

Week 2: Random Variables

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Princeton

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¹Many of these slides are adapted from Adam Glynn and Jens Hainmueller. Some additional individual slides come from Justin Grimmer and Ian Lundberg. Many illustrations by Shay O'Brien.

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- ▶ probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

- 1 Definition of Random Variables
 - What is a Random Variable?
- 2 Discrete Distributions
- 3 Continuous Distribution
- 4 Expectation as a Measure of Central Tendency
- 5 Variance as a Measure of Dispersion
- 6 Fun With Averages
- 7 Joint and Conditional Distributions
- 8 Characterizing Conditional Distributions

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We can use probability to assess the “fairness” of this process.

We will do this by introducing a random variable X to be Barack Obama's position on the 2008 New Hampshire primary ballot.

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We will generally suppress the function notation and just refer to X .

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- Other times the sample space is already numeric so its more obvious (e.g. how many minutes until the train arrives).

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- Is it really easier this way? It seems hard.
random variables are about bridging the abstract math and the concrete world. that can be hard, but it is super important and better than the alternative!

NH Ballot Order Example

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$X = \left\{ \begin{array}{l} \text{Joe Biden} \\ \text{Hillary Clinton} \\ \text{Chris Dodd} \\ \text{John Edwards} \\ \text{Mike Gravel} \\ \text{Dennis Kucinich} \\ \text{Barack Obama} \\ \text{Bill Richardson} \end{array} \right.$$

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

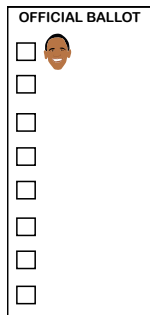
X is a random variable indicating Obama's position on the ballot. Highlighted letters are those leading to a given ballot position. Highlighted individual is first.

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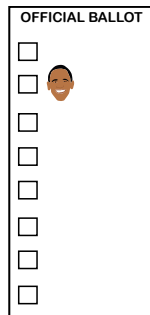
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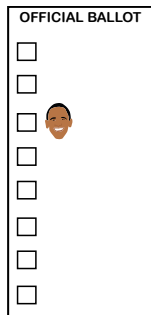
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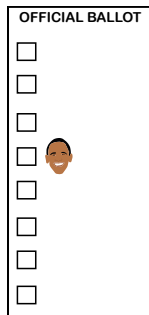
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$$X = \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right.$$



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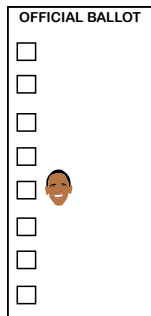
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$$X = \left\{ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right.$$



A,B,C,**D**,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

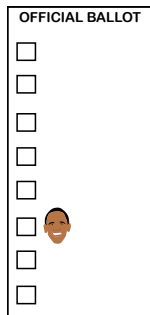
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$$X = \left\{ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \color{red}{6} \end{array} \right.$$



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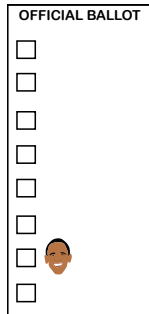
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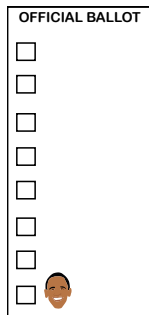
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- A **probability mass function** (PMF) and a **cumulative distribution function** (CDF) are two common ways to define the probability distribution for a discrete random variable.
- Probability mass functions provide a compact way to represent information about **how likely** various outcomes are.

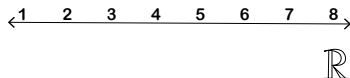
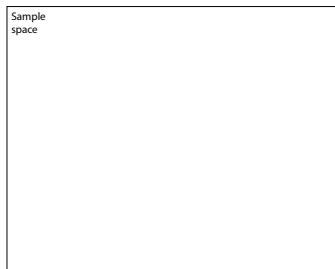
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The probabilities associated with each realization of the random variables come from the underlying stochastic realization of the sample space.

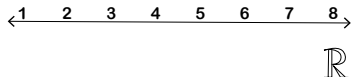
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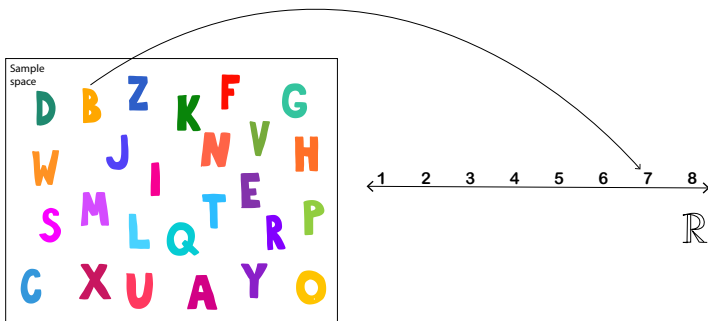
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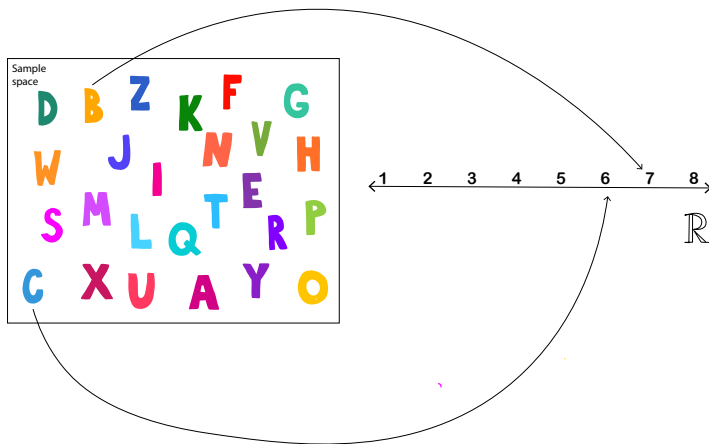
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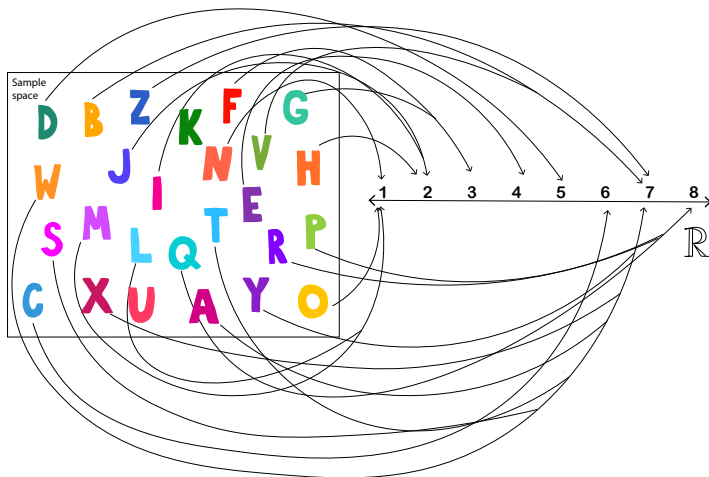
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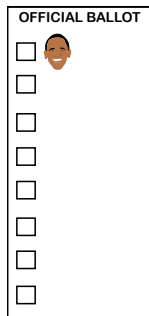


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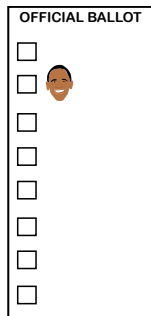
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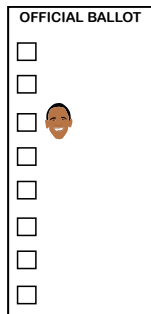
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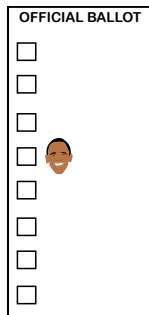
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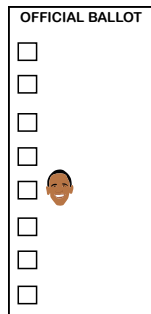
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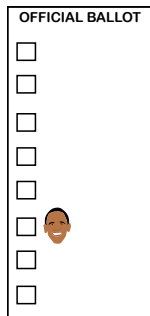
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A, B, **C**, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z

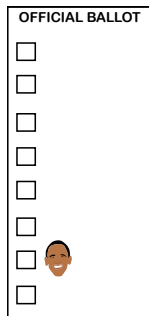
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$$p_X(x) = \begin{cases} 4/26 & x = 1 \\ 4/26 & x = 2 \\ 2/26 & x = 3 \\ 1/26 & x = 4 \\ 1/26 & x = 5 \\ 1/26 & x = 6 \\ 10/26 & x = 7 \end{cases}$$



A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

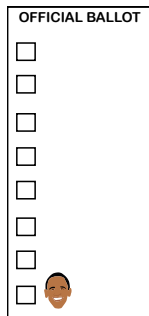
Probability of the random variable equaling a number is just the probability of the underlying event (subset of the sample space).

Example: New Hampshire

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
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Discrete Probability Mass Functions

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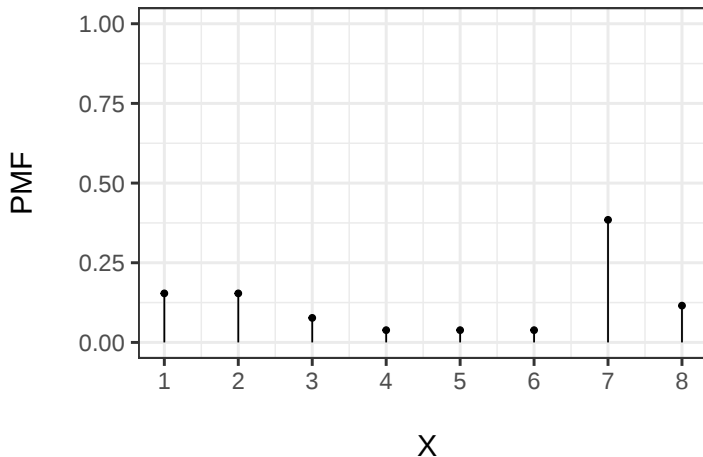
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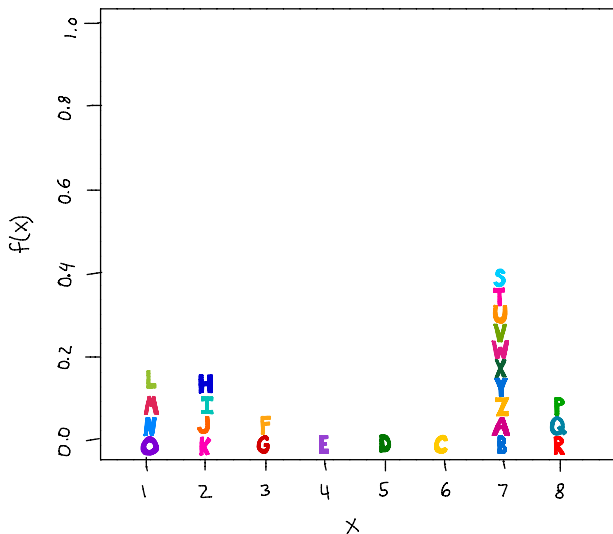
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- the **support** of X is the set of values where the PMF is non-zero.
- $\sum_x p_X(x) = 1$.

NH Obama Ballot Position PMF Plot



NH Obama Ballot Position PMF Plot



Cumulative Distribution Function

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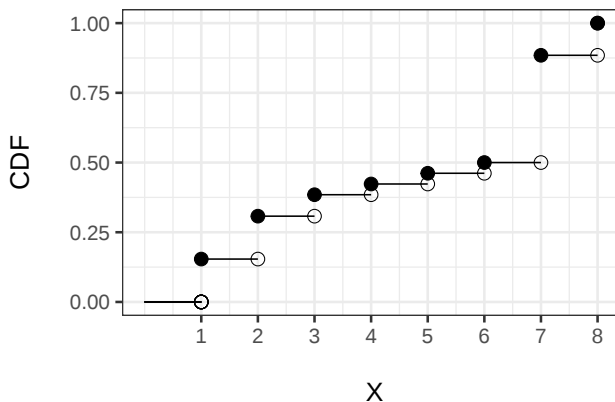
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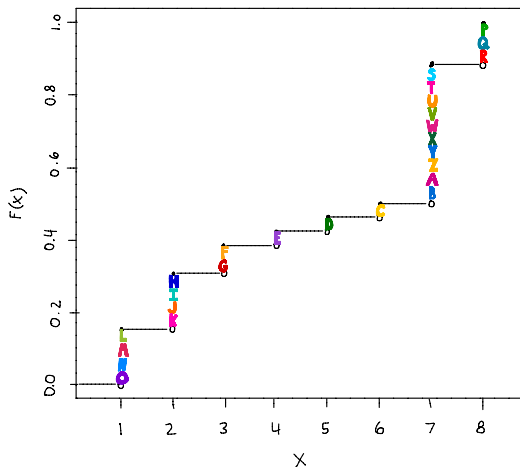
Key properties:

- **non-decreasing**
- **right-continuous**
- converges to 0 and 1 in the limits

NH Obama Ballot Position CDF Plot



NH Obama Ballot Position CDF Plot



Example Discrete Distributions

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- A major advantage of random variables is that they often have a distribution with a known form (that comes with known results!)
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- We can summarize these distributions with one number

- 1 Definition of Random Variables
- 2 Discrete Distributions
- 3 Continuous Distribution
 - Defining a Continuous Random Variable
 - Probability Density Functions and Cumulative Distribution Functions
 - Subtleties of the Continuous Setting
- 4 Expectation as a Measure of Central Tendency
- 5 Variance as a Measure of Dispersion
- 6 Fun With Averages
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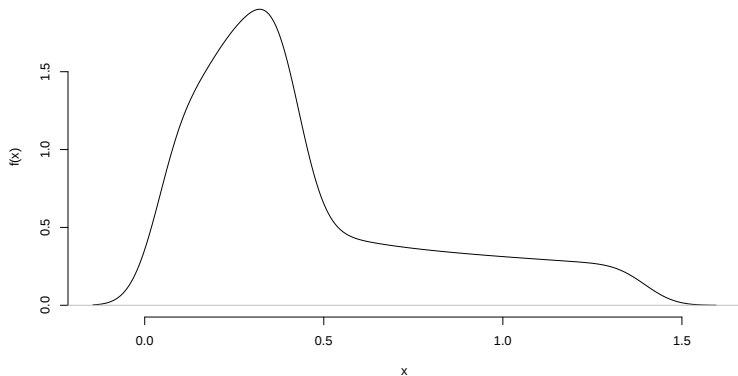
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- They are similar to the discrete case with a few subtle differences.

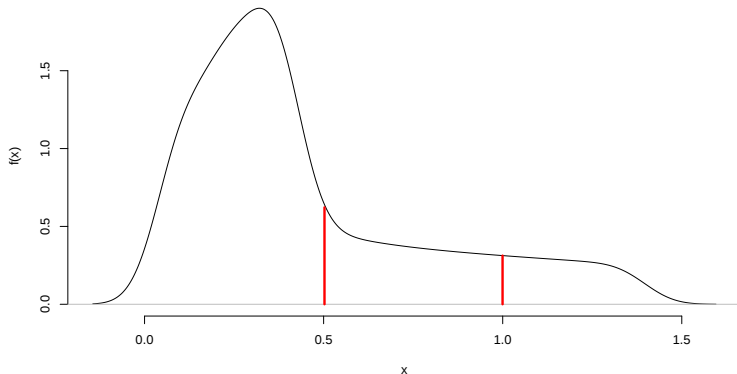
Calculus Review: Integration

Suppose we have some function $f(x)$



Calculus Review: Integration

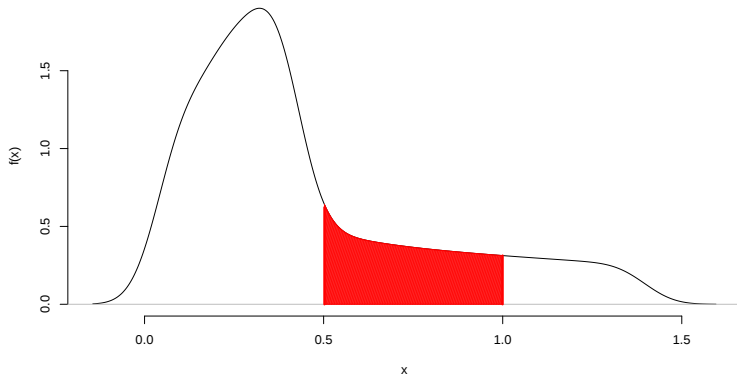
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$$\text{Area under curve} = \int_{1/2}^1 f(x) dx = F(1) - F(1/2)$$

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Definition (Continuous Distribution)

A random variable has a **continuous distribution** if its CDF is differentiable. We also allow there to be endpoints (or finitely many points) where the CDF is continuous but not differentiable, as long as the CDF is differentiable everywhere else. (Blizstein and Hwang Definition 5.1.1)

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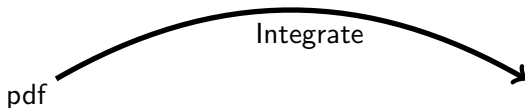
pdf

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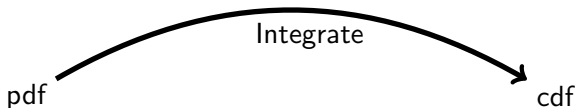


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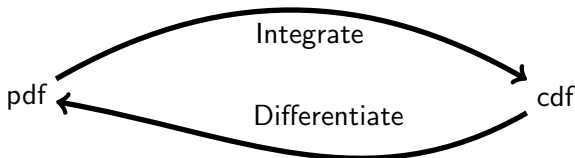


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A Visual Example

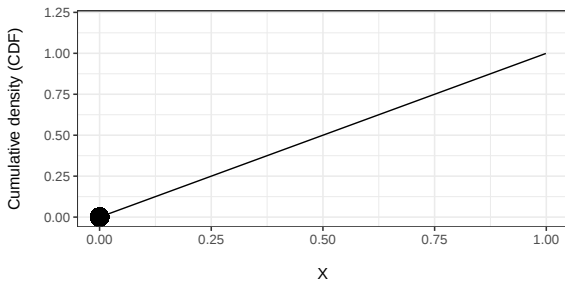
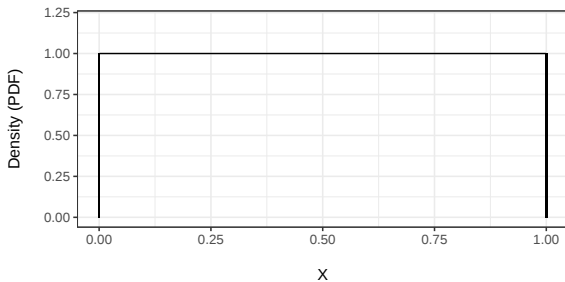
Imagine you choose a number completely at random between 0 and 1 with all equally sized sets of values being equally likely.

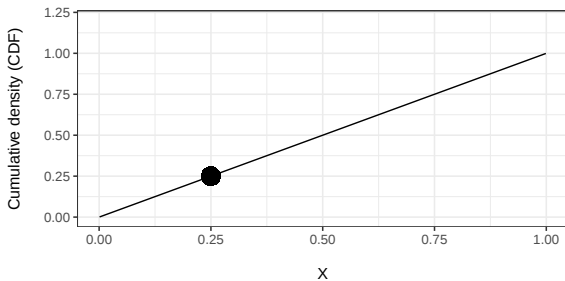
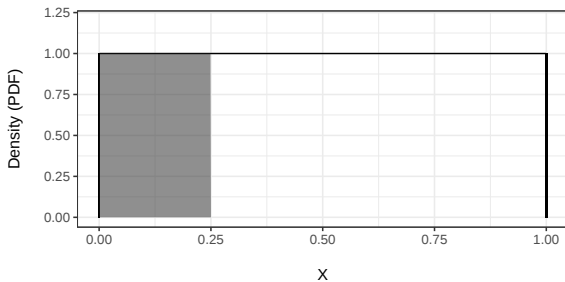
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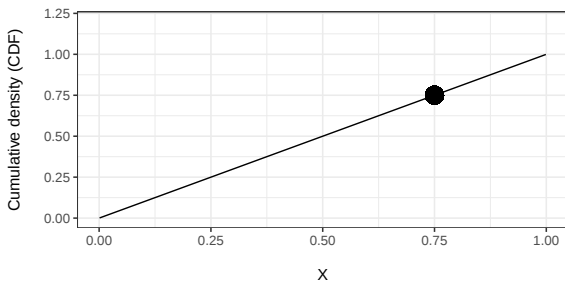
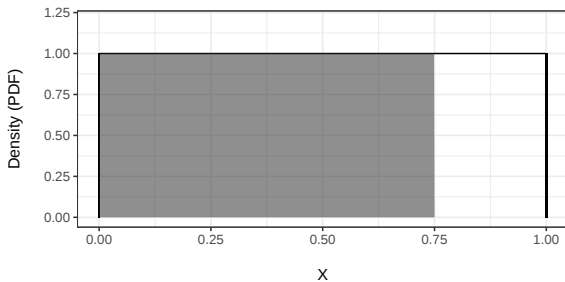
Imagine you choose a number completely at random between 0 and 1 with all equally sized sets of values being equally likely. This is a standard uniform distribution which has the CDF,

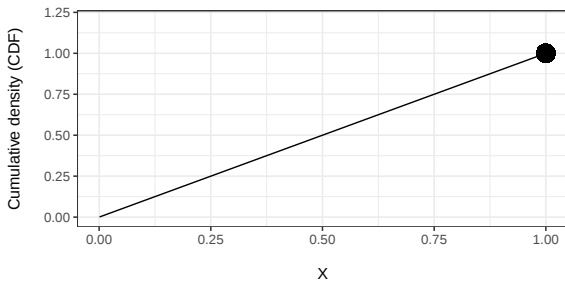
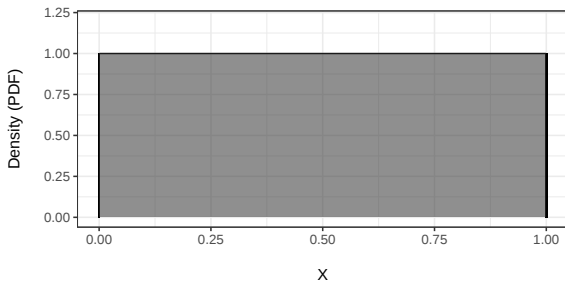
$$F_X(x) = x$$

with support over $[0, 1]$.



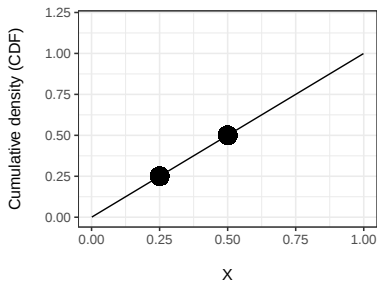
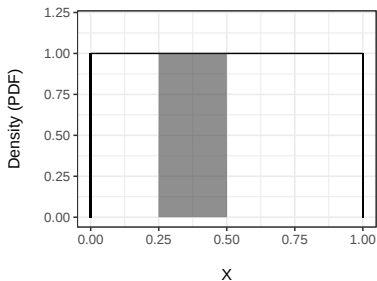




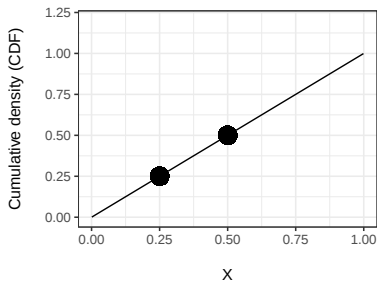
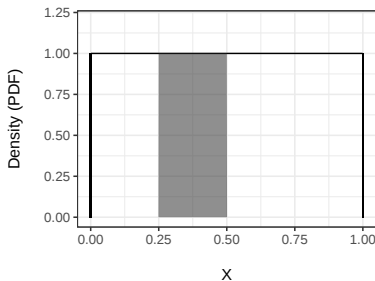


What is the probability that the number is between 0.25 and 0.5?

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$$F_X(.5) - F_X(.25) = .25$$

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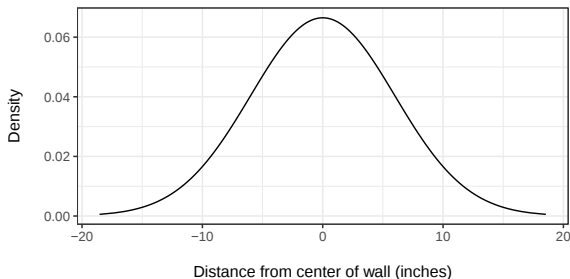
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Let's explain!

A Numerical Example

Let's suppose we have someone throwing darts and we measure how far they are from the center of the wall in inches. In this case, perhaps the darts will be distributed with the following PDF.

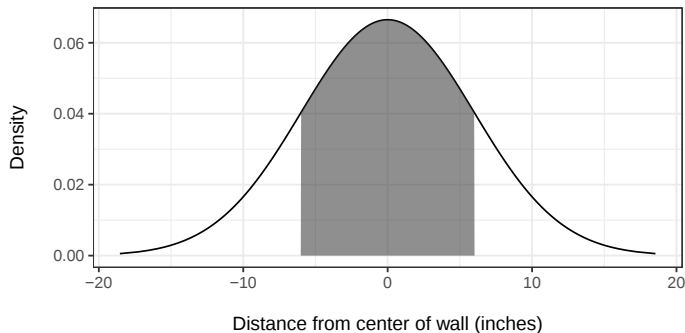


A Numerical Example

How would we calculate the probability that a dart lands within 6 inches of the center of the wall?

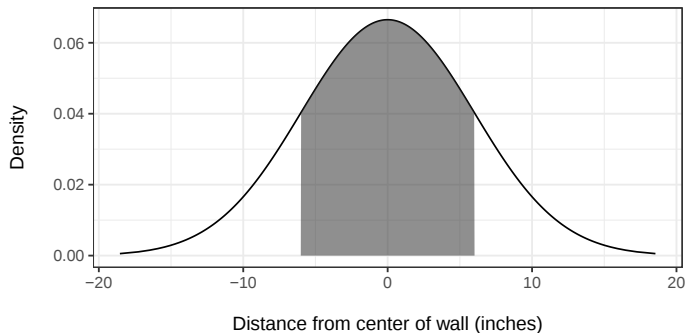
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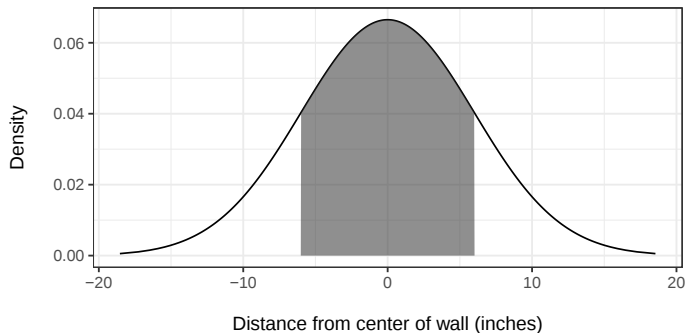
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$$P(X \in (-6, 6)) =$$

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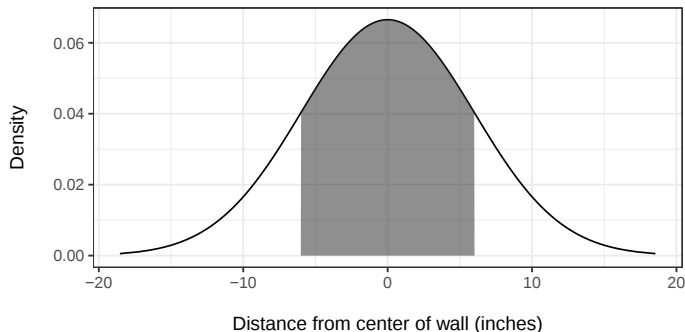
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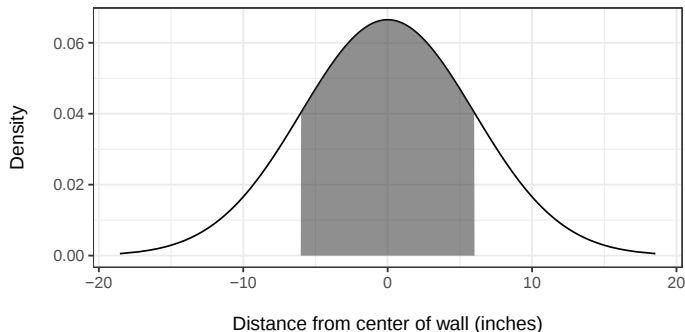
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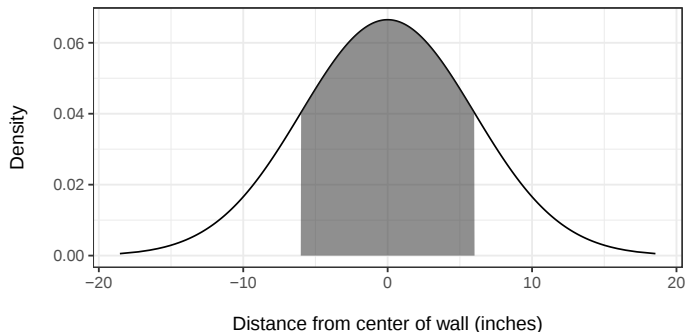
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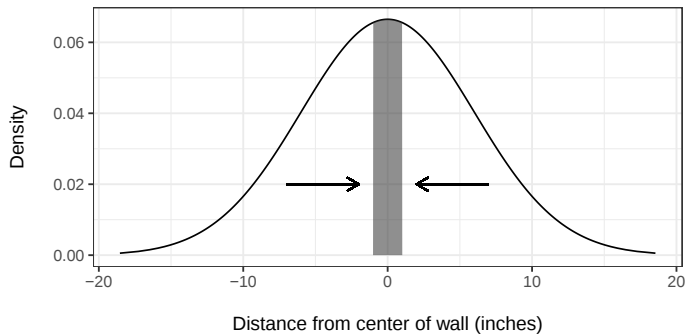
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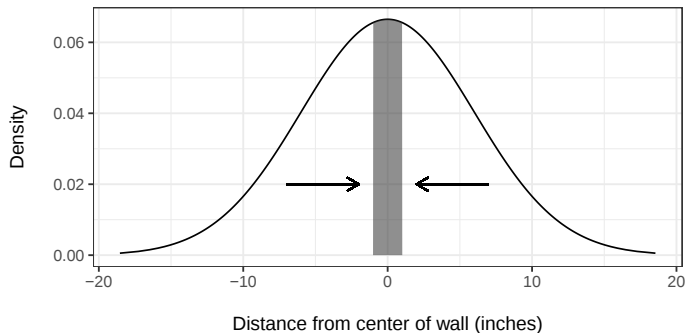


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One inch?

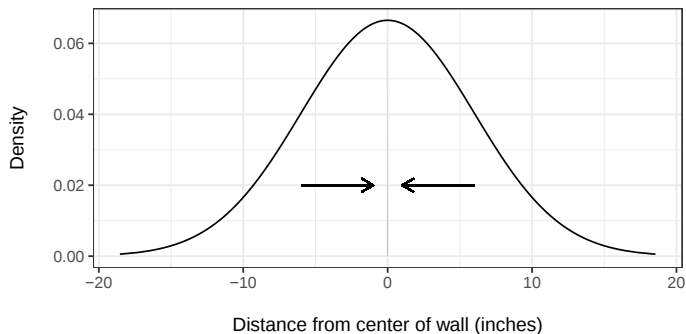


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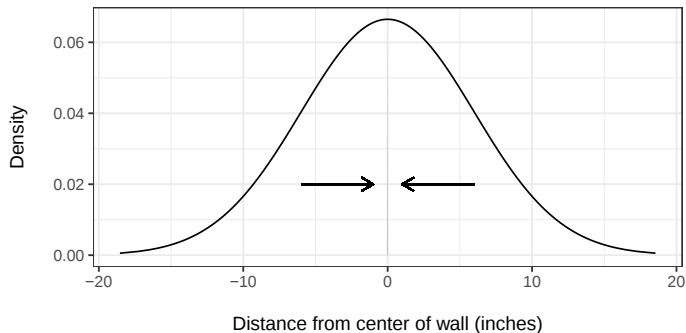


$$P(X \in (-1, 1)) = 0.0664135$$

1/100th of an inch?

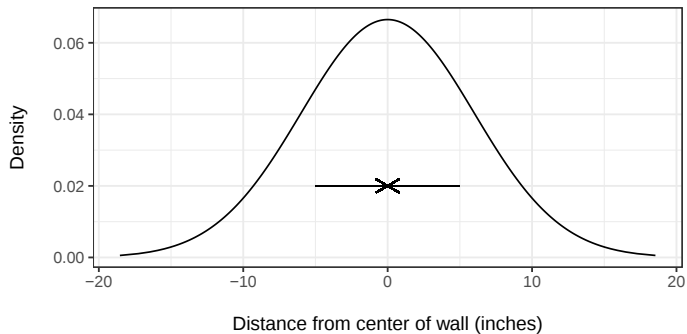


1/100th of an inch?

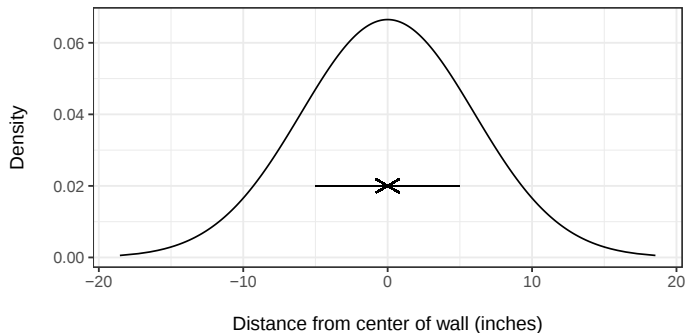


$$P(X \in (-.01, .01)) = 0.0006649037$$

A perfect bullseye?



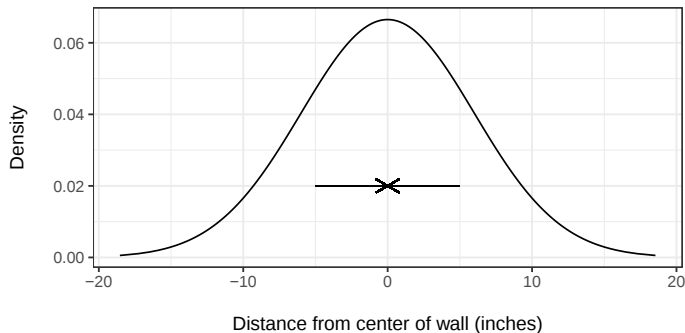
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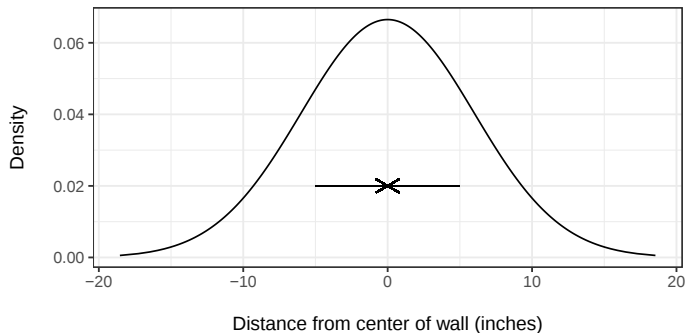
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Why?

A perfect bullseye?



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Because the **width** of the range we are calculating is **zero**, the area is zero.

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- As with discrete random variables there are common families of distributions.

- 1 Definition of Random Variables
- 2 Discrete Distributions
- 3 Continuous Distribution
- 4 Expectation as a Measure of Central Tendency
 - Central Tendency
 - Example: Assessing Racial Prejudice
- 5 Variance as a Measure of Dispersion
- 6 Fun With Averages
- 7 Joint and Conditional Distributions
- 8 Characterizing Conditional Distributions

Characterizing Distributions

Distributions have all kinds of wonky shapes. How do we characterize what they look like?

Expectation

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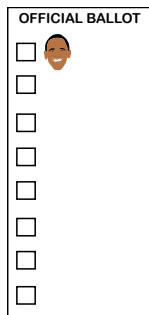
What did we expect for Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

4/26 × 1

+



A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

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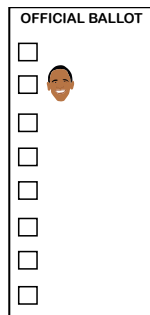
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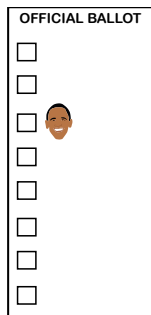
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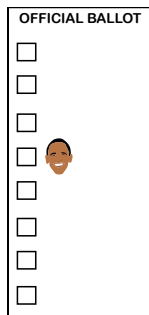


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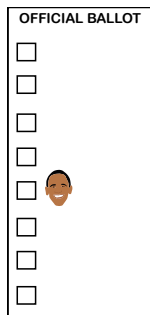


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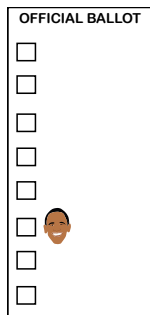


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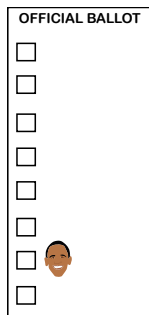


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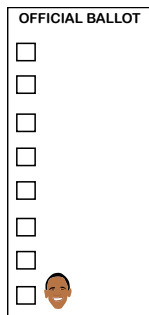


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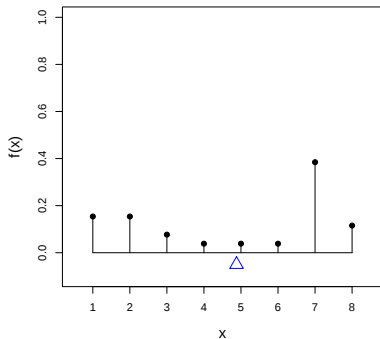
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		4.88

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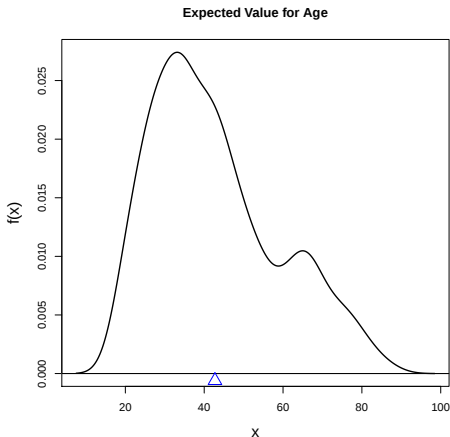
Interpreting Discrete Expected Value

The expected value for a discrete random variable is the balance point of the mass function.



Interpreting Continuous Expected Value

The expected value for a continuous random variable is the balance point of the density function.



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- It has some useful and convenient properties.

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$$\bar{x} = \sum_{\text{all } x_i} x_i f_X(x_i), \text{ where } f_X(x_i) = \frac{1}{N}$$

Three properties of expectation:

- Additivity
- Homogeneity
- LOTUS

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Together properties 1 and 2 are **linearity** (and this is sometimes presented as Linearity of Expectations).

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Why the name LOTUS? “because this can be done very easily and mechanically, perhaps in a state of unconsciousness.” (Blitzstein and Hwang, Sec 4.5)

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- We can use random variables!

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- $A = 0$ if not angered by a black family moving in next door.

$$\begin{aligned}E[Y] &= E[X + A] \\&= E[X] + E[A] \\E[Y] - E[X] &= E[A]\end{aligned}$$

So if we know $E[Y]$ and $E[X]$ we can get the expected proportion angered by our item without knowing the **individual status** of anyone!

Racial Prejudice Example (Kuklinski et al, 1997)

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$X = \#$ of angering items on the **baseline** list for Southerners:

x	0	1	2	3
$p_X(x)$	?	?	?	?
$\hat{p}_X(x)$	0.02	0.27	0.43	0.28
$\hat{F}_X(x)$	0.02	0.29	0.72	1.00

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Y = # of angering items on the **treatment** list for Southerners:

y	0	1	2	3	4
$p_Y(y)$	?	?	?	?	?
$\hat{p}_Y(y)$	0.02	0.20	0.40	0.28	0.10
$\hat{F}_Y(y)$	0.02	0.22	0.62	0.90	1.00

Racial Prejudice Example

$X = \#$ of angering items on the **baseline** list for Southerners:

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$$\widehat{E[A]} = 2.24 - 1.97 = 0.27$$

On List Experiments in Research

When to Worry about Sensitivity Bias: A Social Reference Theory and Evidence from 30 Years of List Experiments

GRAEME BLAIR *University of California, Los Angeles*

ALEXANDER COPPOCK *Yale University*

MARGARET MOOR *Yale University*

Eliciting honest answers to sensitive questions is frustrated if subjects withhold the truth for fear that others will judge or punish them. The resulting bias is commonly referred to as social desirability bias, a subset of what we label sensitivity bias. We make three contributions. First, we propose a social reference theory of sensitivity bias to structure expectations about survey responses on sensitive topics. Second, we explore the bias-variance trade-off inherent in the choice between direct and indirect measurement technologies. Third, to estimate the extent of sensitivity bias, we meta-analyze the set of published and unpublished list experiments (a.k.a., the item count technique) conducted to date and compare the results with direct questions. We find that sensitivity biases are typically smaller than 10 percentage points and in some domains are approximately zero.

- 1 Definition of Random Variables
- 2 Discrete Distributions
- 3 Continuous Distribution
- 4 Expectation as a Measure of Central Tendency
- 5 Variance as a Measure of Dispersion
 - Measures of Dispersion
 - The Mean Squared Error Rationale for Expected Values
- 6 Fun With Averages
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The variance is a special case of this, and the variance of a random variable X (a measure of its dispersion) is given by

$$\begin{aligned} V[X] &= E[(X - E[X])^2] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

It is the expectation of the squared distances from the mean.

For a **discrete** random variable X

$$V[X] = \sum_{\text{all } x} (x - E[X])^2 p_X(x)$$

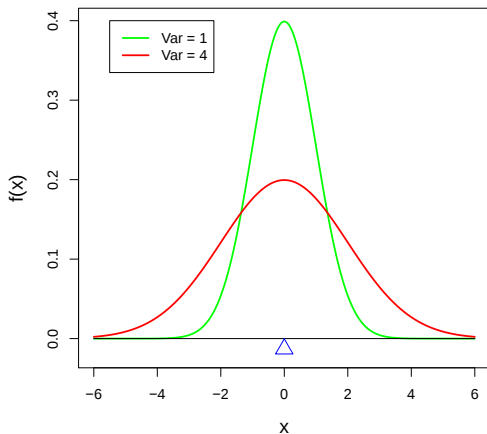
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For a **continuous** random variable X

$$V[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx$$

Variance Measures the Spread of a Distribution



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- It is a reasonable measure for the “spread” of a distribution.
- The Normal distribution—more later this week—is completely determined by its expected value (location) and variance (spread).
- The square root of the variance is the standard deviation.
- The variance and standard deviation have some useful properties.

Property 1 of Variance: Behavior with Constants

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$$V[b] = 0$$

$$V[aX] = a^2 V[X]$$

$$V[aX + b] = a^2 V[X] + 0$$

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Suppose we have k independent random variables $X_1, \dots, X_k$. If $V[X_i]$ exists for all $i = 1, \dots, k$, then

$$V \left[\sum_{i=1}^k X_i \right] = V[X_1] + \dots + V[X_k]$$

NB: Technically independence is sufficient but not necessary.

What was the variance of Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
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$$4/26 \times (1 - 4.88)^2$$

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A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

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Does variance matter for fairness?

One Step Deeper: Moments

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Suppose X is a random variable with pdf f_X . Define,

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

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- We are assuming that the integral converges.
- Another way to characterize distributions is with their moment-generating function.

Expected Value as Mean Squared Error Minimizer

Now we can return to the question of **why expectation?** and offer one technical answer.

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Let's prove the first result (see Blitzstein and Hwang 2014 Theorem 6.1.4 on pg 245 for this proof and the proof on mean absolute error).

Proof of Mean as Mean Squared Error Minimizer

Let X be a random variable and $E[X] = \mu$. We want to show that the value of c that minimizes the mean squared error $E[(X - c)^2]$ is the mean, μ (Blitzstein and Hwang Theorem 6.1.4).

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Now to prove the identity:

$$\begin{aligned} V[X] &= V[X - c] && \text{(Prop 1 of Variance)} \\ &= E[(X - c)^2] - (E[X - c])^2 && \text{(Defn of Variance)} \end{aligned}$$

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$$= E[(X - c)^2] - (\mu - c)^2 \quad (\text{Linearity of Exp})$$

$$V[X] + (\mu - c)^2 = E[(X - c)^2]$$

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Next time: Joint and Conditional Distributions (and their properties!)

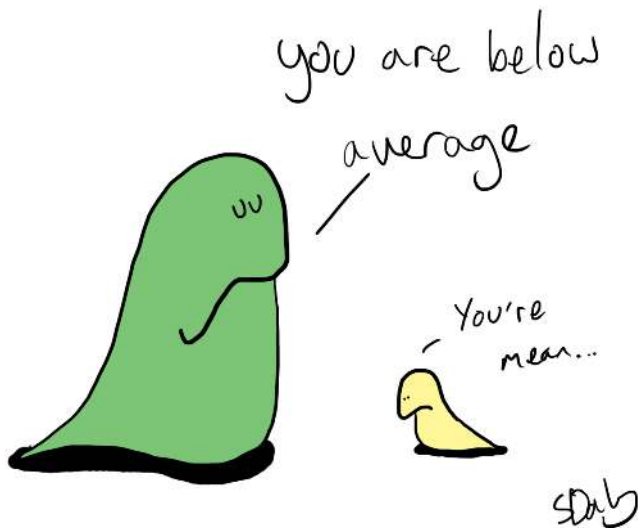
Fun with

Fun with Averages

Fun with Averages

$F(\mu n!)$
WITH

Central Tendency



The Story of Averages



Measurements

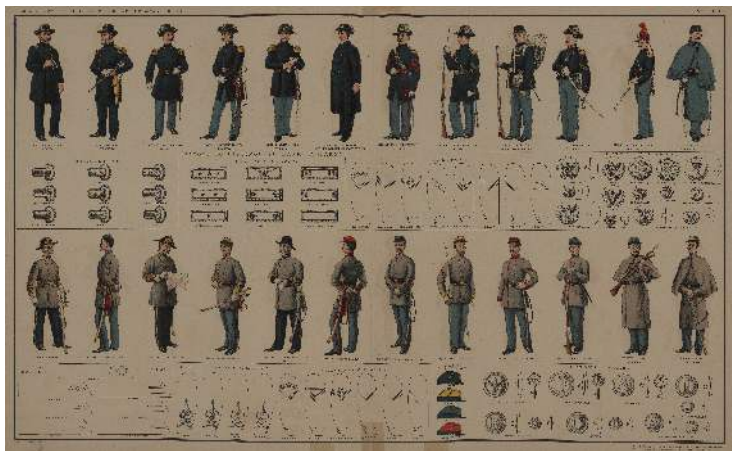
MESURES de la POitrine.	NOMBRE d'hommes.	NOMBRE PROPORTIONNEL.	PROBABILITÉ d'après L'OBSERVATION.	RANG dans LA TABLE.	RANG d'après le CALCUL.	PROBABILITÉ d'après LA TABLE.	NOMBRE D'OBSERVATIONS calculé.
Pouces.							
33	3	5	0,5000			0,5000	7
34	18	31	0,4995	52	50	0,4995	29
35	81	141	0,4964	42,5	42,5	0,4964	110
36	185	322	0,4825	33,5	34,5	0,4854	525
37	420	732	0,4501	26,0	26,5	0,4531	732
38	740	1305	0,3769	18,0	18,5	0,3799	1335
39	1075	1867	0,2464	10,5	10,5	0,2466	1858
			0,0597	2,5	2,5	0,0628	
40	1079	1882	0,1285	5,5	5,5	0,1359	1987
41	954	1628	0,2913	15	15,5	0,3054	1675
42	658	1148	0,4061	21	21,5	0,4130	1096
43	370	645	0,4706	30	29,5	0,4690	560
44	92	160	0,4806	35	37,5	0,4911	221
45	50	87	0,4955	41	45,5	0,4980	69
46	21	38	0,4991	49,5	53,5	0,4996	16
47	4	7	0,4998	56	61,8	0,4999	3
48	1	2	0,5000			0,5000	1
	5758	1,0000					1,0000

Social Physics

The determination of the average man is not merely a matter of speculative curiosity; it may be of the most important service to the science of man and the social system. It ought necessarily to precede every other inquiry into social physics, since it is, as it were, the basis. The average man, indeed, is in a nation what the centre of gravity is in a body; it is by having that central point in view that we arrive at the apprehension of all the phenomena of equilibrium and motion

- Quetelet

The Military Takes to the Idea



The Problem with Averages



THE EDGE **HEALTH**

Stacking Up

Ever wish you just measured up against the average American? Here's a breakdown of how you stack up in 15 categories of fitness, health, and lifestyle.

69%

Percent of men who consider themselves "physically fit"

13%

Percentage who actually are fit

\$874

Average spent on athletic wear last year

45%

Percentage of men who think that fitness is a better way to lose weight than diet alone

52

Number of men who think that fitness is a better way to lose weight than diet alone

2.87

Number of men who think that fitness is a better way to lose weight than diet alone

1.1 lbs.

Average of men who think that fitness is a better way to lose weight than diet alone

1

Average number of men who think that fitness is a better way to lose weight than diet alone

77 yrs.

Average age of men who think that fitness is a better way to lose weight than diet alone

23

Age of the average man who thinks that fitness is a better way to lose weight than diet alone

12:17

Average time to complete a mile

1

Percent of men who think that fitness is a better way to lose weight than diet alone

Size Yourself Up

	5'0"	5'5"	6'0"	6'5"	7'0"
Weight (lb.)	125	150	175	200	225
Height (in.)	60	65	70	75	80
Weight (lb.)	125	150	175	200	225
Height (in.)	60	65	70	75	80
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Height (in.)	60	65	70	75	80

5'9.2"

The average American male's height

40"

Size of the average guy's chest (bust) and the average guy's waist (bust)

17.6%

Percentage of men who think that fitness is a better way to lose weight than diet alone

13"

Size of the average guy's chest (bust) and the average guy's waist (bust)

34"

Average guy's waist (bust)

5.08

The average guy's weight (bust) and the average guy's waist (bust)

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Sex Stats

The average guy's weight (bust) and the average guy's waist (bust)

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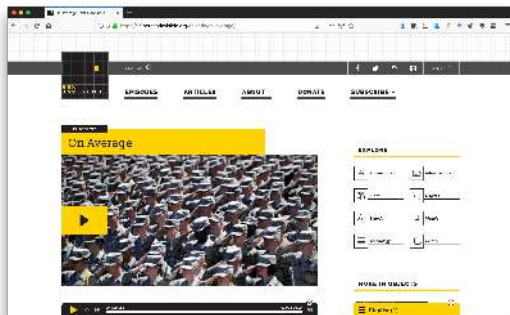
The average guy's weight (bust) and the average guy's waist (bust)

Photo: iStockphoto.com

The Face of the Average Man



On averages



<https://99percentinvisible.org/episode/on-average/>

Where We've Been and Where We're Going...

- Last Week

- ▶ welcome and outline of course
- ▶ described uncertain outcomes with **probability**.

- This Week

- ▶ define **random variables**
- ▶ summarize random variables using **expectation** and **variance**
- ▶ properties of **joint** and **conditional** distributions
- ▶ famous distributions

- Next Week

- ▶ **estimating** these features from data
- ▶ estimating **uncertainty**

- Long Run

- ▶ probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

- 1 Definition of Random Variables
- 2 Discrete Distributions
- 3 Continuous Distribution
- 4 Expectation as a Measure of Central Tendency
- 5 Variance as a Measure of Dispersion
- 6 Fun With Averages
- 7 Joint and Conditional Distributions**
 - First Visual Example
 - Discrete Random Variable
 - Continuous Random Variable

Joint Distributions

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- The **joint distribution** of two (or more) variables describes the pairs of observations that we are more or less likely to see.
- The **conditional distribution** describes one random variable given knowledge of another.
- We will start with a visual preview, then step back to go through the math more concretely.

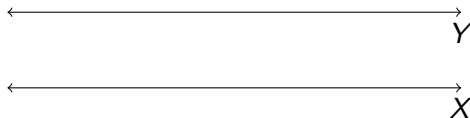
Understanding Joint Distributions Mathematically

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- Consider two random variables now, X and Y , each on the real line, $\mathbb{R}$.

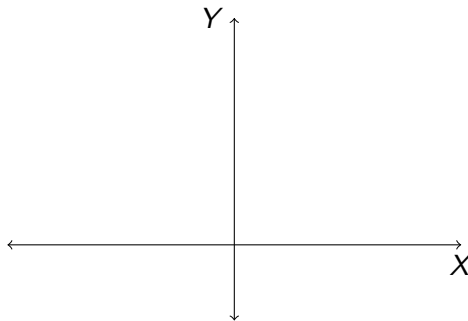
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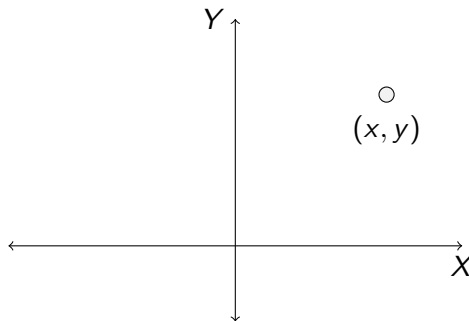
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Understanding Joint Distributions Mathematically

- Consider two random variables now, X and Y , each on the real line, $\mathbb{R}$.
- The pair form a two-dimensional space, or $\mathbb{R} \times \mathbb{R}$
- One realization of the random variable is a point in that space



Example: Racial Prejudice

- Recall the list experiment about racial prejudice. Suppose we define $X = 0$ (Non-southern), 1 (Southern) and $Y =$ “number of angering items” for a randomly selected respondent receiving the treatment list.

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- Furthermore, we define the probability that this respondent will have the values $X = x$ and $Y = y$ to be $p_{Y,X}(y, x) = \pi_{yx}$

$$X = \begin{array}{c} \text{N} \\ \text{W} \downarrow \text{E} \\ \text{S} \end{array}, \begin{array}{c} \text{N} \\ \text{W} \rightarrow \text{E} \\ \text{S} \end{array}$$

$$Y = \begin{array}{cc} \text{😊😊} \\ \text{😊😊} \end{array}, \begin{array}{cc} \text{😡😊} \\ \text{😊😊} \end{array}, \begin{array}{cc} \text{😡😡} \\ \text{😊😊} \end{array}, \begin{array}{cc} \text{😡😡} \\ \text{😡😊} \end{array}, \begin{array}{cc} \text{😡😡} \\ \text{😡😡} \end{array}$$

$$f(\begin{array}{cc} \bullet\bullet \\ \bullet\bullet \end{array}, \begin{array}{c} \text{N} \\ \text{W} \downarrow \text{E} \\ \text{S} \end{array}) = \pi_{\begin{array}{cc} \bullet\bullet \\ \bullet\bullet \end{array} \begin{array}{c} \text{N} \\ \text{W} \downarrow \text{E} \\ \text{S} \end{array}}$$

Example Joint Distribution: Binary X , Discrete Y

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Although we cannot observe the responses for the entire population, we can imagine what they might look like as a joint distribution.

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$f_{X,Y}(x,y)$ y	x		$f_Y(y)$
	0	1	
0	π_{00}	π_{01}	$\pi_{00} + \pi_{01}$
1	π_{10}	π_{11}	$\pi_{00} + \pi_{01}$
2	π_{20}	π_{21}	$\pi_{00} + \pi_{01}$
3	π_{30}	π_{31}	$\pi_{00} + \pi_{01}$
4	π_{40}	π_{41}	$\pi_{00} + \pi_{01}$
$f_X(x)$	$\sum_{y=0}^4 \pi_{y0}$	$\sum_{y=0}^4 \pi_{y1}$	

Example Joint Distribution: Binary X, Discrete Y

Although we cannot observe the responses for the entire population, we can imagine what they might look like as a joint distribution.

$f(\text{☼}, \text{⌚})$	$\mathbf{x} = \text{⌚}$		$f(\text{☼})$
$\mathbf{y}$			
	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$  + $\pi_{\text{☼}, \text{⌚}}$ 
	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$  + $\pi_{\text{☼}, \text{⌚}}$ 
	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$  + $\pi_{\text{☼}, \text{⌚}}$ 
	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$  + $\pi_{\text{☼}, \text{⌚}}$ 
	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$ 	$\pi_{\text{☼}, \text{⌚}}$  + $\pi_{\text{☼}, \text{⌚}}$ 
$f(\text{⌚})$	$\sum_{\text{☼}} \pi_{\text{☼}, \text{⌚}}$ 	$\sum_{\text{☼}} \pi_{\text{☼}, \text{⌚}}$ 	

Discrete Conditional Distribution

Discrete Conditional Distribution

$$f(\text{four dots} | \text{clock}) = \frac{f(\text{four dots}, \text{clock})}{f(\text{clock})}$$

$$y = \text{four dots}$$

$$x = \text{clock}$$



$$\frac{\pi_{\text{four dots}, \text{clock}}}{\sum_{\text{four dots}} \pi_{\text{four dots}, \text{clock}}}$$

$$\frac{\pi_{\text{four dots}, \text{clock}}}{\sum_{\text{four dots}} \pi_{\text{four dots}, \text{clock}}}$$



$$\frac{\pi_{\text{four dots}, \text{clock}}}{\sum_{\text{four dots}} \pi_{\text{four dots}, \text{clock}}}$$

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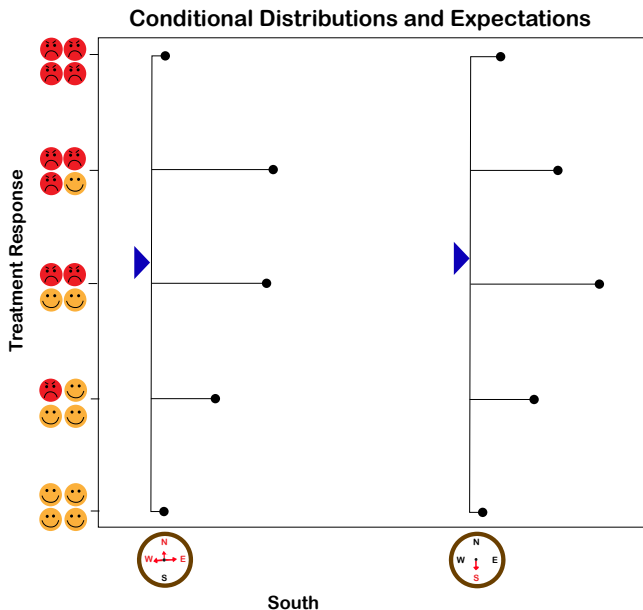
$$\frac{\pi_{\text{four dots}, \text{clock}}}{\sum_{\text{four dots}} \pi_{\text{four dots}, \text{clock}}}$$



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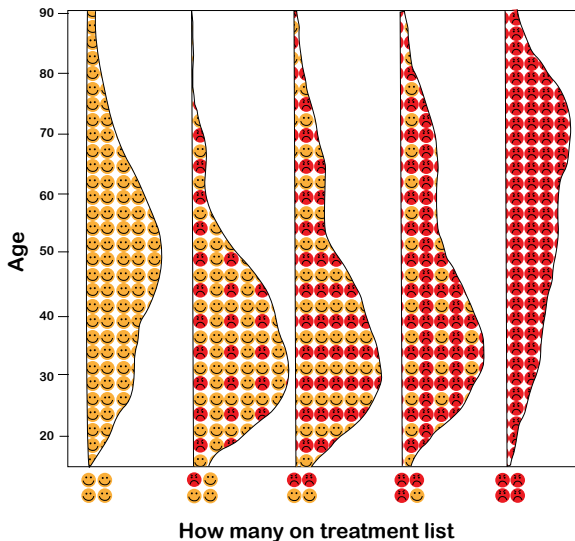
$$\frac{\pi_{\text{four dots}, \text{clock}}}{\sum_{\text{four dots}} \pi_{\text{four dots}, \text{clock}}}$$

Discrete Conditional Distribution

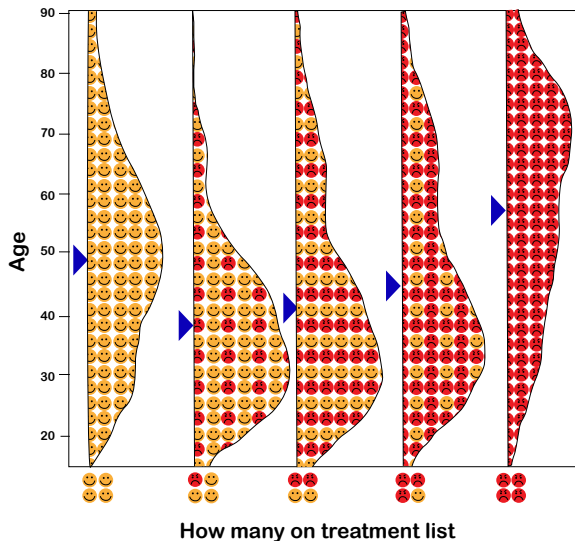


Example: Continuous Conditional Distribution

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Conditional Expectation Function (coming soon!)



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Joint Probability Mass Function

Definition

For two discrete random variables X and Y the **joint** Probability Mass Function (PMF) $P_{X,Y}(x,y)$ gives the probability that $X = x$ and $Y = y$ for all x and y :

$$p_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$$

Restrictions:

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Restrictions:

- $p_{X,Y}(x,y) \geq 0$ and $\sum_x \sum_y p_{X,Y}(x,y) = 1$.

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Should the U.S. allow more immigrants to come and live here?

		X: Education			
		less HS	HS	College	BA
Y: Support	oppose	0.07	0.22	0.18	0.15
	neutral	0.02	0.06	0.05	0.05
	favor	0.01	0.03	0.04	0.11

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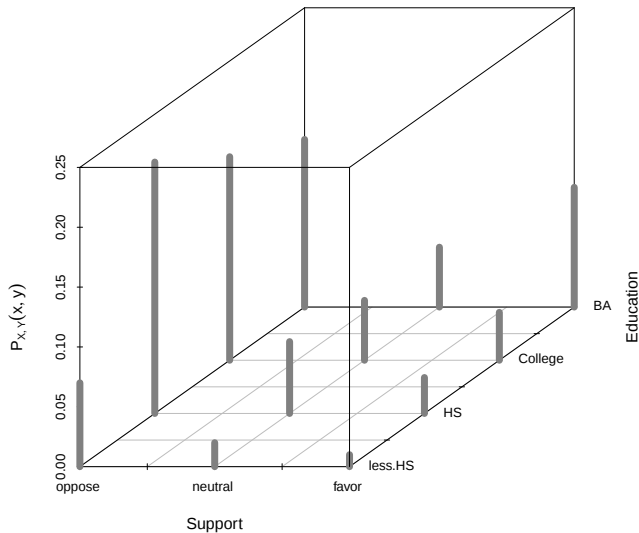
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With discrete random variables this is very similar to thinking about a cross-tab, with frequencies/ probabilities in the cells instead of raw numbers.

Joint Probability Mass Function



From Joint to Marginal PMF

Given the **joint** PMF $p_{X,Y}(x,y)$ can we recover the **marginal** PMF $p_Y(y)$ (distribution over a single variable)?

		X: Education				
		less HS	HS	College	BA	
Y: Support	oppose	0.07	0.21	0.17	0.14	
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From Joint to Marginal PMF

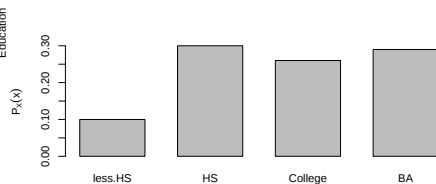
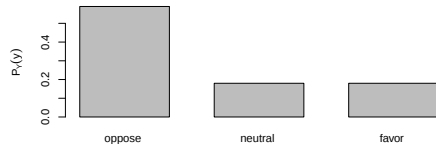
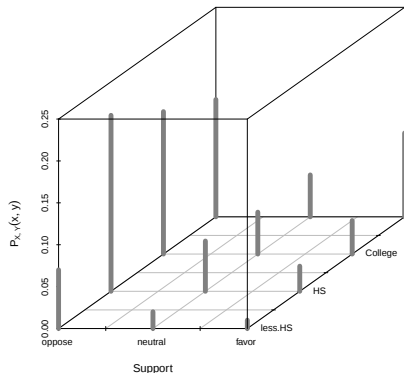
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		X: Education				
		less HS	HS	College	BA	$p_Y(y)$
Y: Support	oppose	0.07	0.21	0.17	0.14	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.10	0.19

To obtain $p_Y(y)$ we **marginalize** the joint probability function $p_{X,Y}(x,y)$ over X :

$$p_Y(y) = \sum_x p_{X,Y}(x,y) = \sum_x P(X = x, Y = y)$$

Joint and Marginal Probability Mass Functions



Why Does Marginalization Work?

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Define the conditional mass function $P(X = x|Y = y)$ as,

$$\begin{aligned} P(X = x|Y = y) &\equiv p_{X|Y}(x|y) \\ &= p_{X,Y}(x, y)/p_Y(y) \end{aligned}$$

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Marginalizing **over** y to get $p_X(x)$ is then,

$$p_X(x_j) = \sum_{i=1}^N p_{X|Y}(x_j|y_i)p_Y(y_i)$$

A Table

	$Y = 0$	$Y = 1$	
$X = 0$	$p(0,0)$	$p(0, 1)$	$p_X(0)$
$X = 1$	$p(1,0)$	$p(1,1)$	$p_X(1)$
	$p_Y(0)$	$p_Y(1)$	

A Table

	$Y = 0$	$Y = 1$	
$X = 0$	0.01	0.05	?
$X = 1$	0.25	0.69	?
	0.26	0.74	

$$\begin{aligned} p_X(0) &= P(X = 0|Y = 0)P(Y = 0) + P(X = 0|Y = 1)P(Y = 1) \\ &= \frac{0.01}{0.26} \times 0.26 + \frac{0.05}{0.74} \times 0.74 \\ &= 0.06 \end{aligned}$$

A Table

	$Y = 0$	$Y = 1$	
$X = 0$	0.01	0.05	?
$X = 1$	0.25	0.69	?
	0.26	0.74	

$$\begin{aligned}p_X(0) &= P(X = 0|Y = 0)P(Y = 0) + P(X = 0|Y = 1)P(Y = 1) \\&= \frac{0.01}{0.26} \times 0.26 + \frac{0.05}{0.74} \times 0.74 \\&= 0.06\end{aligned}$$

$$\begin{aligned}p_X(1) &= P(X = 1|Y = 0)P(Y = 0) + P(X = 1|Y = 1)P(Y = 1) \\&= \frac{0.25}{0.26} \times 0.26 + \frac{0.69}{0.74} \times 0.74 \\&= 0.94\end{aligned}$$

Conditional PMF

Conditional PMF

Definition

The **conditional** PMF of Y given X , $p_{Y|X}(y|x)$, is the PMF of Y when X is known to be at a particular value $X = x$:

$$p_{Y|X}(y|x) = \frac{P(X = x \text{ and } Y = y)}{P(X = x)} = \frac{p_{X,Y}(x, y)}{p_X(x)}$$

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Key relationships:

- $p_{X,Y}(x, y) = p_{Y|X}(y|x)p_X(x)$ (multiplicative rule)
- $p_{Y|X}(y|x) = p_{X|Y}(x|y)p_Y(y)/p_X(x)$ (Bayes' rule)

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Conditional PMFs are just like ordinary PMFs, but refer to a universe where the “conditioning event” ($X = x$) is known to have occurred.

Conditional distributions are key in statistical modeling because they inform us how the distribution of Y varies across different levels of X .

From Joint to Conditional: $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$

Table: Joint PMF $p_{X,Y}(x,y)$ and Marginal PMFs $p_X(x), p_Y(y)$

		Education				
	$p_{X,Y}(x,y)$	less HS	HS	College	BA	$p_Y(y)$
Support	oppose	0.07	0.22	0.18	0.15	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.11	0.19
	$p_X(x)$	0.11	0.32	0.27	0.31	1.00

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Table: Conditional PMF $p_{Y|X}(y|x)$

		Education				
	$p_{Y X}(y x)$	less HS	HS	College	BA	
Support	oppose	0.70	0.70	0.65	0.48	0.62
	neutral	0.20	0.20	0.19	0.17	0.19
	favor	0.10	0.10	0.15	0.34	0.19
		1.00	1.00	1.00	1.00	1.00

Joint and Conditional Probability Mass Functions

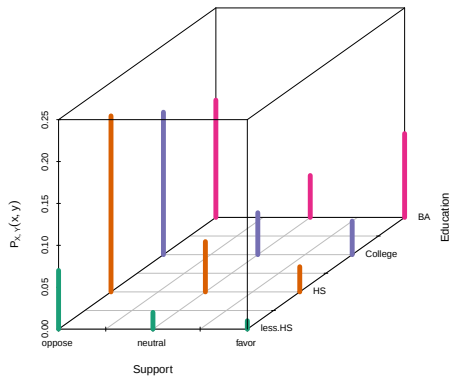


Figure: Joint

Joint and Conditional Probability Mass Functions

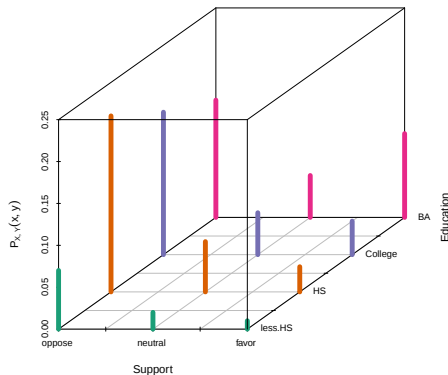


Figure: Joint

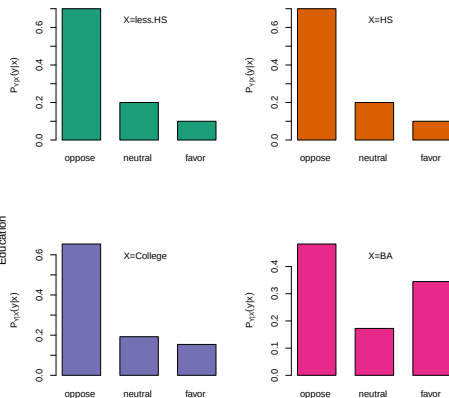


Figure: Conditional

- 1 Definition of Random Variables
- 2 Discrete Distributions
- 3 Continuous Distribution
- 4 Expectation as a Measure of Central Tendency
- 5 Variance as a Measure of Dispersion
- 6 Fun With Averages
- 7 Joint and Conditional Distributions**
 - First Visual Example
 - Discrete Random Variable
 - Continuous Random Variable

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Definition

For two **continuous** random variables X and Y the **joint** PDF $f_{X,Y}(x,y)$ gives the density height where $X = x$ and $Y = y$ for all x and y .

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$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$$

where

- $f_{Y|X}(y|x)$: **Conditional** PDF of Y given $X = x$
- $f_X(x)$: **Marginal** PDF of X

Restrictions:

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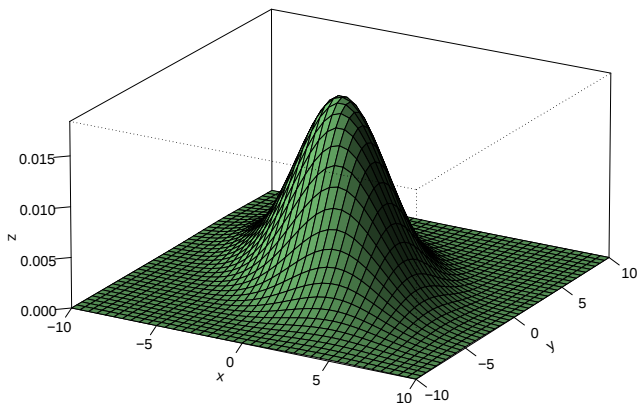
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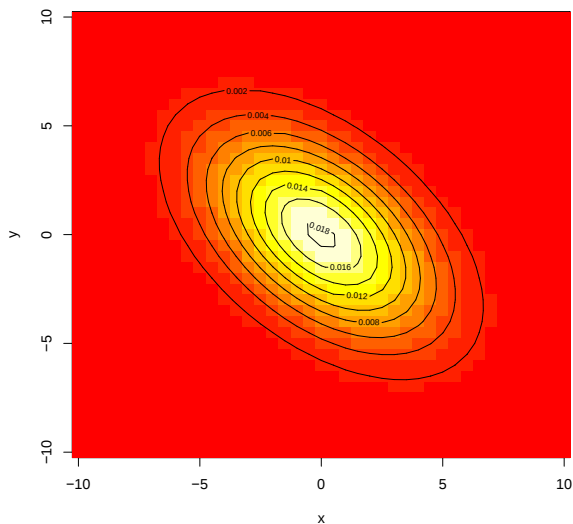
- $\int_x \int_y f_{X,Y}(x,y) dy dx = 1$

3D Plot of a Joint Probability Density Function

Bivariate Normal Distribution: $z = f_{X,Y}(x, y)$



Contour Plot of a Joint Probability Density Function



From Joint to Marginal PDF

How can we obtain $f_Y(y)$ from $f_{X,Y}(x,y)$?

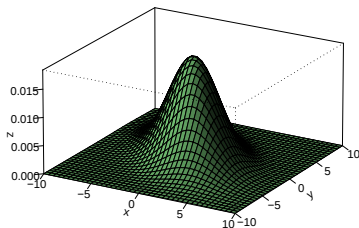
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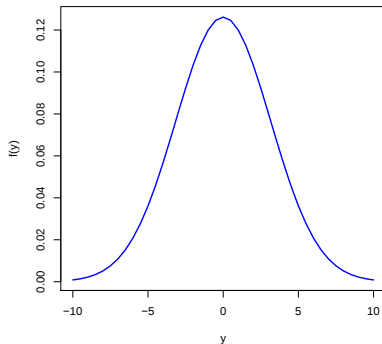
We marginalize the joint probability function $f_{X,Y}(x,y)$ over X :

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

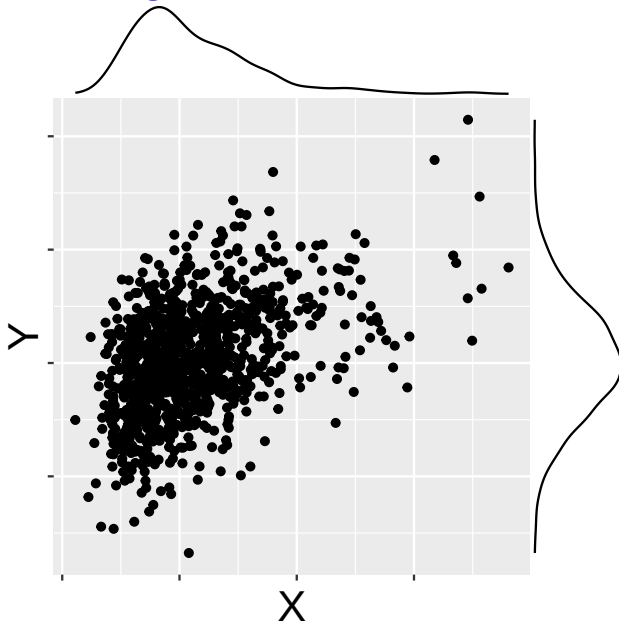
Joint PDF $f(x,y)$



Marginal PDF $f(y)$

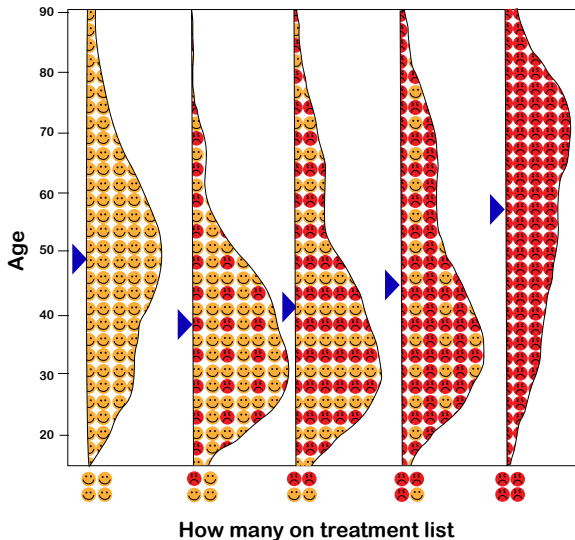


From Joint to Marginal PDF



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 - Conditional Expectation
 - Conditional Variance

Remember this?



Conditioning on X

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- A common goal in statistical modeling is to characterize the conditional distribution of the outcome variable $f_{Y|X}(y|x)$ across different levels of $X = x$.
- Typically, we summarize the conditional distributions with a few parameters such as the **conditional mean** of $E[Y|X = x]$ and the **conditional variance** $V[Y|X = x]$
- Moreover, we are often interested in estimating $E[Y|X]$, i.e. the **conditional expectation function** that describes how the conditional mean of Y varies across all possible values of X .

Conditional Expectation

Definition (Conditional Expectation (Discrete))

Let Y and X be discrete random variables. The conditional expectation of Y given $X = x$ is defined as:

$$E[Y|X = x] = \sum_y y P(Y = y|X = x) = \sum_y y p_{Y|X}(y|x)$$

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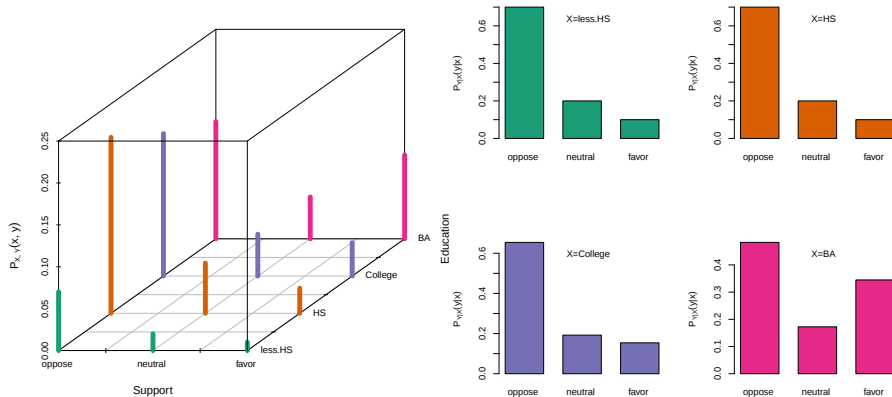
$$E[Y|X = x] = \sum_y y P(Y = y|X = x) = \sum_y y p_{Y|X}(y|x)$$

Definition (Conditional Expectation (Continuous))

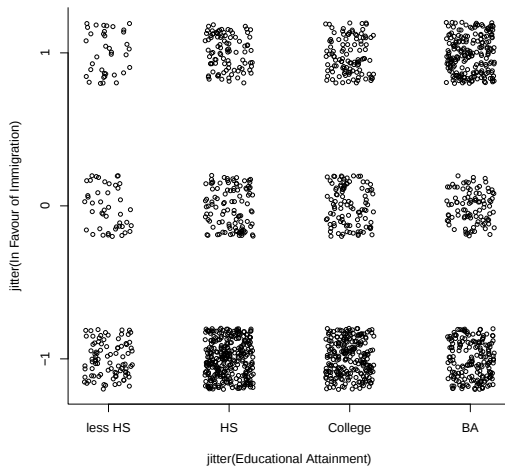
Let Y and X be continuous random variables. The conditional expectation of Y given $X = x$ is given by:

$$E[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy$$

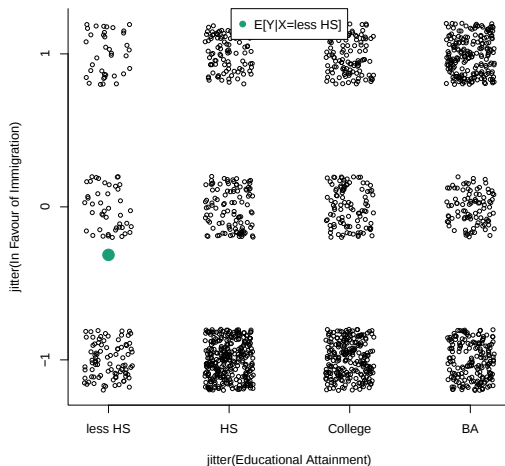
Joint and Conditional Probability Mass Functions



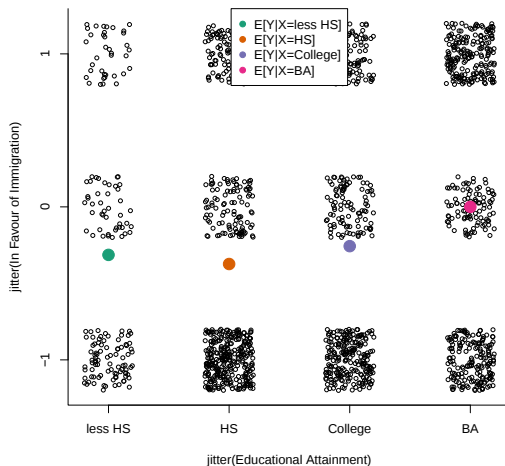
Conditional PMF $P_{Y|X}(y|x)$



Conditional Expectation $E[Y|X = 1]$



Conditional Expectation Function $E[Y|X]$



Law of Iterated Expectations

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Theorem (Law of Iterated Expectations/Adam's Law)

For two random variables X and Y ,

$$E[Y] = E[E[Y|X]] = \begin{cases} \sum E[Y|X = x] \cdot p_X(x) & (\text{discrete } X) \\ \int_{-\infty}^{\infty} E[Y|X = x] \cdot f_X(x) dx & (\text{continuous } X) \end{cases}$$

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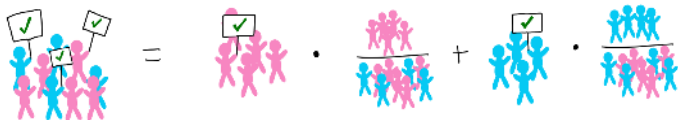
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Note that the outer expectation is taken with respect to the distribution of X .

Example: Y (support) and $X \in \{1, 0\}$ (AfAm). Then, the LIE tells us:

$$\underbrace{E[Y]}_{\text{Average Support}} = \underbrace{E[E[Y|X]]}_{\text{Average Support|AfAm}^c} = \underbrace{E[Y|X=1]}_{\text{Average Support|AfAm}^c} \cdot \underbrace{p_X(1)}_{P(\text{AfAm}^c)} + \underbrace{E[Y|X=0]}_{\text{Average Support|AfAm}} \cdot \underbrace{p_X(0)}_{P(\text{AfAm})}$$



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 - ▶ This says that the conditional expectation is the function of X that **minimizes the squared prediction error** for Y across any possible function of X .
 - ▶ This is analogous to the result we saw about the mean.

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Remember, the conditional distribution of $Y|X$ is basically like any other probability distribution, so we are going to want to summarize the **center and spread**.

Conditional Variance

Definition

The **conditional variance** of Y given $X = x$ is defined as:

$$V[Y|X = x] = \begin{cases} \sum_{\text{all } y} (y - E[Y|X = x])^2 P_{Y|X}(y|x) & \text{(discrete } Y) \\ \int_{-\infty}^{\infty} (y - E[Y|X = x])^2 f_{Y|X}(y|x) dy & \text{(continuous } Y) \end{cases}$$

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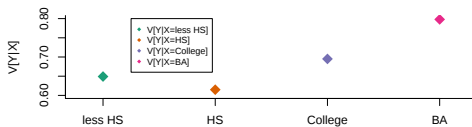
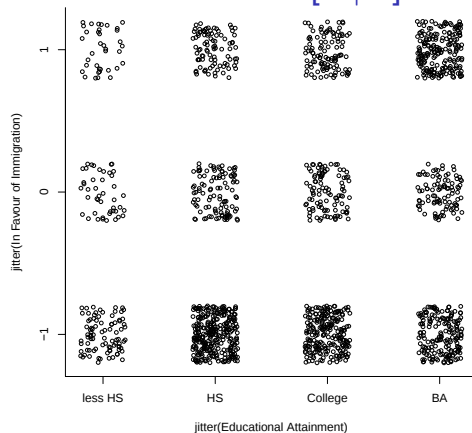
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Example: Y (support) and $X \in \{1, 0\}$ (group). The LTV says that the total variance in support can be decomposed into two parts:

- 1 On average, how much support varies within groups (**within variance**)
- 2 How much average support varies between groups (**between variance**)

Conditional Variance Function $V[Y|X]$



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Let's look at this in pictures.

(If you want to know more: Blitzstein and Hwang pg 392-393)

Important Subtleties in Pictures



Sample space

Important Subtleties in Pictures



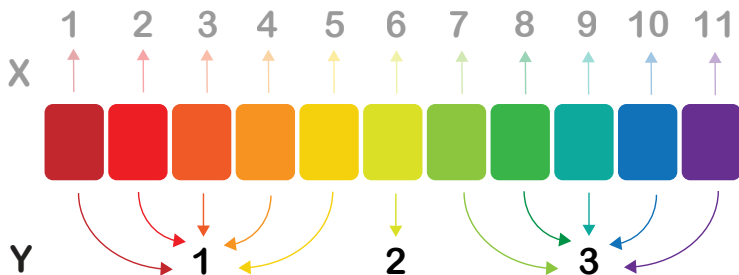
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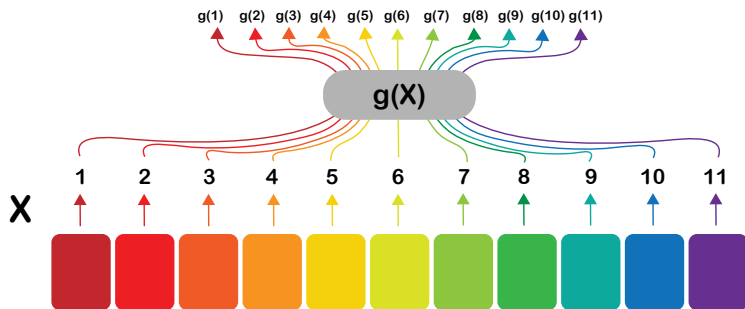
Random variable

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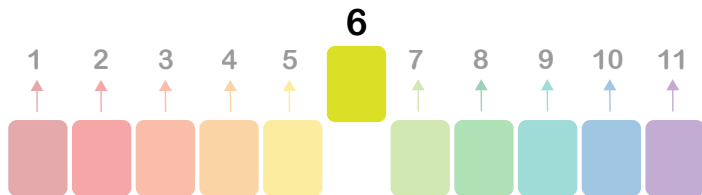
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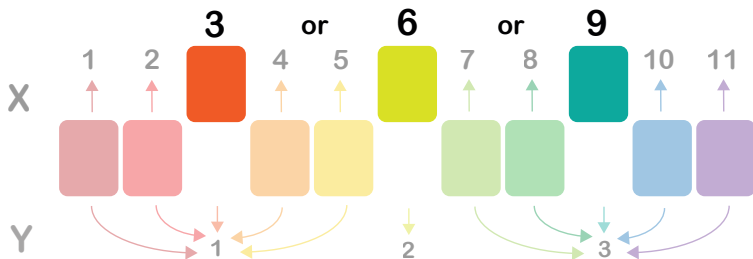
Function of a random variable is a random variable

Important Subtleties in Pictures



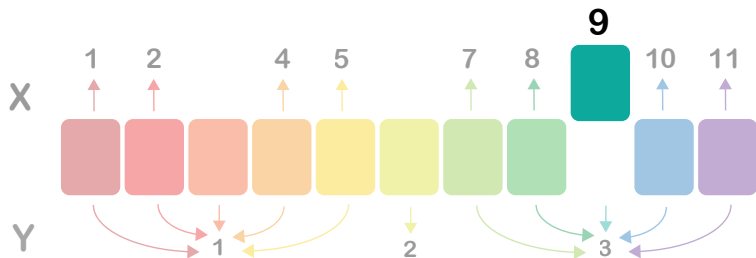
$$E[X]$$

Important Subtleties in Pictures



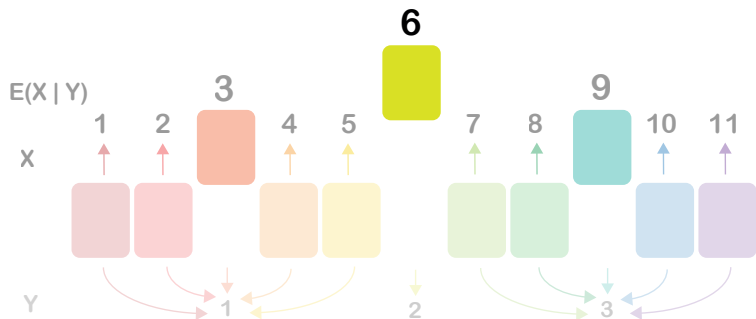
$$E[X|Y]$$

Important Subtleties in Pictures



$$E[X|Y = 3]$$

Important Subtleties in Pictures



$$E[E(X|Y)] = E[X]$$

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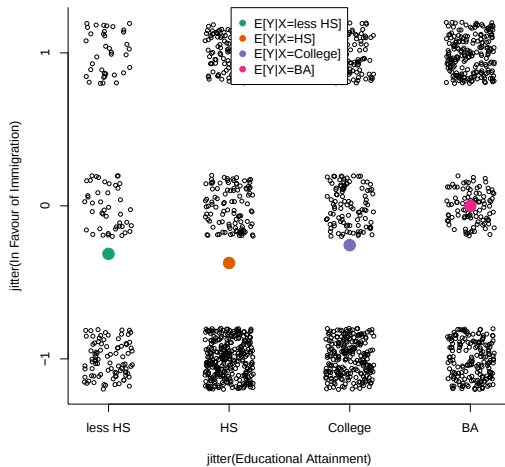
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$$E[Y|X = x] = E[Y]$$

Is $Y \perp\!\!\!\perp X$?



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We can prove the continuous case by following the same steps, with $\sum$ replaced by $\int$.

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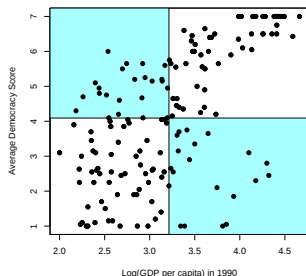
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- Points in upper right and lower left quadrants (relative to the means) add to the covariance.
- Points in the upper left and lower right quadrants subtract from the covariance.



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Therefore, $X \perp\!\!\!\perp Y \implies \text{Cov}[X, Y] = 0$, but not vice versa.

Important Identities for Variances and Covariances

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Proof: Plug in to the definition of variance and expand (try it yourself!)

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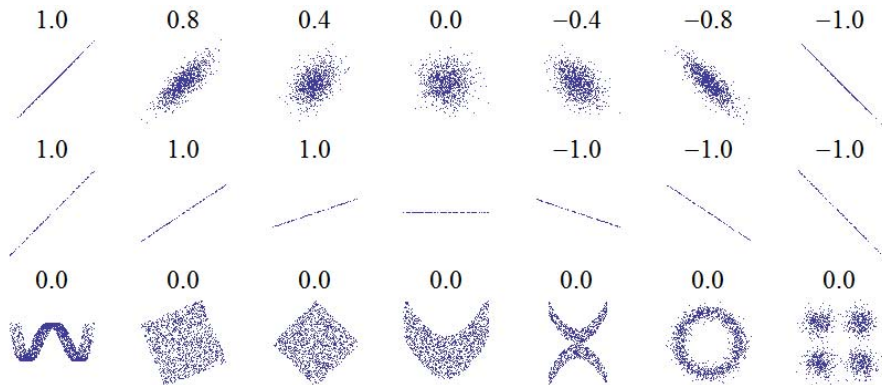
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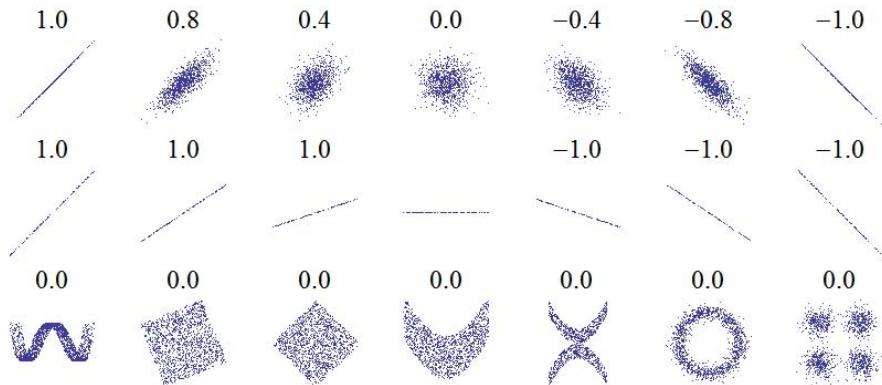
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- Always satisfies: $-1 \leq \text{Cor}[X, Y] \leq 1$.

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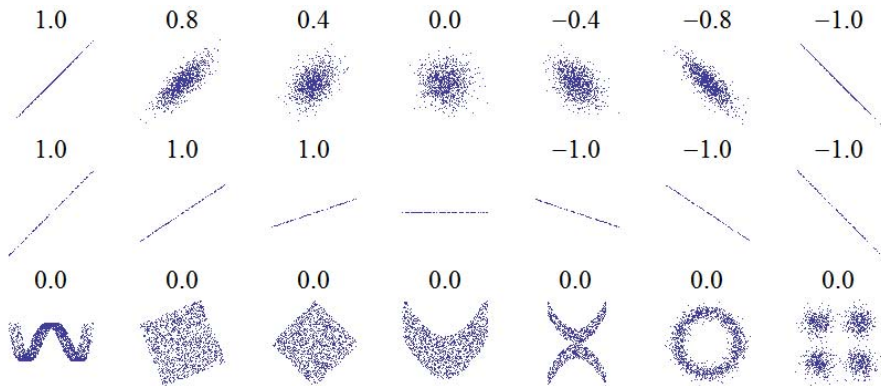


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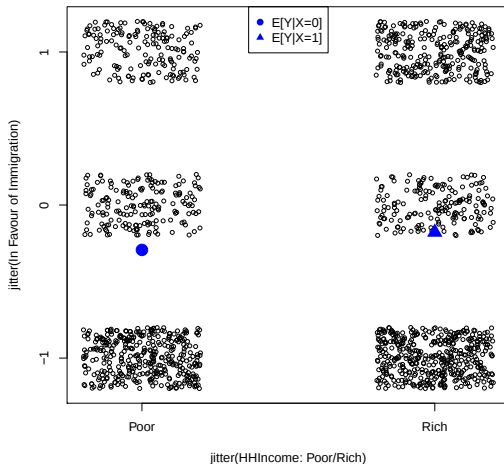
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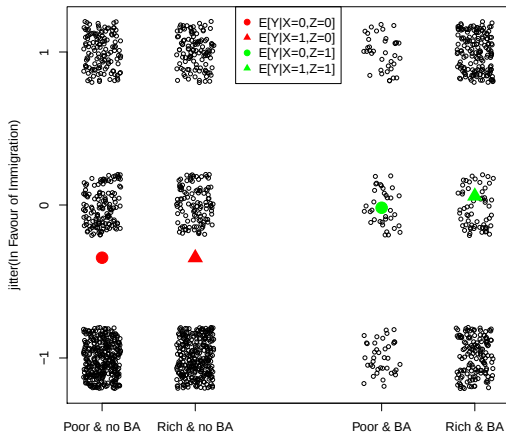
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Example: X = wealth, Y = support for immigration, Z = education.



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Where We've Been and Where We're Going...

- Last Week

- ▶ welcome and outline of course
- ▶ described uncertain outcomes with **probability**.

- This Week

- ▶ define **random variables**
- ▶ summarize random variables using **expectation** and **variance**
- ▶ properties of **joint** and **conditional** distributions
- ▶ famous distributions

- Next Week

- ▶ **estimating** these features from data
- ▶ estimating **uncertainty**

- Long Run

- ▶ probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

- 1 Definition of Random Variables
- 2 Discrete Distributions
- 3 Continuous Distribution
- 4 Expectation as a Measure of Central Tendency
- 5 Variance as a Measure of Dispersion
- 6 Fun With Averages
- 7 Joint and Conditional Distributions
- 8 Characterizing Conditional Distributions
- 9 Independence and Covariance

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- Examples: Bernoulli, Binomial, Gamma, Normal, Poisson, t -distribution



Bernoulli Random Variable

Definition

Suppose X is a random variable, with $X \in \{0, 1\}$ and $P(X = 1) = \pi$. Then we will say that X is **Bernoulli** random variable,

$$P(X = x) = \pi^x(1 - \pi)^{1-x}$$

for $x \in \{0, 1\}$ and $P(X = x) = 0$ otherwise.

We will (equivalently) say that

$$X \sim \text{Bernoulli}(\pi)$$

$\sim$ means equality in distribution (not values!). Often $X \sim \text{Bernoulli}(\pi)$ would be read 'X is distributed Bernoulli with parameter π '

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$$E[X] = \pi$$

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Bernoulli Random Variable Mean and Variance

Suppose $X \sim \text{Bernoulli}(\pi)$

$$\begin{aligned} E[X] &= 1 \times P(X = 1) + 0 \times P(X = 0) \\ &= \pi + 0(1 - \pi) = \pi \end{aligned}$$

$$\text{var}(X) = E[X^2] - E[X]^2$$

$$\begin{aligned} E[X^2] &= 1^2 P(X = 1) + 0^2 P(X = 0) \\ &= \pi \end{aligned}$$

$$\begin{aligned} \text{var}(X) &= \pi - \pi^2 \\ &= \pi(1 - \pi) \end{aligned}$$

$$E[X] = \pi$$

$$\text{var}(X) = \pi(1 - \pi)$$

Importantly, we can also just look this up!

Normal/Gaussian Random Variables

Definition

Suppose X is a random variable with $X \in \mathbb{R}$ and **density**

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Then X is a **normally** distributed random variable with parameters μ and σ^2 .

Equivalently, we'll write

$$X \sim \text{Normal}(\mu, \sigma^2)$$

Expected Value/Variance of Normal Distribution

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Proposition

Scale/Location. If $Z \sim N(0, 1)$, then $X = aZ + b$ is,

$$X \sim \text{Normal}(b, a^2)$$

Intuition

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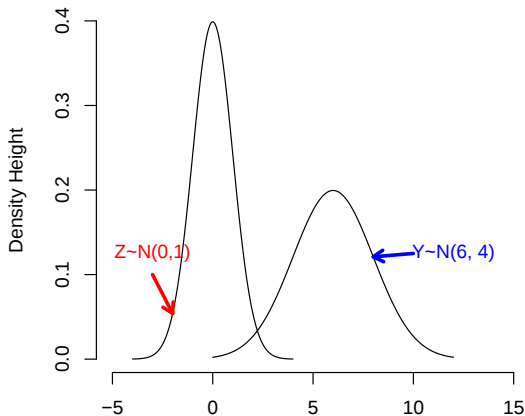
$$Y = 2Z + 6$$

Intuition

Suppose $Z \sim \text{Normal}(0, 1)$.

$$Y = 2Z + 6$$

$Y \sim \text{Normal}(6, 4)$



Expectation and Variance

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$$\text{Var}(Z) = 1$$

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Multivariate Normal

Definition

Suppose $\mathbf{X} = (X_1, X_2, \dots, X_N)$ is a vector of random variables. If $\mathbf{X}$ has pdf

$$f_{X_1, X_2}(\mathbf{x}) = (2\pi)^{-N/2} \det(\mathbf{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \mathbf{\Sigma}(\mathbf{x} - \boldsymbol{\mu})\right)$$

Then we will say $\mathbf{X}$ has a **Multivariate Normal** Distribution,

$$\mathbf{X} \sim \text{Multivariate Normal}(\boldsymbol{\mu}, \mathbf{\Sigma})$$

Multivariate Normal Distribution

Consider the (bivariate) special case where $\boldsymbol{\mu} = (0, 0)$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$f(x_1, x_2) = (2\pi)^{-2/2} 1^{-1/2} \exp \left(-\frac{1}{2} \left((\mathbf{x} - \mathbf{0})' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{x} - \mathbf{0}) \right) \right)$$

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$\rightsquigarrow$ product of univariate standard normally distributed random variables

Properties of the Multivariate Normal Distribution

Suppose $\mathbf{X} = (X_1, X_2, \dots, X_N)$

$$\begin{aligned} E[\mathbf{X}] &= \boldsymbol{\mu} \\ \text{cov}(\mathbf{X}) &= \boldsymbol{\Sigma} \end{aligned}$$

So that,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \dots & \text{cov}(X_1, X_N) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \dots & \text{cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \dots & \text{var}(X_N) \end{pmatrix}$$

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- We can characterize distributions in terms of their **expectation** (location) and **variance** (spread).
- **Joint** and **conditional** distributions capture the relationship between random variables.
- There is a common set of famous distributions such as the **Normal** distribution.

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Next week: inference!