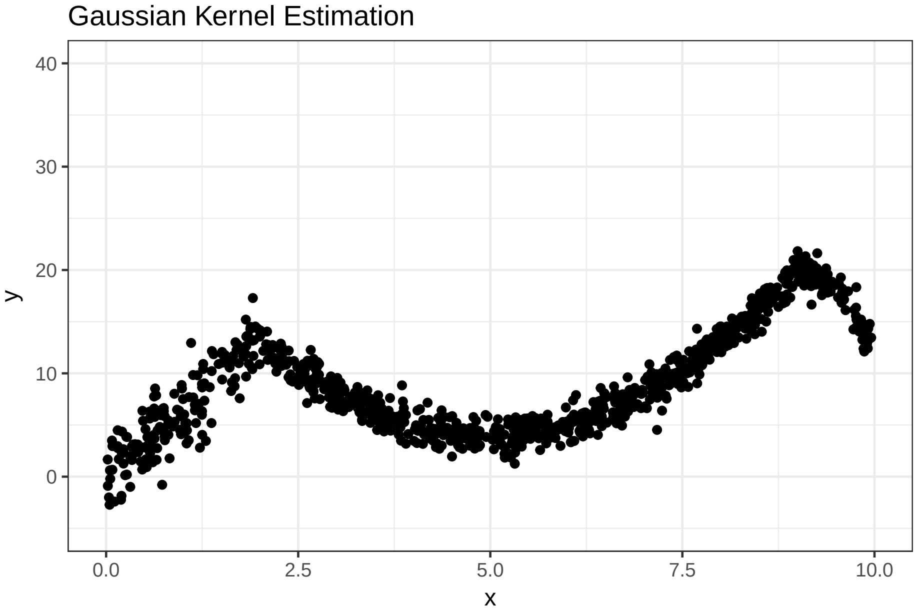

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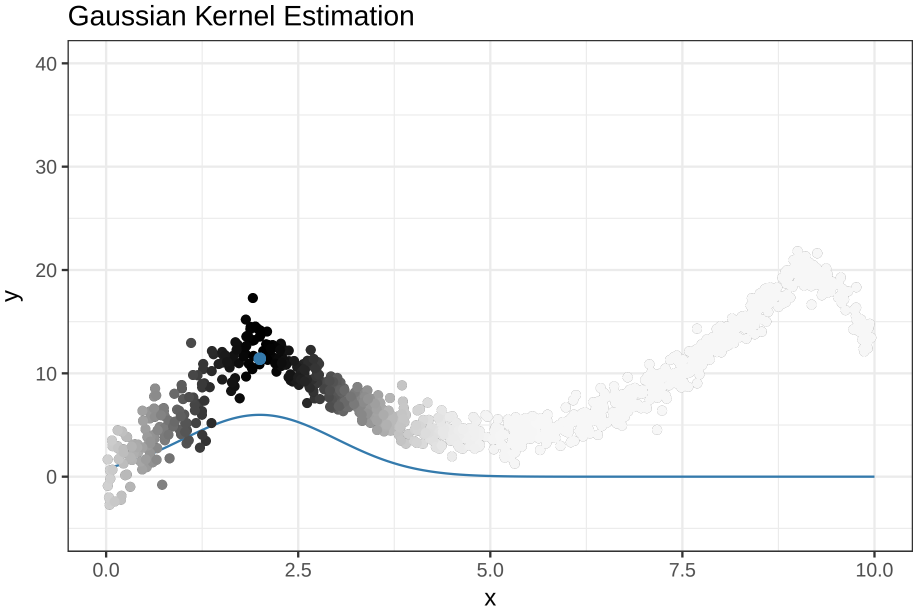

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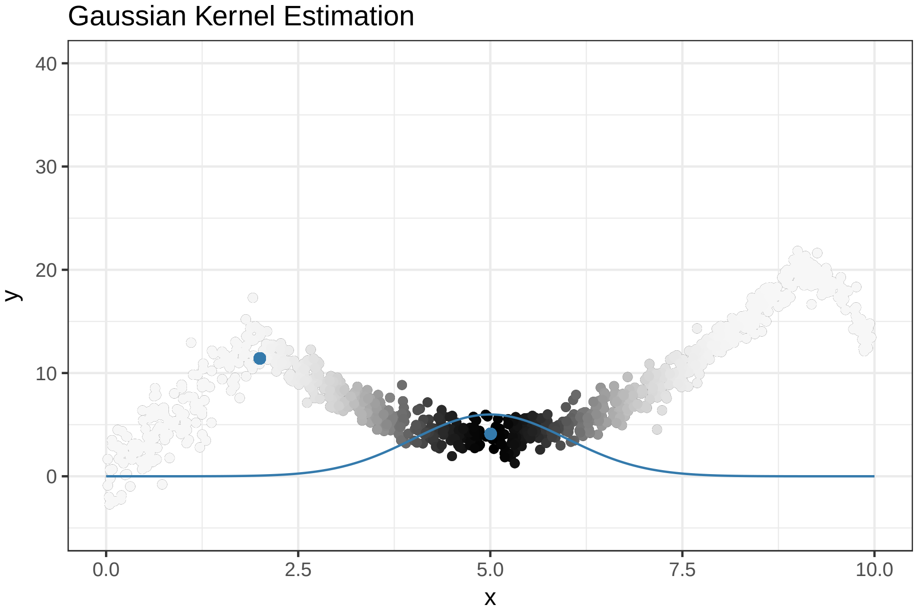

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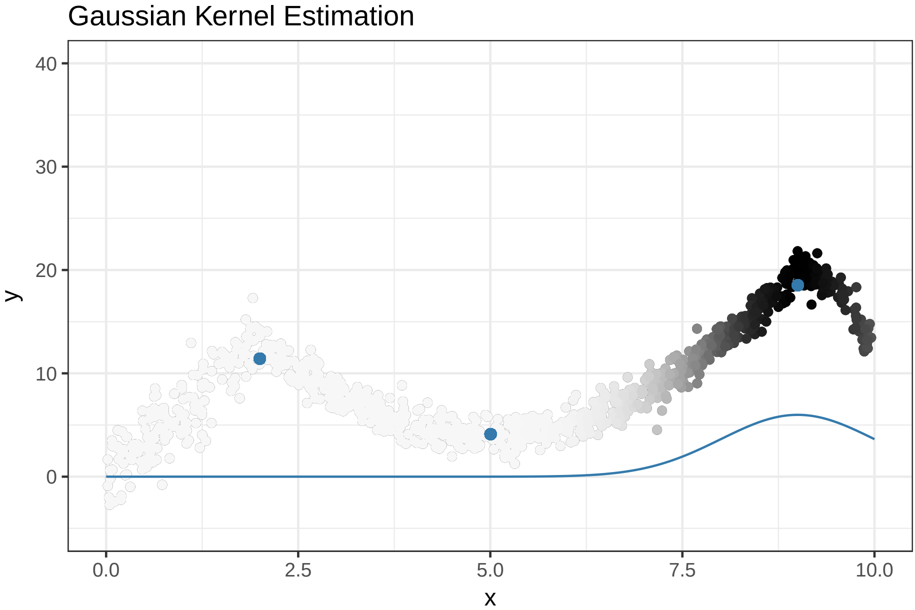

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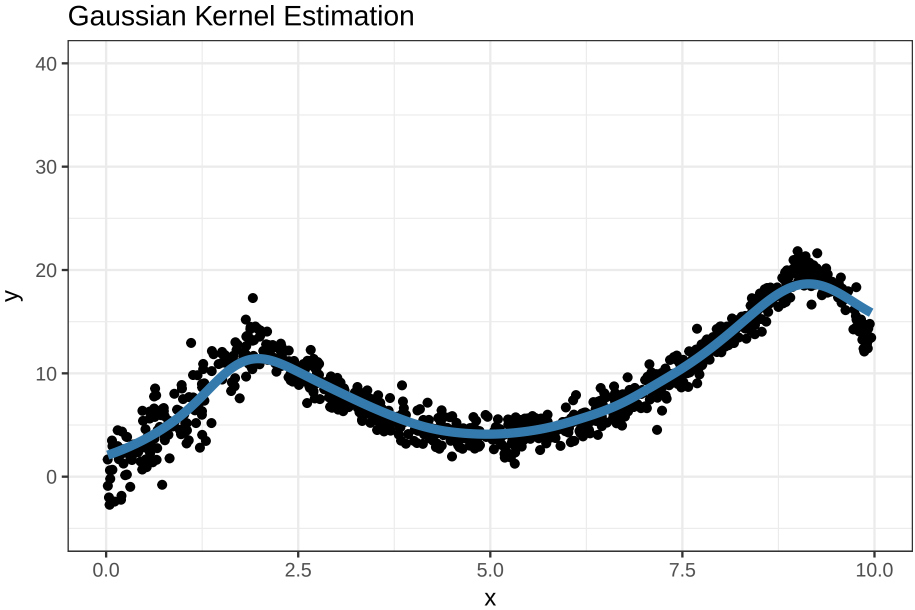

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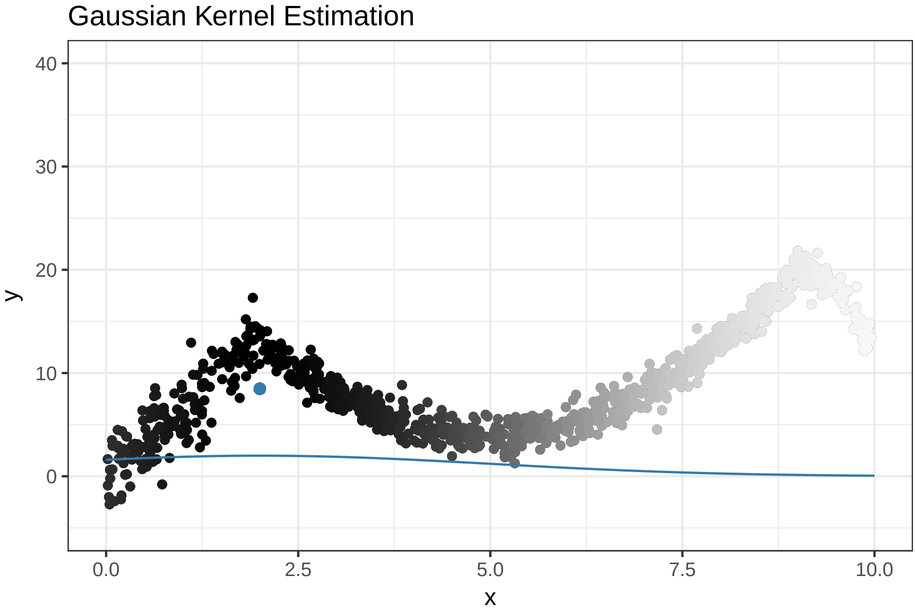

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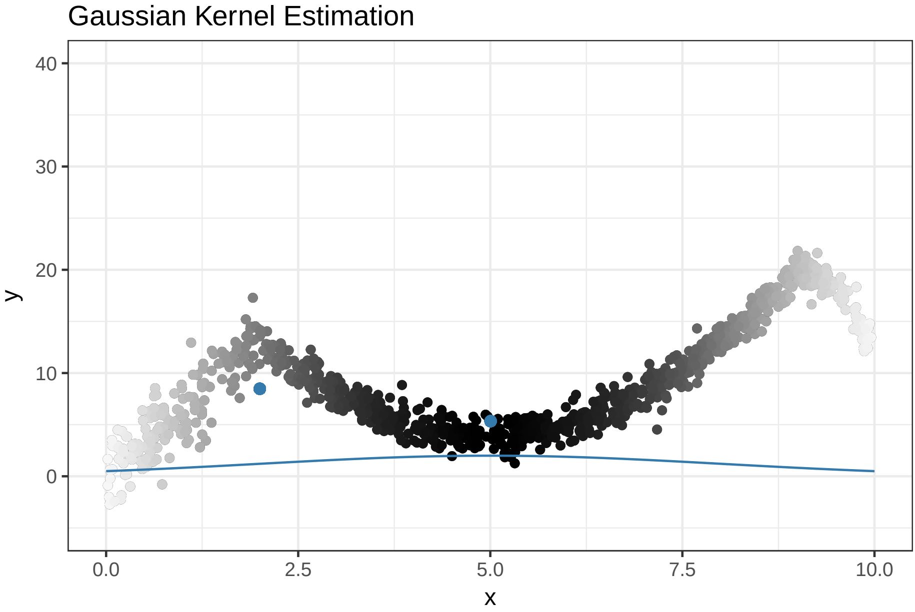

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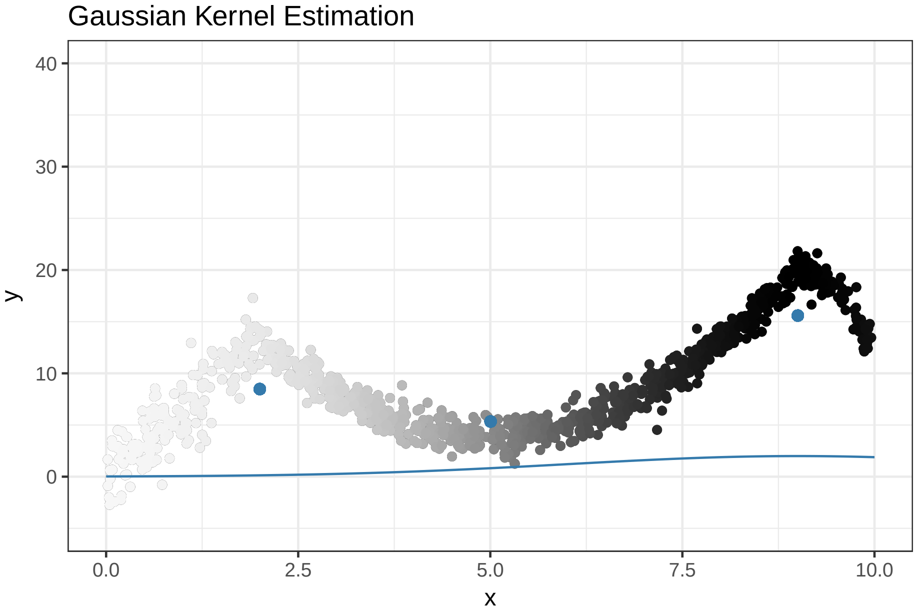

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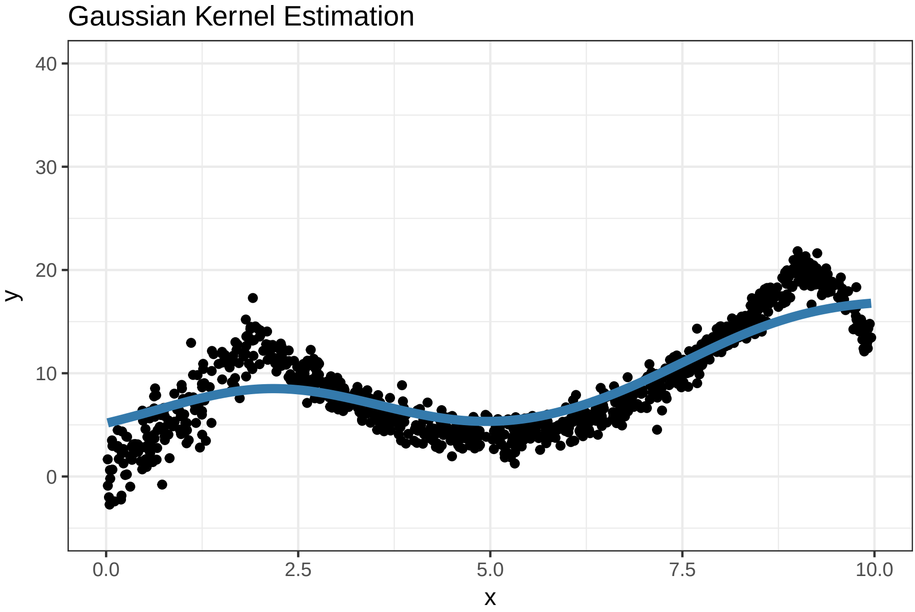

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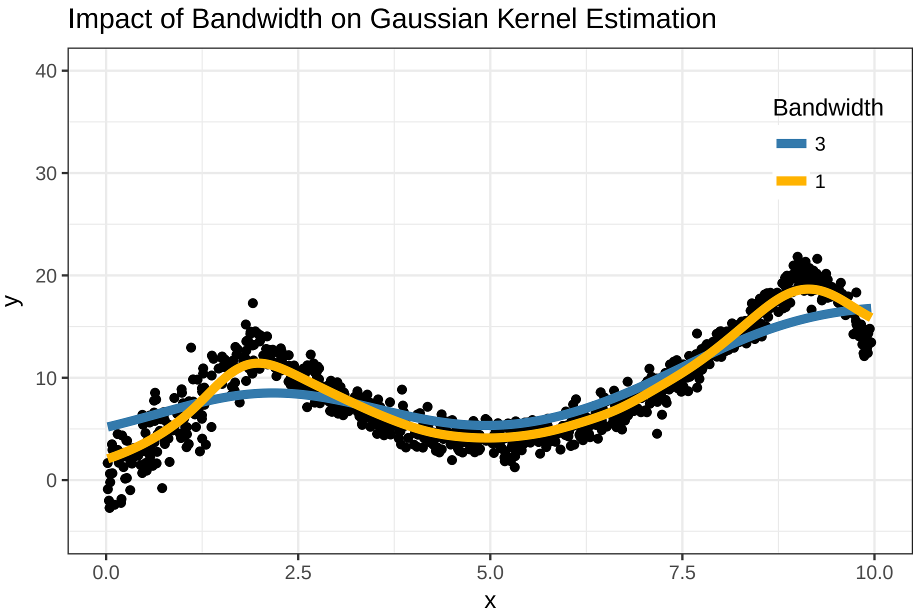

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  - A very flexible estimator allows the shape of the function to vary (e.g. a kernel regression with a small bandwidth)
  - A very inflexible estimator restricts the shape of the function to a particular form
    (e.g. a kernel regression with a very wide bandwidth)

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- What function minimizes MSE (in that sense is 'best'), among this class of functional forms?

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Theorem (Aronow and Miller Thm 2.2.21)

For a random variable X and Y, if $V[X] > 0$, then the best (min MSE) linear predictor (BLP) of Y given X is $g(X) = \beta_0 + \beta_1 X$ where,

$$\beta_0 = E[Y] - \frac{Cov[X, Y]}{V[X]} E[X] \qquad\qquad \beta_1 = \frac{Cov(X, Y)}{V(X)}$$

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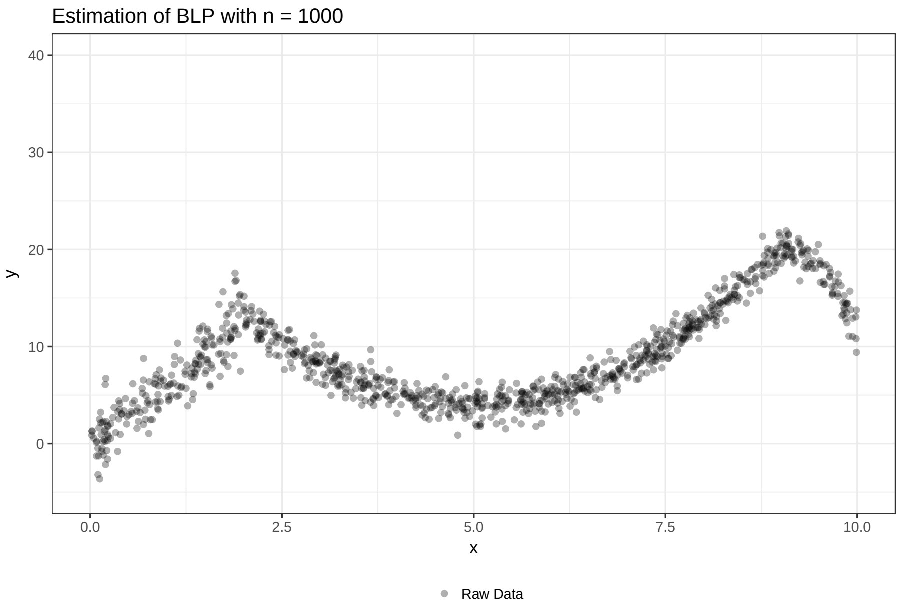
Corollary (Aronow and Miller Thm 2.2.22)

- *The BLP is the best linear predictor of the CEF. I.e. setting $b_0 = \beta_0$ and $b_1 = \beta_1$ minimizes $E\left[(E[Y \mid X] - (b_0 + b_1 X))^2\right]$*
- *If the CEF is linear, the CEF is the BLP*
- *Defining $\epsilon = Y - (\beta_0 + \beta_1 X)$,*
  - $E[\epsilon] = 0$
  - $E[X\epsilon] = 0$
  - $Cov[X, \epsilon] = 0$

Best Linear Predictor



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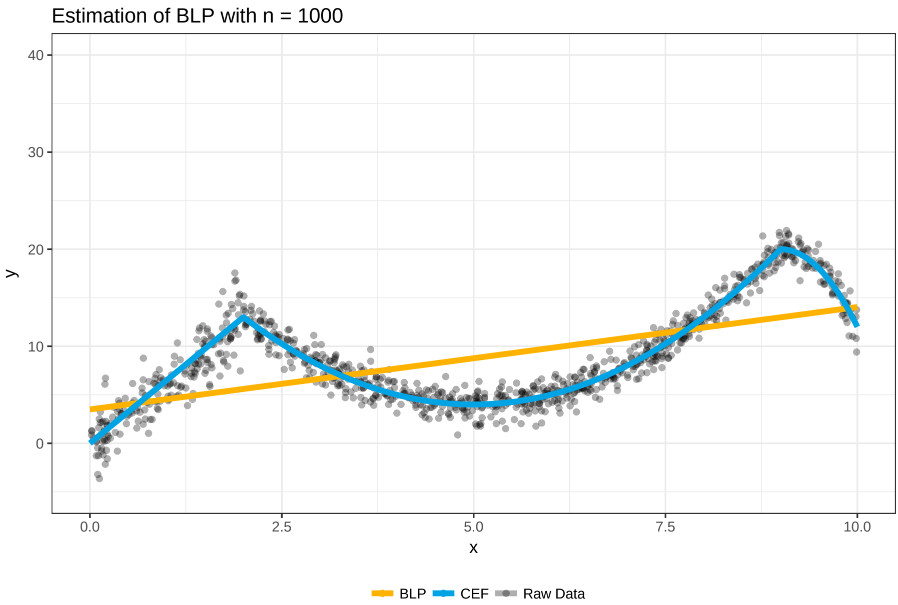

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- In general, this is distinct from the CEF:
  - CEF is the best predictor of Y among all functions.
  - Linear projection is the best predictor among linear functions.
- The nice thing about the linear projection is that it exists and is well-defined even if the CEF is non-linear.

BLP Approximations Depend on $p(X)$

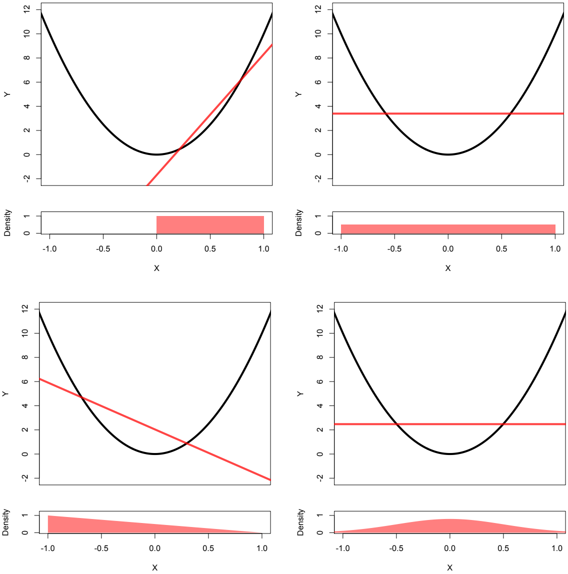

Figure 2.2.2: Plotting the CEF and the BLP over Different Distributions of X

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The BLP is always a line regardless of the data.

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  - We can make the model more flexible, even in a linear framework (e.g. we can add polynomials, use log transformations, etc.)
- Perhaps the biggest reason is that it extends easily to the case where X is a vector of random variables.

Interpretation of the BLP slope

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  2. **Slope**: for units one higher in X we predict Y to be higher by β_1

  $$E[Y|X=x+1] - E[Y|X=x] \approx (\beta_0 + \beta_1(x+1)) - (\beta_0 + \beta_1 x)$$

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$$\begin{aligned} E[Y|X=x+1] - E[Y|X=x] &\approx (\beta_0 + \beta_1(x+1)) - (\beta_0 + \beta_1 x) \\ &= \beta_0 + \beta_1 x + \beta_1 - \beta_0 - \beta_1 x \end{aligned}$$

Interpretation of the BLP slope

- When we model the CEF as a line, we can interpret the parameters of the line in appealing ways:
  1. **Intercept**: the average outcome among units with $X = 0$ is β_0:
  $$E[Y|X=0] \approx \beta_0 + \beta_1 0 = \ \beta_0$$
  2. **Slope**: for units one higher in X we predict Y to be higher by β_1
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- This helps clarify that it is an approximation to the CEF and that the units being described are different.

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- While the line is defined over all regions of the data we may be concerned about:
  - interpolation
  - extrapolation
  - predicting in ranges of X with sparse data
  - changes in the distribution of X

Which Predictions Do You Trust?

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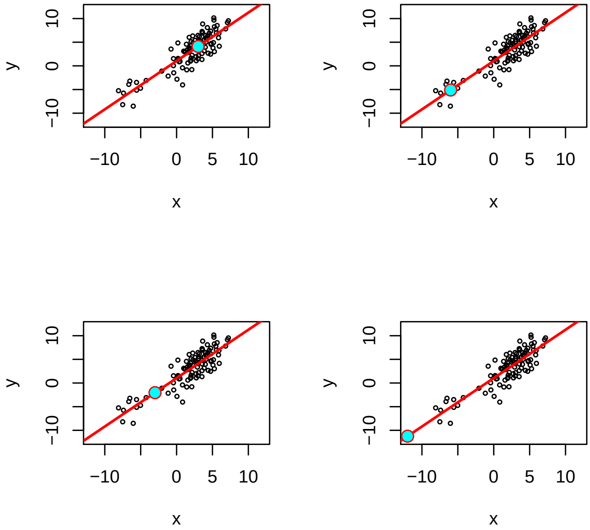

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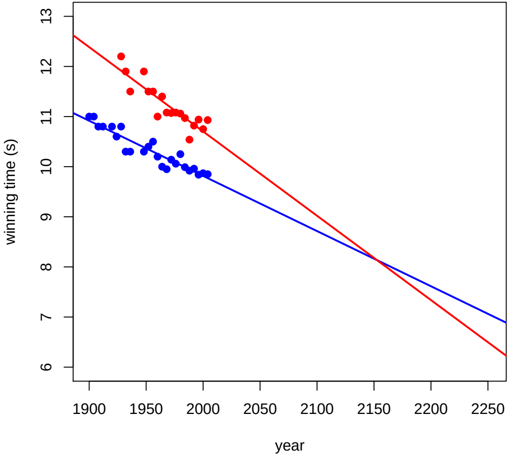
How do the authors make this conclusion?

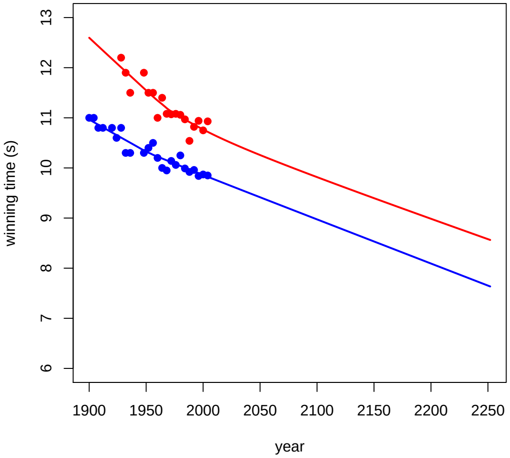
Using data from 1900 to 2004, they fit linear regression models of the winning 100 meter time on year for both men and women. They then use the estimates from these models to extrapolate 152 years into the future.

Tatem et al. Extrapolation



Tatem et al.'s predictions. Men's times are in blue, women's times are in red.

Alternate Models Fit Well, Yield Different Predictions



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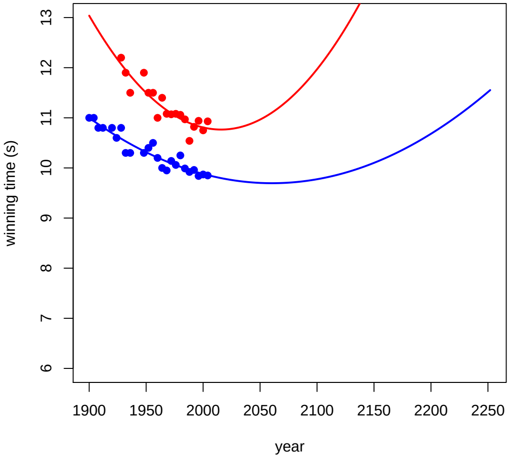

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A More Subtle Example

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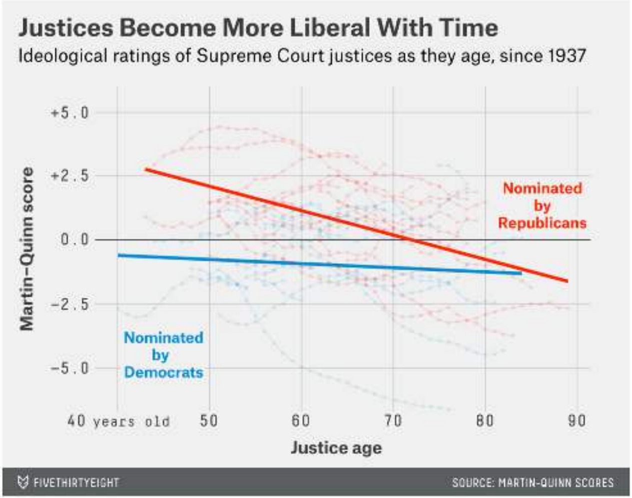

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Regression as a Causal Model (A Preview)

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- We will return to this later in the course
- For now, it is safest to treat β_1 as a purely descriptive/predictive quantity

Fun with Linearity

“The Siren’s Song of Linearity”

Fun with Linearity

Psychonomic Bulletin & Review
2007, 14 (2), 288-294

Iterated learning: Intergenerational knowledge transmission reveals inductive biases

MICHAEL L. KALISH
University of Louisiana, Lafayette, Louisiana

THOMAS L. GRIFFITHS
University of California, Berkeley, California

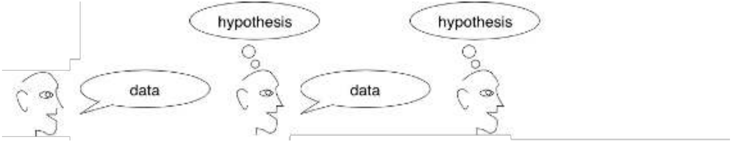
AND

STEPHAN LEWANDOWSKY
University of Western Australia, Perth, Australia

Cultural transmission of information plays a central role in shaping human knowledge. Some of the most complex knowledge that people acquire, such as languages or cultural norms, can only be learned from other people, who themselves learned from previous generations. The prevalence of this process of "iterated learning" as a mode of cultural transmission raises the question of how it affects the information being transmitted. Analyses of iterated learning utilizing the assumption that the learners are Bayesian agents predict that this process should converge to an equilibrium that reflects the inductive biases of the learners. An experiment in iterated function learning with human participants confirmed this prediction, providing insight into the consequences of intergenerational knowledge transmission and a method for discovering the inductive biases that guide human inferences.

Images on following slides courtesy of Tom Griffiths

Fun with Linearity



The Design

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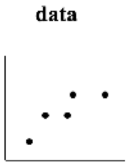

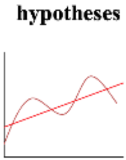

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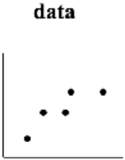

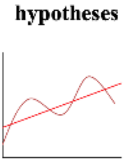

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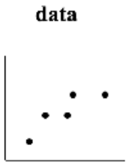

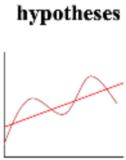

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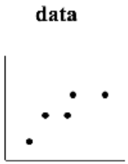

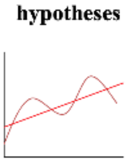

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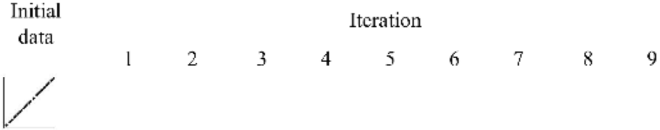
- Predictions are data for the next learner

Results



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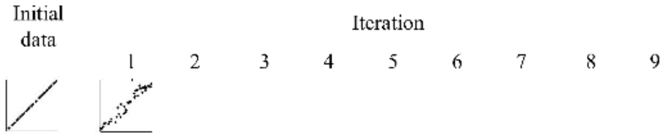

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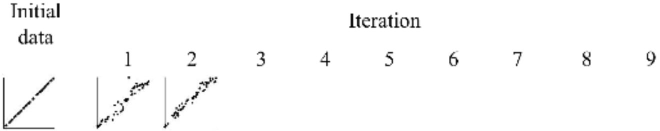

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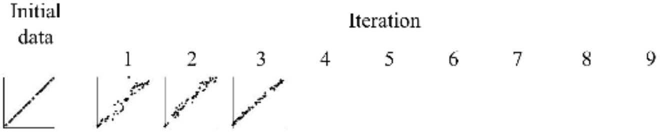

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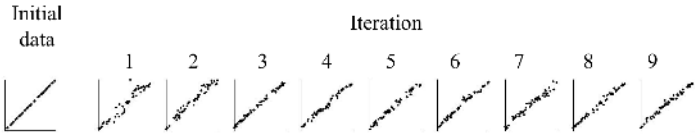

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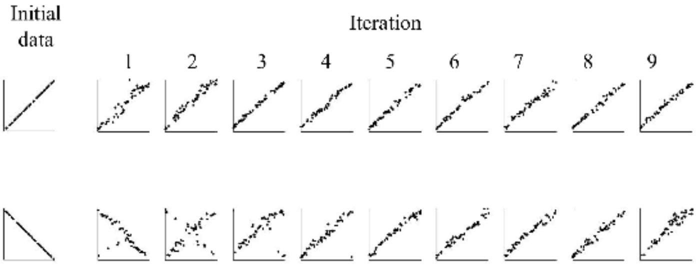

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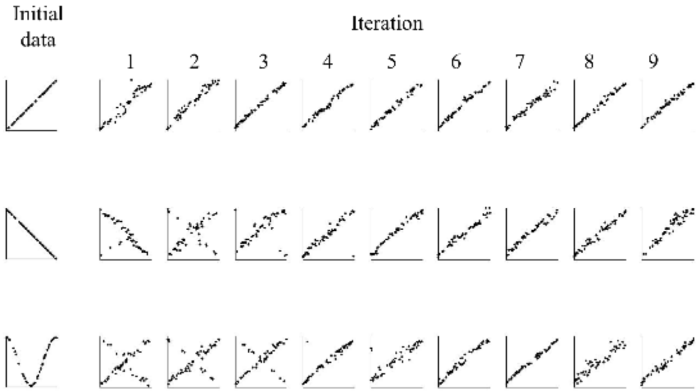

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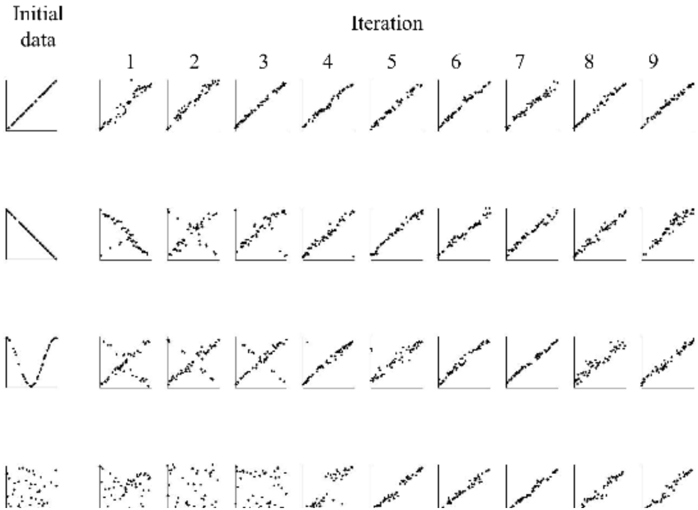

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- This is called the ordinary least squares (OLS) estimator.

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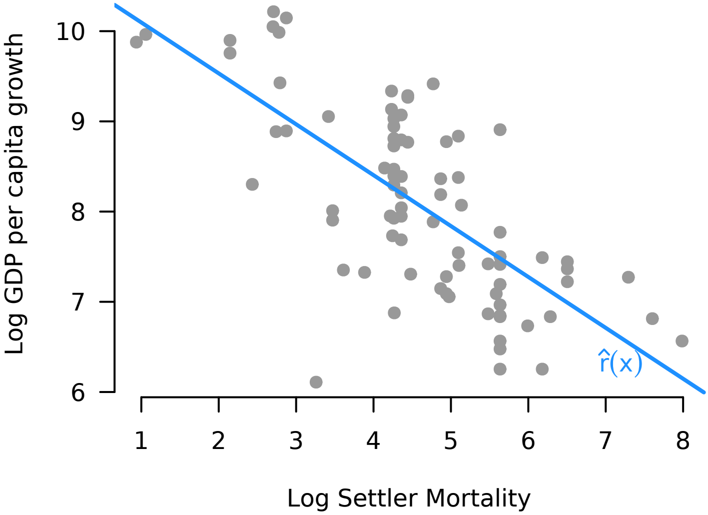
This corresponds to the linear projection which minimizes the sum of squared errors.

Fitted linear CEF/regression function



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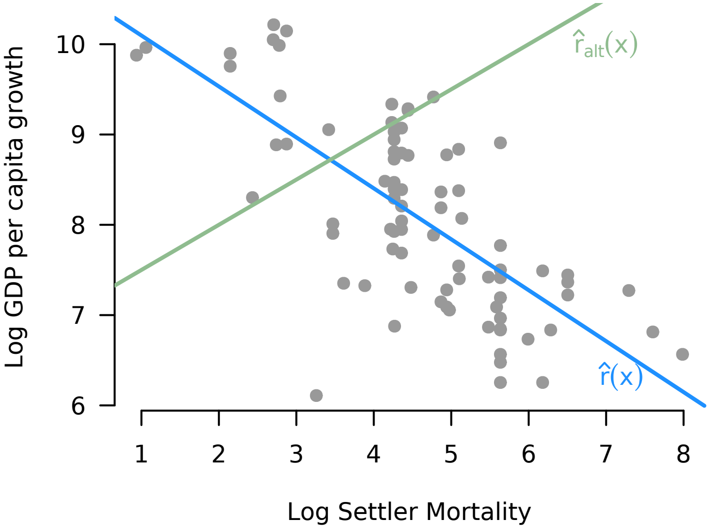

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Definition (Residual)

The **residual** is the difference between the actual value of Y_i and the predicted value, $\widehat{Y}_i$:

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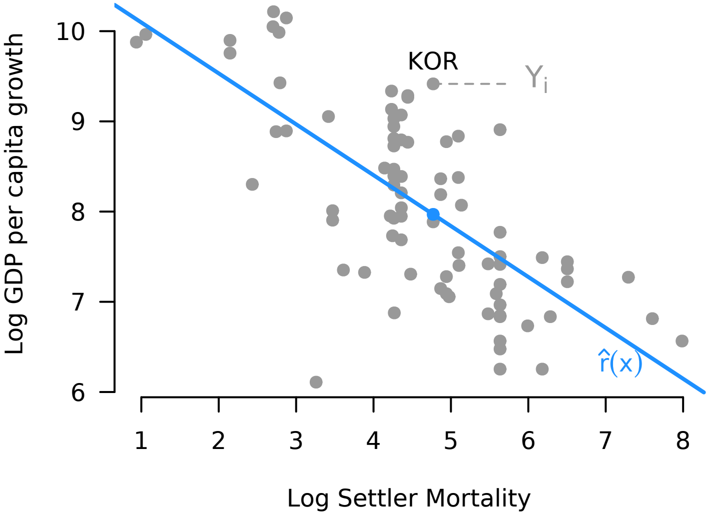
Fitted linear CEF/regression function



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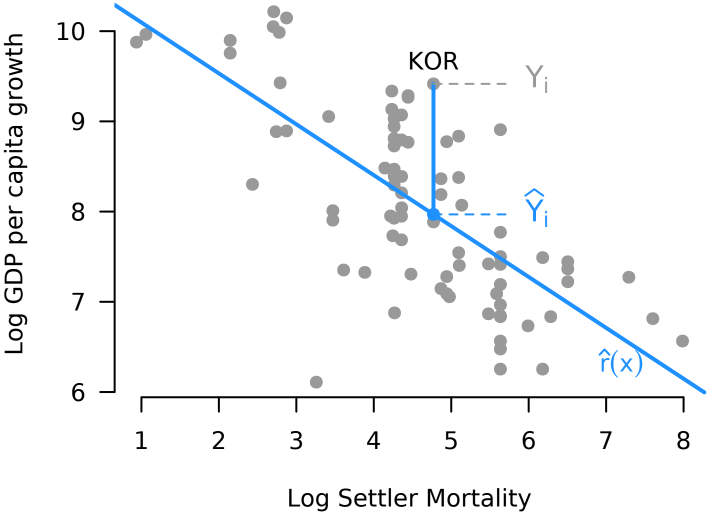

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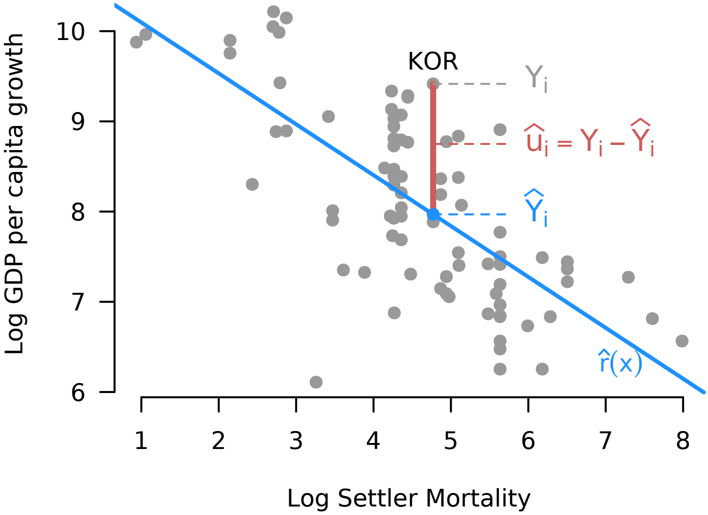
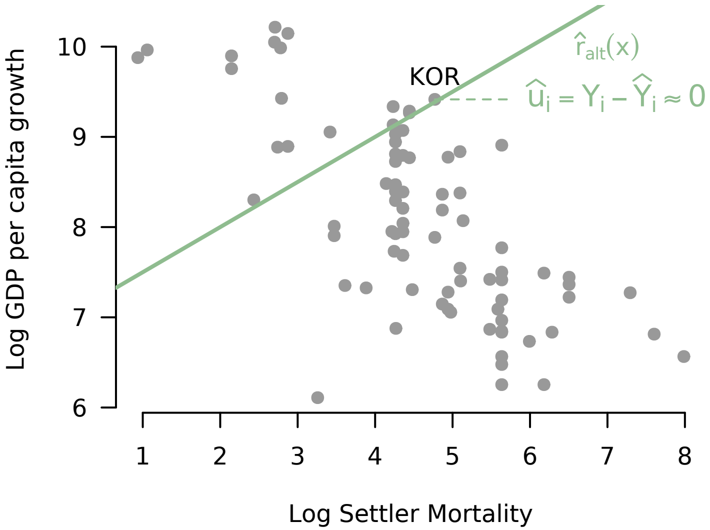


Why not this line?



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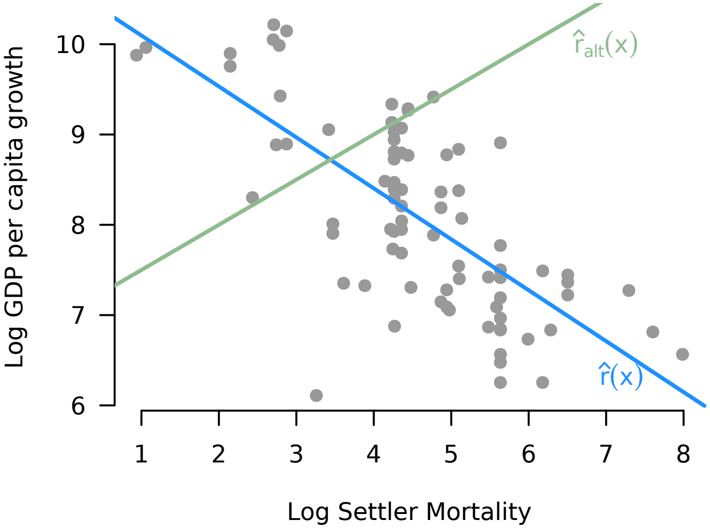
- The smaller the magnitude of the residuals, the better we are doing at predicting Y
- Choose the line that minimizes the residuals

Which is better at minimizing residuals?



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- Let $\{b_0, b_1\}$ be possible values for $\{\beta_0, \beta_1\}$

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- Let's think about n pairs of sample observations: $(Y_1, X_1), (Y_2, X_2), \ldots, (Y_n, X_n)$
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- Derivations are on the next few slides if you are interested.

Step 1: Take Partial Derivatives

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Solving for the Intercept

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A Helpful Lemma on Deviations from Means

Lemmas are like helper results that are often invoked repeatedly.

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&= \left(\sum_{i=1}^{n} X_i\right) - \sum_{i=1}^{n} X_i \\
&= 0
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The OLS estimator

- Now we're done! Here are the **OLS estimators**:

$$\widehat{\beta}_0 = \overline{Y} - \widehat{\beta}_1 \overline{X}$$

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

Intuition of the OLS estimator

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$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{\text{Sample Covariance between } X \text{ and } Y}{\text{Sample Variance of } X}$$

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- The higher the covariance between X and Y, the higher the slope will be.
- Negative covariances $\rightarrow$ negative slopes;
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- If X_i doesn't vary, the denominator is undefined.
- If Y_i doesn't vary, you get a flat line.

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3. The residuals will be uncorrelated with the fitted values:

$$\sum_{i=1}^{n} \widehat{Y}_i\widehat{u}_i = 0 \implies \widehat{\text{Cov}}(\widehat{Y}_i, \widehat{u}_i) = 0$$

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- This is important for two reasons. First, it'll make derivations later much easier. And second, it shows that is just the sum of a random variable. Therefore it is also a random variable.

Lemma 2: OLS as a Weighted Sum of Outcomes

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$$W_i = \frac{(X_i - \overline{X})}{\sum_{i=1}^n (X_i - \overline{X})^2}$$

Sampling distribution of the OLS estimator

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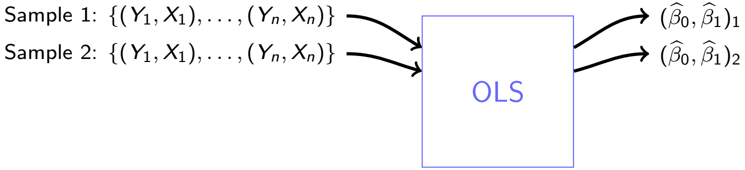

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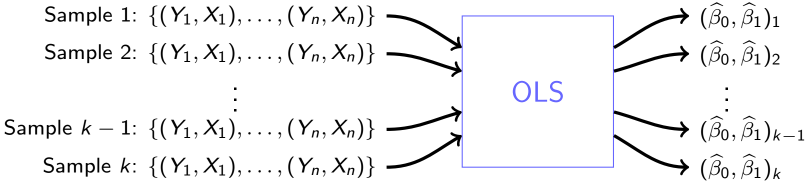

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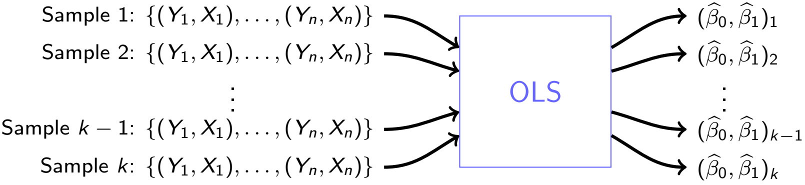

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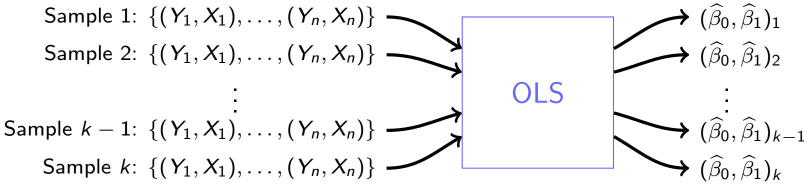

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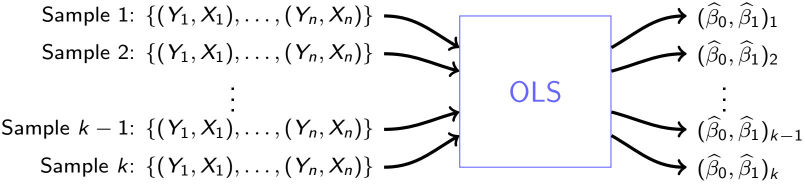

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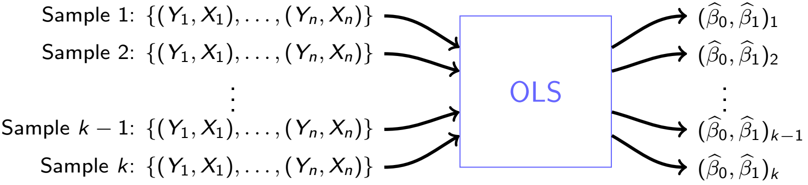

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$\vdots$

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Sample k: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$ → OLS → $(\widehat{\beta}_0, \widehat{\beta}_1)_k$

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  - See how the line varies from sample to sample

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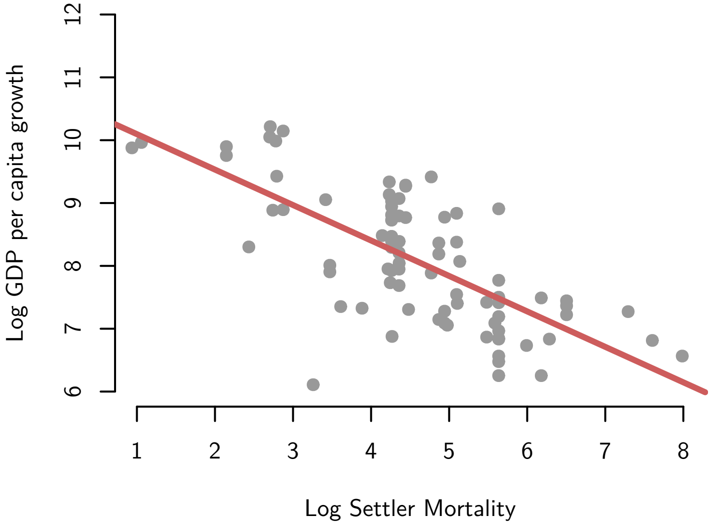
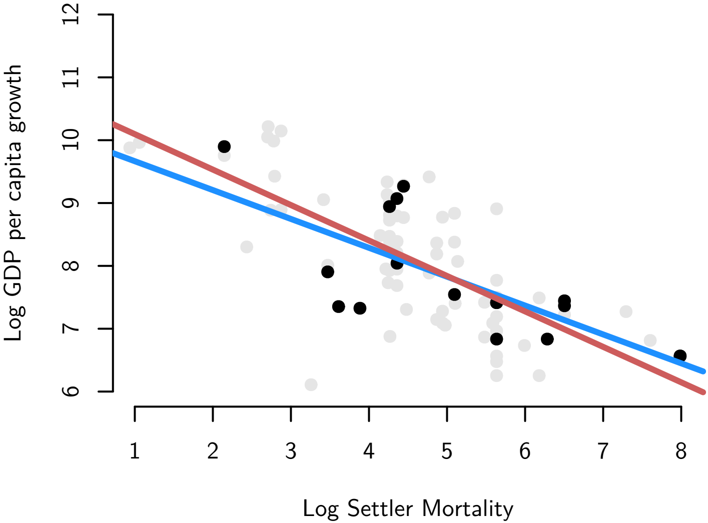
Population Regression



Randomly sample from AJR



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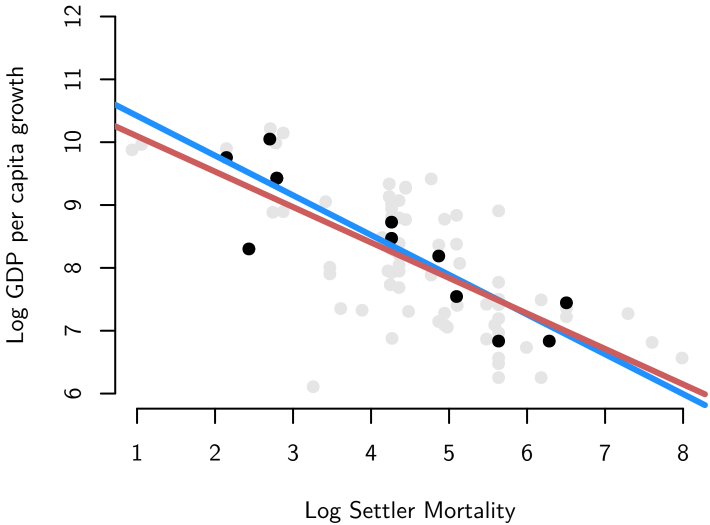

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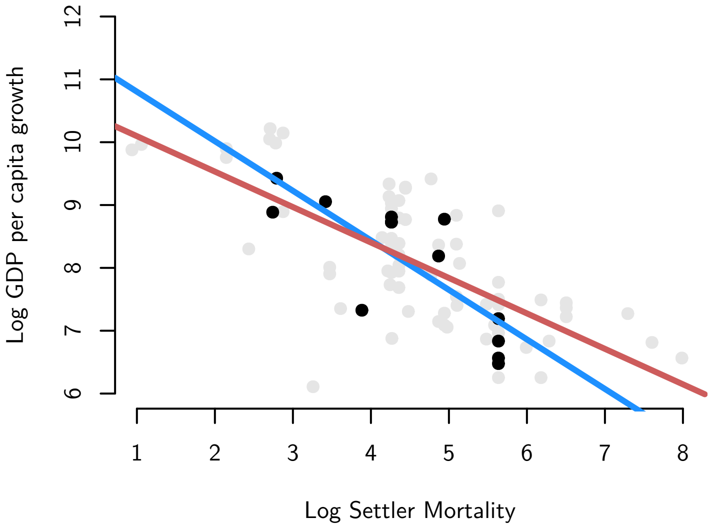

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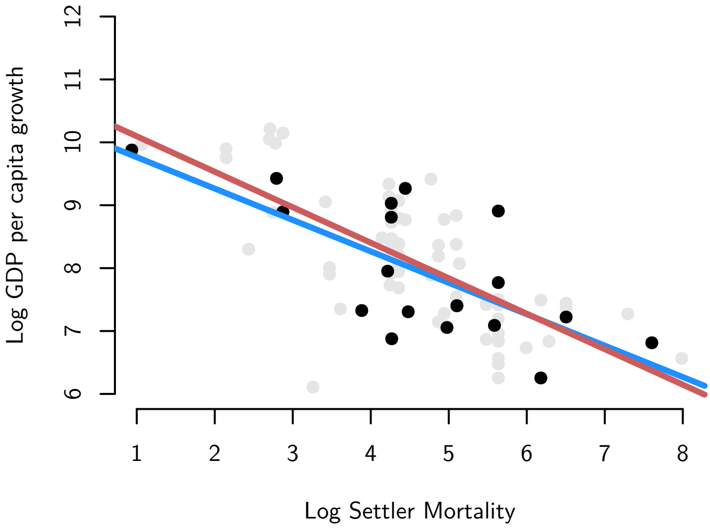

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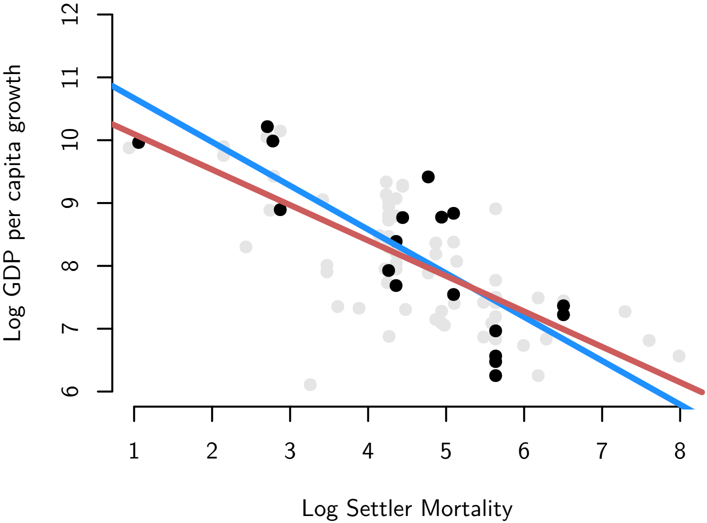

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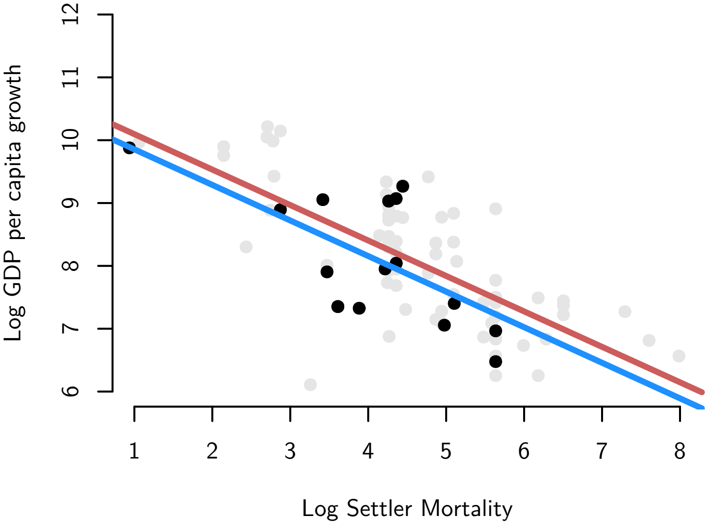

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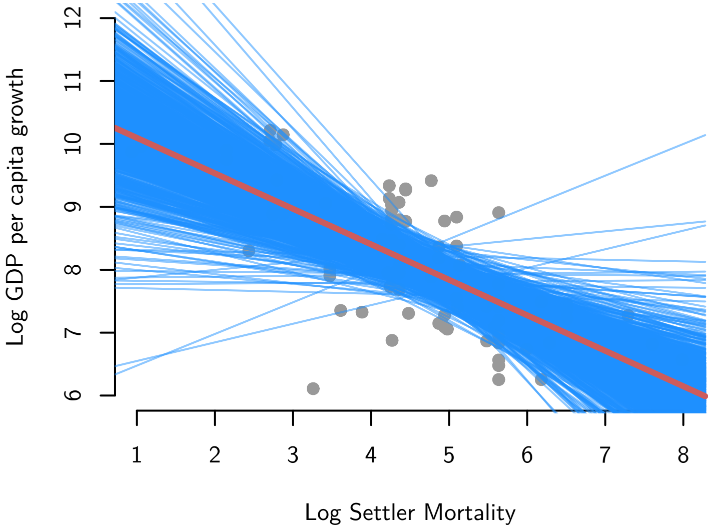

Sampling distribution of OLS

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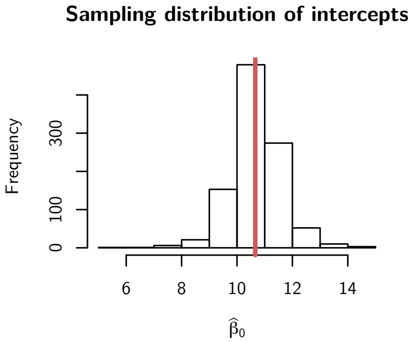
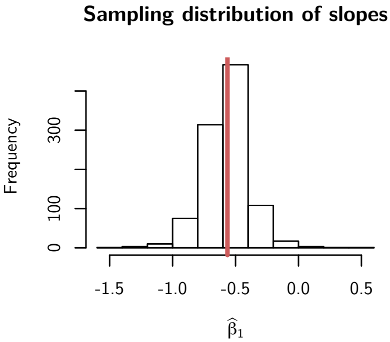
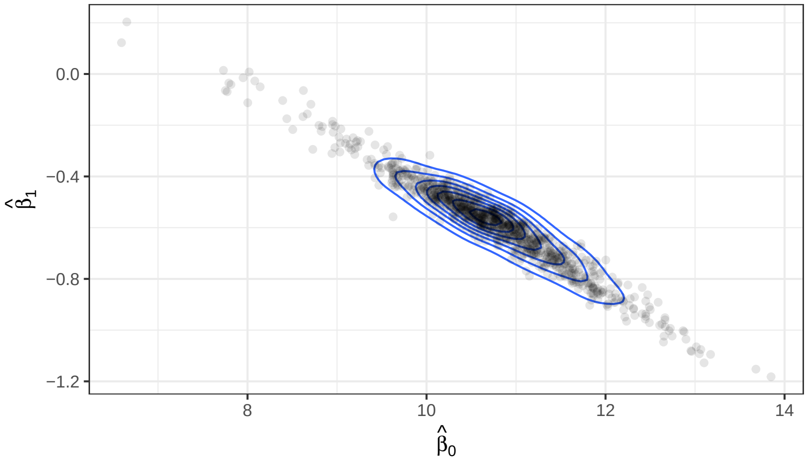
- You can see that the estimated slopes and intercepts vary from sample to sample, but that the "average" of the lines looks about right.





The Sampling Distribution is a Joint Distribution!

While both the intercept and the slope vary, they vary together.



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- We will use the same strategy here!

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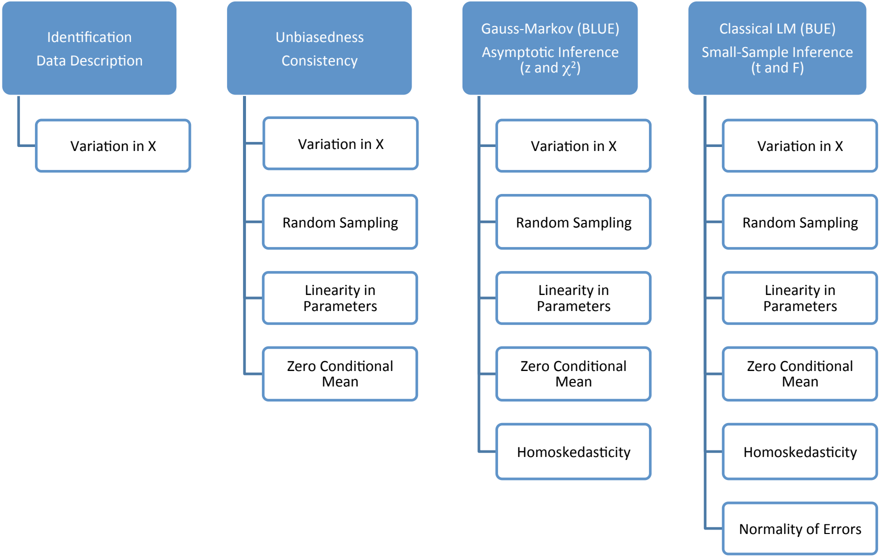

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- We assume this to be the structural model, i.e., the model describing the true process generating Y_i

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The observed data:

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represent an i.i.d. random sample of size n following the population model.

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- Sample selection problems (sample not representative of the population)

OLS Assumption III

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Assumption (III. Variation in X; a.k.a. No Perfect Collinearity)

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Only assumption needed for using OLS as a pure data summary.

Stuck in a moment

- Why does this matter?

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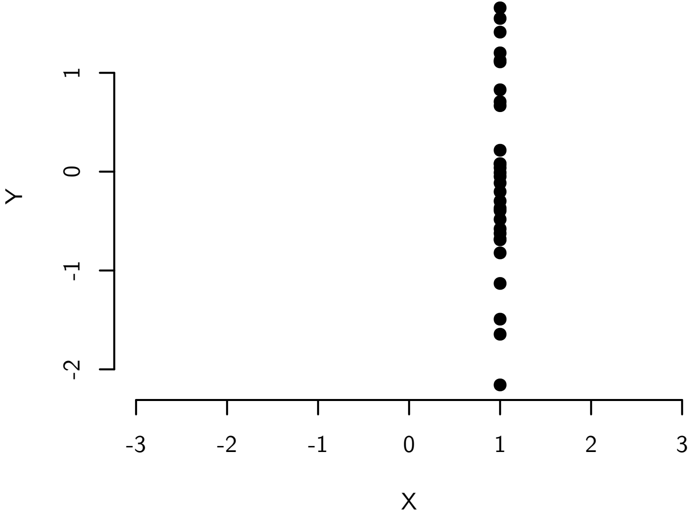
- Why does this matter? How would you draw the line of best fit through this scatterplot, which is a violation of this assumption?



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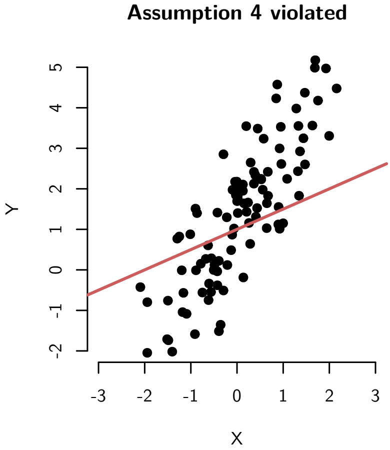

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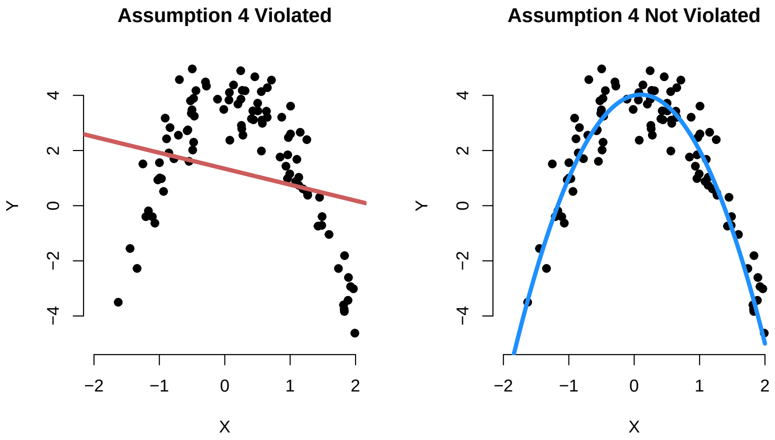

Unbiasedness

With Assumptions 1-4, we can show that the OLS estimator for the slope is unbiased, that is $E[\widehat{\beta}_1] = \beta_1$.

Let's prove it!

Lemma 3: Weighted Combinations of X_i

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\hat{\beta}_1 - \beta_1 &= \sum_{i=1}^{n} W_i u_i
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$$E[\hat{\beta}_1 - \beta_1|X] = E\left[\sum_{i=1}^{n} W_i u_i|X\right]$$

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Using iterated expectations we can show that it is also unconditionally biased $E[\hat{\beta}_1] = E[E[\hat{\beta}_1|X]] = E[\beta_1] = \beta_1$.

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- Under A4 (zero conditional mean error) we have the slightly weaker property $Cov[X_i, u_i] = 0$ so as long as $V[X] > 0$, then we have,

$$\hat{\beta}_1 \xrightarrow{p} \beta_1$$

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- That is we know that the sampling distribution is centered on the true population slope, but we don't know the population variance.

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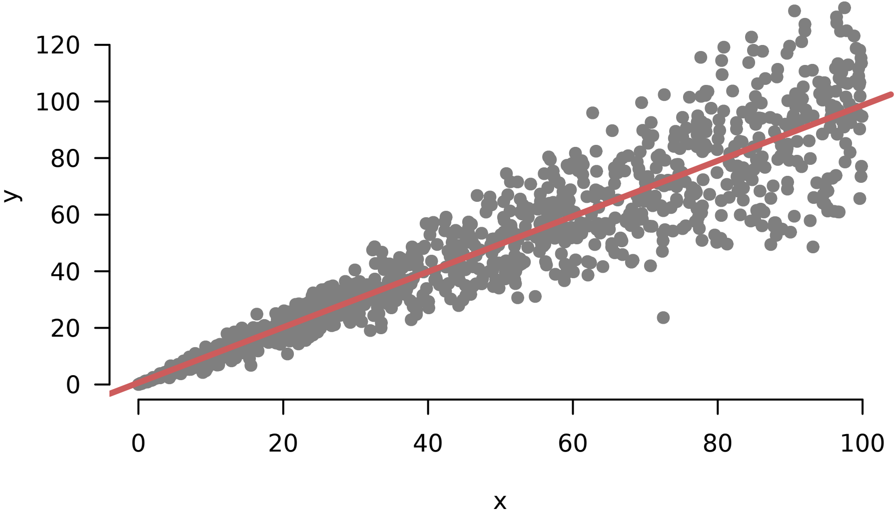
- Assumptions I–V are collectively known as the Gauss-Markov assumptions

Heteroskedasticity



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Variance of OLS Estimators

Theorem (Variance of OLS Estimators)

Given OLS Assumptions I–V (Gauss-Markov Assumptions):

$$V[\hat{\beta}_1 \mid X] = \frac{\sigma_u^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$V[\hat{\beta}_0 \mid X] = \sigma_u^2 \left\{ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right\}$$

where $V[u \mid X] = \sigma_u^2$ *(the error variance).*

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    - As we increase n, the denominator gets large, while the numerator is fixed and so the sampling variance shrinks to 0.

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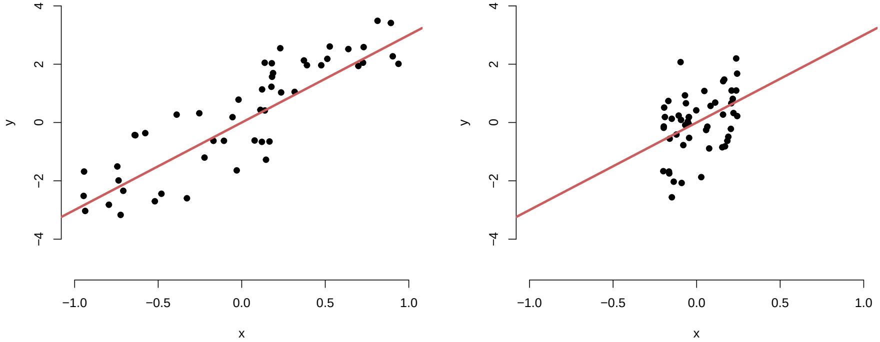

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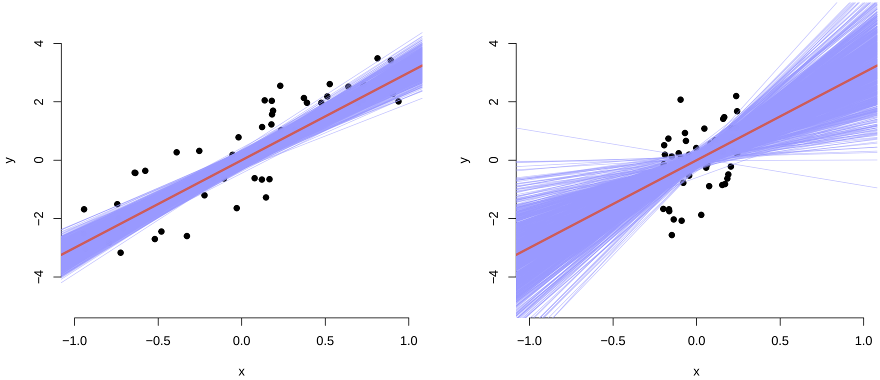

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- Thus, an unbiased estimator for the error variance is:

$$\hat{\sigma}_u^2 = \frac{n}{n-2} MSD(\hat{u}) = \frac{n}{n-2}\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2$$

We plug this estimate into the variance estimators for $\hat{\beta}_0$ and $\hat{\beta}_1$.

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- Now we know the mean and sampling variance of the sampling distribution.
- How does this compare to other estimators for the population slope?

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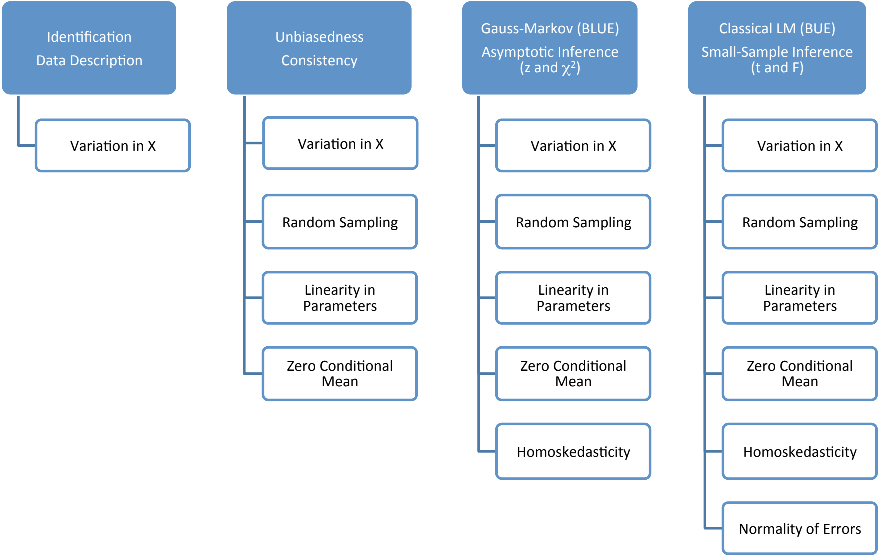
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- Result fails to hold when the assumptions are violated!

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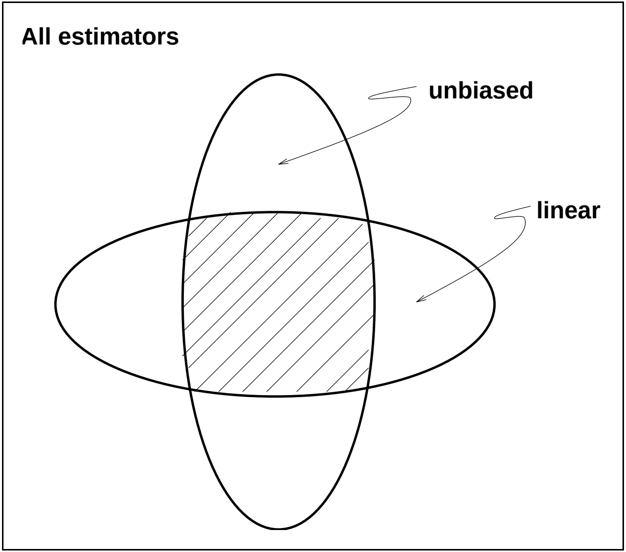

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- What about the last question mark? What's the form of the distribution?

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- Again, we will come back to this in a couple of weeks.

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  - any size sample with normal errors: $T \sim t_{n-2}$
  - conservative to use t_{n-2} anyways since t_{n-2} is approximately normal in large samples.

Test statistic

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- Thus, under the null, we know the distribution of T and can use that to formulate a rejection region and calculate p-values.
- By default, R shows you the test statistic for $\beta_1 = 0$ and uses the t distribution.

Rejection region

- Choose a level of the test, α, and find rejection regions that correspond to that value under the null distribution:

$$\mathbb{P}(-t_{\alpha/2,n-2} < T < t_{\alpha/2,n-2}) = 1 - \alpha$$

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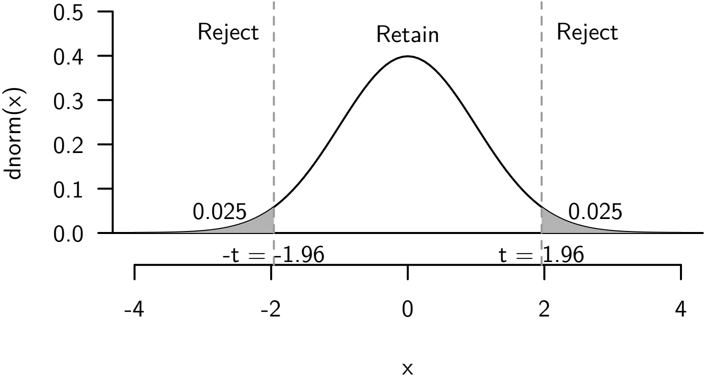
- This is exactly the same as with sample means and sample differences in means, except that the degrees of freedom on the t distribution have changed.



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- If the p-value is less than α we would reject the null at the α level.

Confidence intervals

- Very similar to the approach with sample means. By the sampling distribution of the OLS estimator, we know that we can find t-values such that:

$$\mathbb{P}\left(-t_{\alpha/2,n-2} \leq \frac{\widehat{\beta}_1 - \beta_1}{\widehat{SE}[\widehat{\beta}_1]} \leq t_{\alpha/2,n-2}\right) = 1 - \alpha$$

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- We can derive these for the intercept as well:

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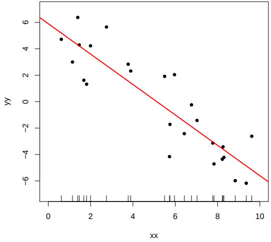
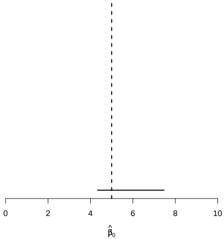
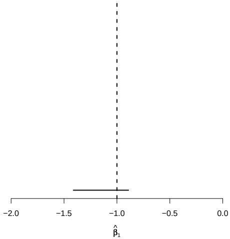
CIs Simulation Example

Returning to our simulation example we can simulate the sampling distributions of the 95 % confidence interval estimates for $\widehat{\beta}_1$ and $\widehat{\beta}_0$







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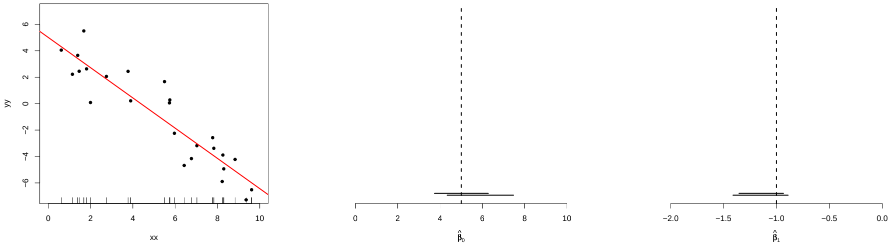

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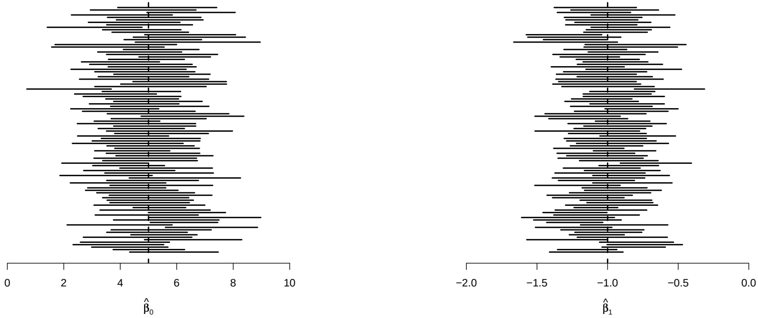

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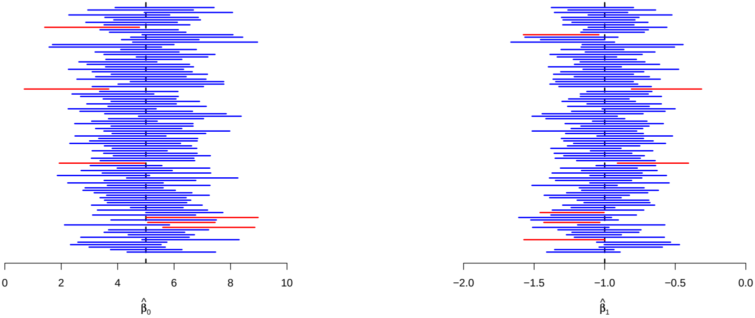

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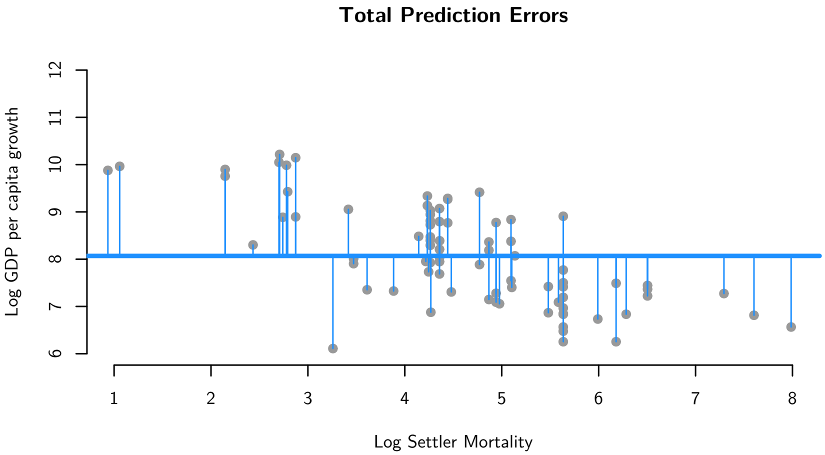
- Once we have estimated our model, we have new prediction errors, which are just the sum of the squared residuals or SS_{res}:

$$SS_{res} = \sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2$$

Sum of Squares



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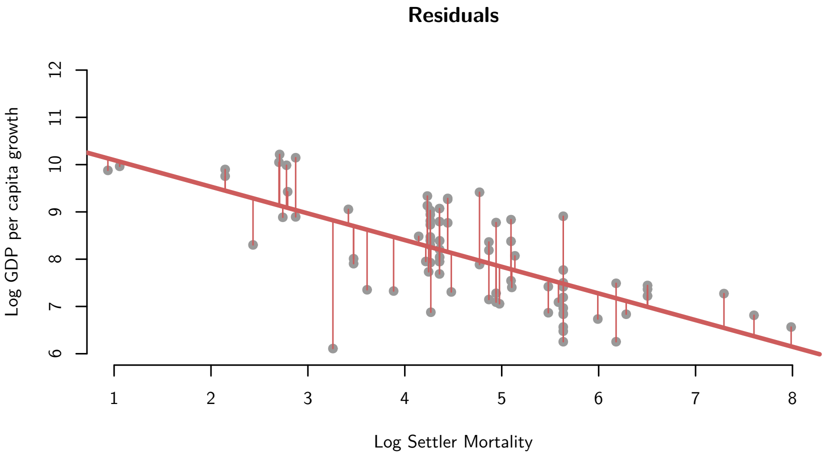

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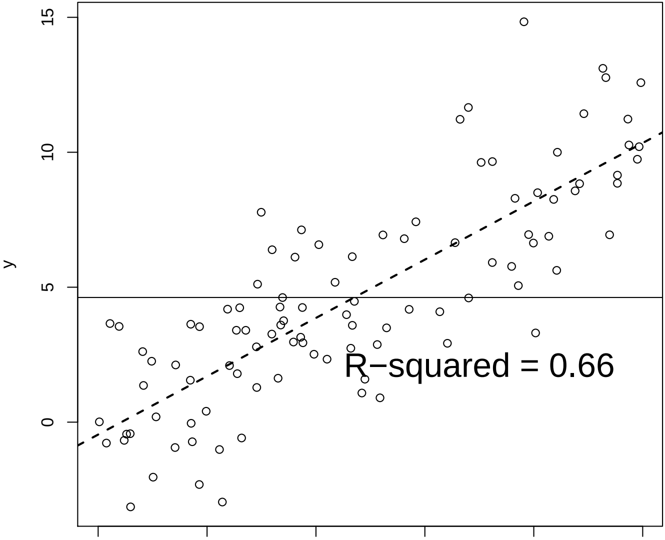
- $R^2 = 1$ implies perfect linear fit

Is R-squared useful?



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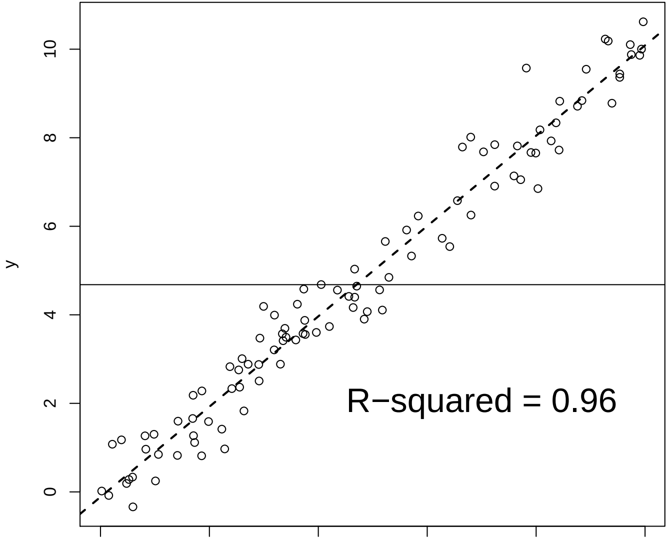

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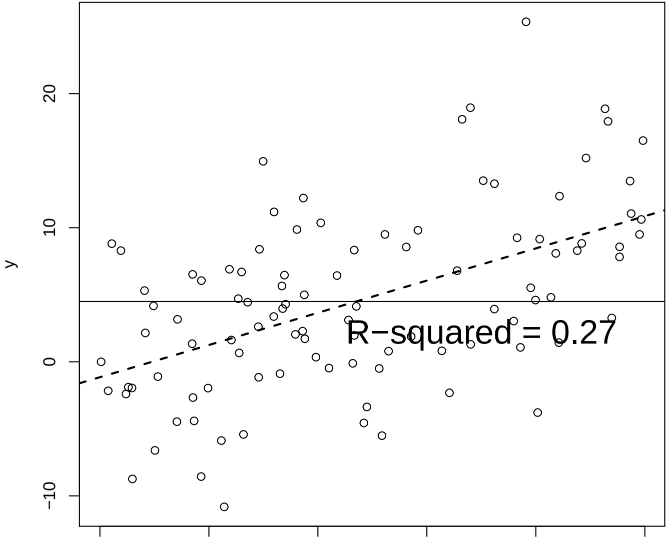

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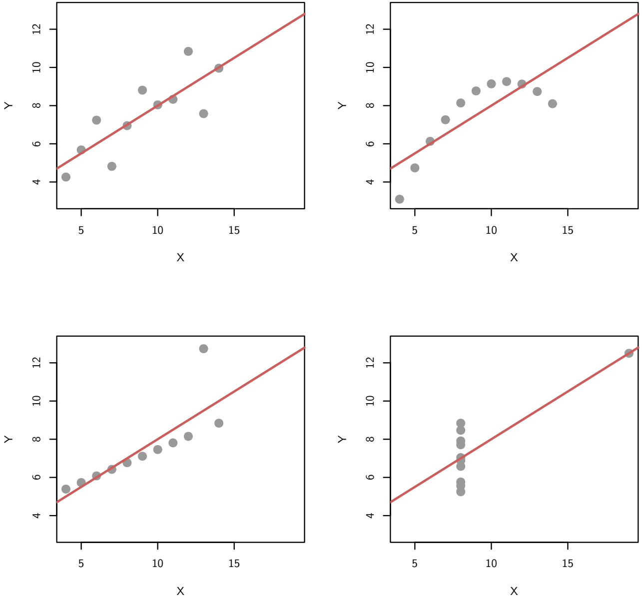
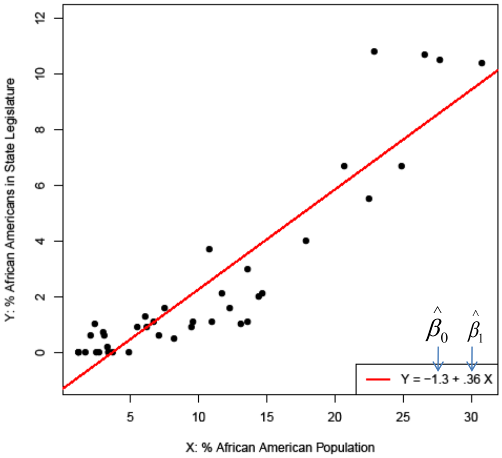


Interpreting a Regression

Let's have a quick chat about interpretation.



State Legislators and African American Population

Interpretations of increasing quality:

```
> summary(lm(beo ~ bpop, data = D))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.31489    0.32775  -4.012 0.000264 ***
bpop         0.35848    0.02519  14.232  < 2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.317 on 39 degrees of freedom
Multiple R-squared: 0.8385,Adjusted R-squared: 0.8344
F-statistic: 202.6 on 1 and 39 DF,  p-value: < 2.2e-16
```

"African American population is statistically significant ($p < 0.001$)"

(no effect size or direction)

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"Percent African American legislators increases with African American population ($p < 0.001$)"

(direction, but no effect size)

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"A one percentage point increase in the African American population causes a 0.35 percentage point increase in the fraction of African American state legislators ($p < 0.001$)."

(unwarranted causal language)

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"A one percentage point increase in the African American population is associated with a 0.35 percentage point increase in the fraction of African American state legislators ($p < 0.001$)."

(hints at causality)

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"In states where an additional percentage point of the population is African American, we observe on average 0.35 percentage point more African American state legislators ($p < 0.001$)."

(p value doesn't help people with uncertainty)

State Legislators and African American Population

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"In states where an additional percentage point of the population is African American, we observe on average 0.35 percentage points more African American state legislators (between .35 and .40 with 95% confidence)."

(still not perfect, the best will be subject matter specific. is fairly clear it is non-causal, gives uncertainty.)

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2. If the association has a causal interpretation explain why, otherwise do not imply a causal interpretation.
3. Provide a meaningful sense of uncertainty
4. Indicate the practical significance of the finding for your argument.

Goal Check: Understand `lm()` Output

```
Call:
lm(formula = sr ~ pop15, data = LifeCycleSavings)

Residuals:
   Min     1Q Median     3Q    Max
-8.637 -2.374  0.349  2.022 11.155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.49660    2.27972   7.675 6.85e-10 ***
pop15       -0.22302    0.06291  -3.545 0.000887 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.03 on 48 degrees of freedom
Multiple R-squared:  0.2075,Adjusted R-squared:  0.191
F-statistic: 12.57 on 1 and 48 DF,  p-value: 0.0008866
```

This Week in Review

- CEFs!
- OLS!
- Classical regression assumptions!
- Inference!

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Going Deeper:

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Next week: Linear Regression with Two Variables!