

Final review

Soc 500: Applied Social Statistics

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Today's Agenda

- Learning from samples
- Bias-variance tradeoff with simulation and robust estimation
- Linear regression interpretation
- OLS assumptions, diagnostics, fixes
- Causal inference with observed variables
- Causal inference with unobserved confounders - Instrumental variables
- What if my code has a bug?
- Questions!

The problem

Suppose that you find yourself in front of this claw machine with an immense tank filled with PokeBalls



You want to know the average quality of the Pokemon in this machine. A sign tells you that the quality of the Pokemon in this machine is normally distributed with mean μ and variance σ^2 (but do not tell you what μ and σ^2 are).

Candidate estimators

You have a bit of pocket change to draw some PokeBalls from the machine. You want to decide how to make inferences about average quality based on the PokeBalls you draw, so you consider the following estimators for μ :

- $\hat{\mu}_1 = \bar{X} = \frac{\sum_{i=1}^n X_i}{n}$
- $\hat{\mu}_2 = c$, where c is a constant
- $\hat{\mu}_3 = \frac{\sum_{i=1}^n X_i}{n+1}$

How do you decide which estimator to use?

Estimator Properties

Concept	Criteria	Intuition
Unbiasedness	$E[\hat{\mu}] = \mu$	Right on average
Efficiency	$V[\hat{\mu}_1] < V[\hat{\mu}_2]$	Low variance
Consistency	$\hat{\mu}_n \xrightarrow{P} \mu$	Converge to estimand as $n \rightarrow \infty$
Asymptotic Normality	$\hat{\mu}_n \overset{\text{approx.}}{\sim} N(\mu, \frac{\sigma^2}{n})$	Approximately normal in large n

- Finite sample properties (unbiasedness, efficiency): what happens across repeated sampling given a fixed n ?
- Asymptotic properties (consistency, asymptotic normality): what happens as n grows large?

Let's derive the sampling distributions

To the board!

- $\hat{\mu}_1 = \bar{X} = \frac{\sum_{i=1}^n X_i}{n}$
- $\hat{\mu}_2 = c$, where c is a constant
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What do we know?

$$X_i \sim N(\mu, \sigma^2)$$

$$\text{so } E(X_i) = \mu \text{ and } \text{Var}(X_i) = \sigma^2$$

(Some) useful properties of expectations:

- $E(X + Y) = E(X) + E(Y)$
- $E(a) = a$
- $E(aX) = aE(X)$
- $E(aX + bY) = aE(X) + bE(Y)$

(Some) useful properties of variances:

- $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$
Note $\text{Cov}(X,Y) = 0$ if X and Y are independent
- $\text{Var}(a) = 0$
- $\text{Var}(X + a) = \text{Var}(X)$
- $\text{Var}(aX) = a^2\text{Var}(X)$
- $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X,Y)$

The sampling distributions

- $\hat{\mu}_1 = \frac{\sum_{i=1}^n X_i}{n} \sim N(\mu, \frac{\sigma^2}{n})$
- $\hat{\mu}_2 = c \sim E(\hat{\mu}_2) = c, \text{Var}(\hat{\mu}_2) = 0$
- $\hat{\mu}_3 = \frac{\sum_{i=1}^n X_i}{n+1} \sim N(\frac{n}{n+1}\mu, \frac{n}{(n+1)^2}\sigma^2)$

Let's evaluate by bias:

- $E(\hat{\mu}_1) - \mu = \mu - \mu = 0$
- $E(\hat{\mu}_2) - \mu = c - \mu$
- $E(\hat{\mu}_3) - \mu = \frac{n}{n+1}\mu - \mu = \frac{-1}{n+1}\mu$

μ_1 is the only unbiased estimator.

Interpretation

Suppose you get a 95% confidence interval for the mean quality of the Pokemon in the machine of (12 points, 21 points).

How would you explain what this confidence interval represents?

- 1 Learning From Samples
- 2 Simulation and the bias-variance tradeoff
- 3 Linear regression interpretation
- 4 OLS Assumptions, Diagnostics, Fixes
- 5 Instrumental Variables
- 6 Causal inference with observed variables
- 7 What if my code has a bug?

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Simulation: Big ideas

Steps in simulation:

- ① Set up 1 simulation
- ② Enclose that in a function
- ③ Replicate the function lots of times

Simulation example: Robust regression vs. OLS

- Robust regression reduces the influence of influential points
- But it may yield biased estimates if those outliers are warranted
- Given a population, can we simulate the distribution of these two estimators for β ?

Making the population

(There is nothing here to study. This is just how we made the dataset, so that if you want to play with it you can use this code and have the data on your computer)

```
set.seed(08544)
N <- 10000
x <- rexp(N, rate = .001) / 1000
error <- rnorm(N, mean = 0, sd = exp(x / 2))
y <- exp(x / 2) + error
pop <- data.frame(y,x)
```

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One simulation

```
set.seed(12345)
```

One simulation

```
set.seed(12345)  
samp <- sample_n(pop, size = 20)
```

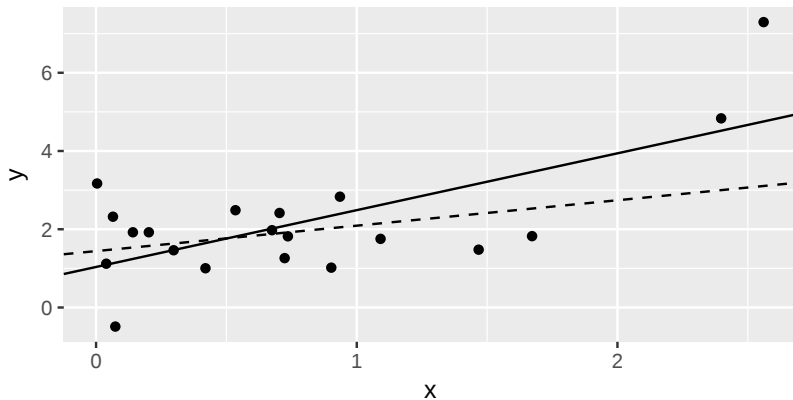

One simulation

```
set.seed(12345)
samp <- sample_n(pop, size = 20)
lm.fit <- lm(y ~ x, data = samp)
mm.fit <- rlm(y ~ x, method = "MM",
              maxit = 40, data = samp)
```


One simulation

MM reduces the influence of the upper right points

OLS and MM fits
(MM is dashed)



Steps in simulation:

- ① Set up 1 simulation
- ② **Enclose that in a function**
- ③ Replicate the function lots of times

```

get.fit <- function(population) {
  samp <- sample_n(pop, size = 20)
  lm.fit <- lm(y ~ x, data = samp)
  mm.fit <- rlm(y ~ x, method = "MM",
               maxit = 40, data = samp)
  beta.lm <- coef(lm.fit)[2]
  beta.mm <- coef(mm.fit)[2]
  return(c(beta.lm = beta.lm,
           beta.mm = beta.mm))
}

```

Steps in simulation:

- ① Set up 1 simulation
- ② Enclose that in a function
- ③ **Replicate the function lots of times**

```
set.seed(08544)
sims <- replicate(10000, get.fit(population = pop))
```

Properties of estimators, bias-variance tradeoff

Bias:

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$$\text{Var}(\hat{\beta})$$

Mean squared error:

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$$\text{Var}(\hat{\beta})$$

Mean squared error:

$$\begin{aligned} \text{MSE}(\hat{\beta}) &= E[(\hat{\beta} - \beta)^2] \\ &= \left(E[\hat{\beta} - \beta] \right)^2 + \text{Var}(\hat{\beta}) \\ &= \text{Bias}^2 + \text{Variance} \end{aligned}$$

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- Mean squared error is a measure of predictive validity.

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- Mean squared error is a measure of predictive validity.
- It is affected by both bias and variance.

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- Mean squared error is a measure of predictive validity.
- It is affected by both bias and variance.
- In some cases, we are willing to choose a biased estimator because it has lower variance, and thus lower mean squared error. This is the **bias-variance tradeoff**.

Evaluate bias and variance

```
## Note the truth
```

```
truth <- coef(lm(y ~ x, data = pop))[2]
```

Evaluate bias and variance

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## Note the truth
truth <- coef(lm(y ~ x, data = pop))[2]
## Look at the bias of the estimators
bias <- apply(sims, 1, mean) - truth
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Evaluate bias and variance

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variance <- apply(sims, 1, var)
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## Mean squared error
errors <- sims - truth
```


Evaluate bias and variance

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truth <- coef(lm(y ~ x, data = pop))[2]
## Look at the bias of the estimators
bias <- apply(sims, 1, mean) - truth
## Variance of the estimators
variance <- apply(sims, 1, var)
## Mean squared error
errors <- sims - truth
mse <- apply(errors, 1, function(x) mean(x ^ 2))
```

	Bias	Variance	Mean squared error
OLS	-0.19	2.88	2.92
MM estimation	-0.58	1.03	1.37

The MM estimator is biased but has lower variance and lower MSE.

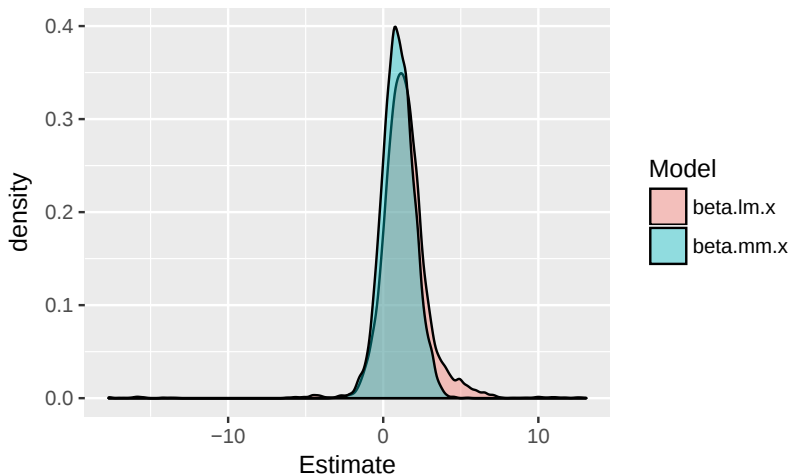
```
## Plot the distribution of each estimator
t(sims) %>%
```

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t(sims) %>%  
  melt(value.name = "Estimate") %>%
```

```
## Plot the distribution of each estimator
t(sims) %>%
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```

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## Plot the distribution of each estimator
t(sims) %>%
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  ggplot(aes(x = Estimate, fill = Model)) +
```

```
## Plot the distribution of each estimator
t(sims) %>%
  melt(value.name = "Estimate") %>%
  mutate(Model = Var2) %>%
  ggplot(aes(x = Estimate, fill = Model)) +
  geom_density(alpha = .4)
```



Linear regression interpretation: Big ideas

- Interpreting coefficients
- Test difference in coefficients
- Interaction terms
- Plotting an interaction

Scenario

- You have a tutoring business.

Scenario

- You have a tutoring business.
- You want to show with data that your business is effective.

Interpreting coefficients

Suppose we fit a model

$$\text{SAT score} = \beta_0 + \beta_1(\text{Days studying}) + \beta_2(\text{Having a tutor}) + u$$

We estimate $\hat{\beta}_1 = 50$, $\hat{\beta}_2 = 300$.

Interpreting coefficients

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- ① Every additional day spent studying causes one to score 50 points higher on the SAT. **Incorrect.**
- ② Each day spent studying increases one's SAT score by 50 points. **Incorrect.**

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- ② Each day spent studying increases one's SAT score by 50 points. **Incorrect.**
- ③ Each day spent studying is associated with a 50 point increase in SAT score.

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- ② Each day spent studying increases one's SAT score by 50 points. **Incorrect.**
- ③ Each day spent studying is associated with a 50 point increase in SAT score. **Better, but still not great.**
- ④ We expect that a student who spends 7 days studying will score 50 points higher than a student who spends 6 days studying. **Incorrect.**
- ⑤ We expect that a student who spends 7 days studying will score 50 points higher than a student who spends 6 days studying, assuming they either both have a tutor or both

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Test difference in coefficients

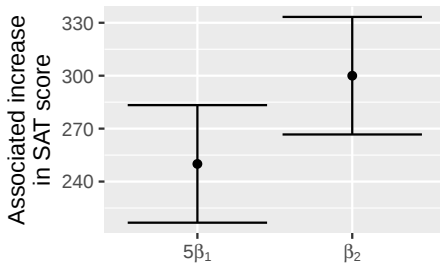
$$\text{SAT score} = \beta_0 + \beta_1(\text{Days studying}) + \beta_2(\text{Having a tutor}) + u$$

- A critic of your business says, “Tutoring is no more beneficial than spending 5 days studying.”

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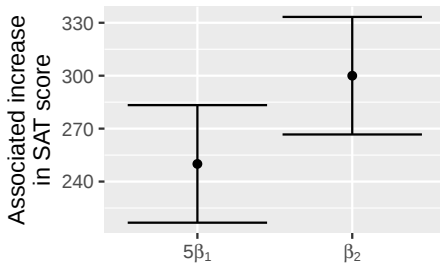
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- Have you disproven your critic?

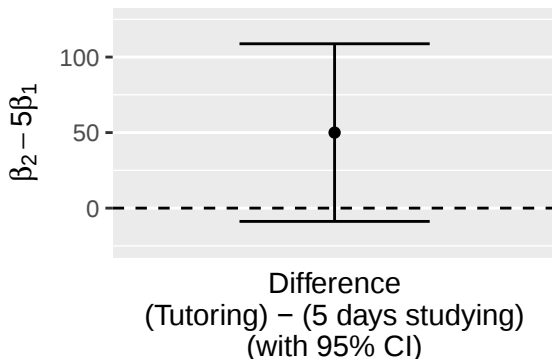
No. You need explicitly look at uncertainty about the difference.

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Test difference in coefficients

- A critic of your business says, “Tutoring is no more beneficial than spending 5 days studying.”
- Now you produce this graph:



- Have you disproven your critic?

No. 0 is within the 95% confidence interval, so we cannot reject the null hypothesis that tutoring and 5 days studying are equally beneficial. The test is inconclusive.

You think your critic is totally wrong, since studying is more beneficial if you have a tutor!

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Your paid consultant proposes the model:

$$+ \beta_2(\text{Days studying})(\text{Having a tutor}) + u$$

We don't like this model. It assumes that

$$= \beta_0$$

This is an unreasonable assumption! When we fit a model with an interaction, it should always include the lower-order terms. We need a term for having a tutor.

Interaction terms

Suppose we fit a new model

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- Among students **without a tutor**, we expect that a student who studies one more day than another student will have a:
 - 25 point higher SAT score.
 - 50 point higher SAT score.
 - 75 point higher SAT score.

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- How do we interpret $\hat{\beta}_3 = 25$?
 - ① Among those with tutors, each day studying increases SAT score by 25 points.

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 - ② Among those with tutors, we expect that students who study one more day than other students will have SAT scores that are 25 points higher.

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 - ② Among those with tutors, we expect that students who study one more day than other students will have SAT scores that are 25 points higher.
 - ③ The positive association between having a tutor and SAT score is 25 points greater for each additional day of studying.

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- How do we interpret $\hat{\beta}_3 = 25$?
 - ① Among those with tutors, each day studying increases SAT score by 75 points.
 - ② Among those with tutors, we expect that students who study one more day than other students will have SAT scores that are 75 points higher.
 - ③ The positive association between having a tutor and SAT score is 25 points greater for each additional day of studying.

Plotting interactions

Your model:

$$\text{SAT score} = \beta_0 + \beta_1(\text{Days studying}) + \beta_2(\text{Having a tutor}) \\ + \beta_3(\text{Having a tutor})(\text{Days studying}) + u$$

How could we convey this interaction in a plot?

- ① (go to board)
- ② What would the y -axis be?
- ③ The x -axis?

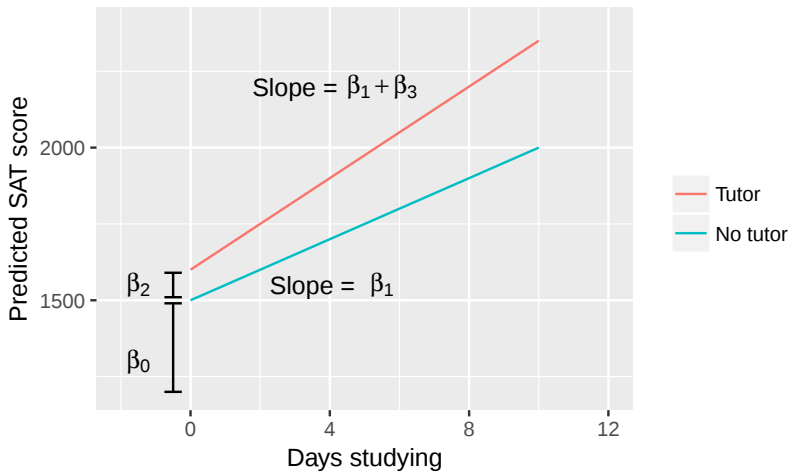
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- ② What would the y -axis be?
- ③ The x -axis?
- ④ How does having a tutor appear on the plot?



Plotting interactions with lots of variables

Your model:

$$\begin{aligned} \text{SAT score} = & \beta_0 + \beta_1(\text{Days studying}) + \beta_2(\text{Having a tutor}) \\ & + \beta_3(\text{Having a tutor})(\text{Days studying}) \\ & + \beta_4(\text{Air temperature on SAT day}) \\ & + \beta_5(\text{Number of sharpened pencils}) + u \end{aligned}$$

- How could we convey this interaction in a plot?

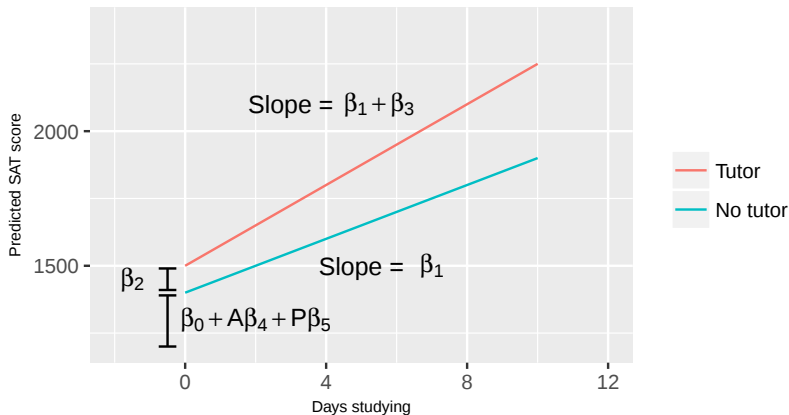
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- How could we convey this interaction in a plot?
- We could set the other covariates to their means when making predicted values.

Predicted values holding air temperature and sharpened pencils at their means
(A = mean air temperature, P = mean # sharpened pencils)



Linear regression interpretation: Big ideas

- Interpreting coefficients
- Test difference in coefficients
- Interaction terms
- Plotting an interaction

OLS Assumptions, Diagnostics, and Fixes: Big ideas

- OLS assumptions: what they mean, why we need them
- Strategies for diagnosing OLS assumption violations and correcting for them

Assumptions Needed for Unbiasedness and Consistency

Random/iid sample The observed data represent a random sample from the population described by the model. This can be violated if we have correlation across observations (spatial, multiple observations from same cluster, same unit at multiple times)

- Result: no bias in parameters with appropriate alternative assumptions (structured dependence)
- Result (ii): violations imply heteroskedasticity
- Result (iii): outliers from different distributions can cause inefficiency/bias
- Diagnose: plotting residuals, checking out extreme values, Durbin-Watson for serial correlation
- Correct: including fixed effects/random effects, aggregate data to cluster level, cluster-robust standard errors, HAC standard errors for serial correlation, etc.

Assumptions Needed for Unbiasedness and Consistency

No perfect collinearity No explanatory variable is the linear combination of the other explanatory variables

- Result: We don't get an estimate!
- Diagnose/correct: drop one collinear term
- Note we also cannot get an estimate if we have no variation in X .

Zero conditional mean Expected value of the error term is zero conditional on all values of the explanatory variable: $E(u | X) = 0$. Note it implies that $Cov(X, u) = 0$.

- Result: biased/inconsistent estimates
- Diagnose: very difficult. Ask yourself: are there any unobserved confounders?
- Correct: instrumental variables and other methods that deal unmeasured confounding Note: Not to be confused with the mean of the residuals!

Homoskedasticity

Needed for large-sample inference: The error term has the same variance conditional on all values of the explanatory variable.

- Result: Our estimates are unbiased but our standard errors are biased
- Diagnostics:
 - Visual: plot residuals vs fitted values (is there fanning?), spread-location plots
 - Formal tests: Breusch-Pagan, Cook-Weisberg, etc.
- Correcting: transform dependent variable, adjust using weights, **use an estimator of $\text{Var}(\hat{\beta})$ that's robust to heteroskedasticity**, admit we have the wrong model

Heteroskedasticity Consistent Estimator

- Under non-constant error variance:

$$\text{Var}[\mathbf{u}] = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

- When $\mathbf{\Sigma} \neq \sigma^2 \mathbf{I}$, we are stuck with this expression:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{\Sigma}\mathbf{X} (\mathbf{X}'\mathbf{X})^{-1}$$

- Idea: If we can consistently estimate the components of $\mathbf{\Sigma}$, we could directly use this expression by replacing $\mathbf{\Sigma}$ with its estimate, $\hat{\mathbf{\Sigma}}$.

White's Heteroskedasticity Consistent Estimator

Robust standard errors are easily computed with the “sandwich” formula:

- ① Fit the regression and obtain the residuals $\hat{\mathbf{u}}$
- ② Construct the “meat” matrix $\hat{\mathbf{\Sigma}}$ with squared residuals in diagonal:

$$\hat{\mathbf{\Sigma}} = \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{u}_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \hat{u}_n^2 \end{bmatrix}$$

- ③ Plug $\hat{\mathbf{\Sigma}}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix

$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\hat{\mathbf{\Sigma}}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

- There are various **small sample corrections** to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|\mathbf{X}] = \frac{n}{n-k-1} \cdot (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\hat{\mathbf{\Sigma}}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

Instrumental Variables - The Big Ideas

- Identifying assumptions - what are they? Do they hold up?
- Estimation - Wald, 2SLS
- Interpreting results - what have we identified?

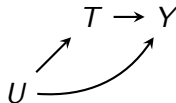
Causal effect of juvenile detention

Let's work through this example:

Aizer, Anna, and Joseph Doyle. 2015. "Juvenile Incarceration, Human Capital, and Future Crime: Evidence from Randomly Assigned Judges." *The Quarterly Journal of Economics* 759-804.

Research Question

- Question: What is the causal effect of juvenile detention on high school completion and adult incarceration?
- What are possible unobserved confounders? In what direction might those confounders bias our results?



The Natural Experiment

The initial judge that a child encounters is randomly assigned. It is “a function of the sequence with which cases happen to enter into the system and the judge availability that is set in advance.” 96% of people who come before the court are found guilty, but judges exercise discretion over whether a child is placed on probation or detained. This paper exploits variation in judge sentencing behaviour.

- Data: 10 years of administrative data from the Chicago Public Schools Student Database, Juvenile Court of Cook County Delinquency Database, Illinois Department of Corrections Adult Admissions and Exits Database

What's the identification strategy?

- **What's the instrument (Z)?**

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- **What are the outcomes (Y)?**

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Being assigned a judge with a particular propensity to incarcerate (calculated based on all other cases before the judge)

- **What's the treatment (T)?**

Being incarcerated as a juvenile

- **What are the outcomes (Y)?**

1) High school completion, 2) adult incarceration

Let's evaluate the assumptions

- 1 **Exogeneity of the Instrument**
Are judges randomly assigned?
- 2 **Exclusion Restriction**

Let's evaluate the assumptions

① **Exogeneity of the Instrument**

Are judges randomly assigned?

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Can a judge's propensity to incarcerate affect high school completion/adult incarceration through causal channels other than through juvenile incarceration?

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Does a judge's propensity to incarcerate induce variation in sentences? (checkable by regressing treatment on the instrument)

④ **Monotonicity**

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Are there defiers?

Results

"...juvenile detention is estimated to decrease high school graduation by 13 percentage points and increase adult incarceration by 23 percentage points"

Wait, but what effect did we identify?

We identified the **complier average treatment effect**.

What does this mean substantively? To whom does this effect generalize?

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We identified the **complier average treatment effect**.

What does this mean substantively? To whom does this effect generalize?

We have estimated the effect of incarceration for marginal juvenile cases in which the sentencing outcome is affected by the judge assigned. We might believe that these effects might be different for children who commit particularly mild or particularly severe crimes.

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Running example

Does marriage cause men to earn more per hour?

The experimental ideal

What experiment would answer this question?

Assumption: Selection on observables

$$(Y_i(1), Y_i(0) \perp\!\!\!\perp D_i \mid X_i)$$

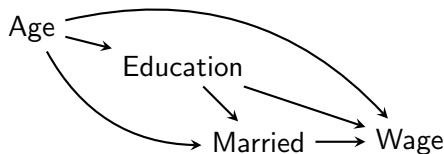
Example:

Net of education and age, marriage is independent of potential wages.

Can we draw in a DAG?

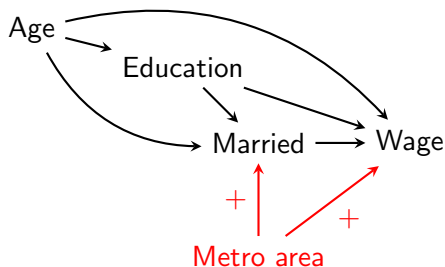
DAGs: Confounders

Net of education and age, marriage is independent of potential wages.



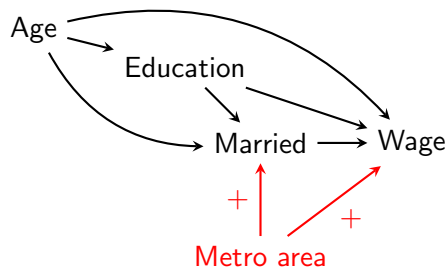
Sensitivity and expected direction of bias

Potential violation: Which way is the estimated effect of T on Y biased?



Sensitivity and expected direction of bias

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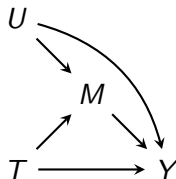


Answer: Upwardly biased (+ times + = +).

For every critic, you can try to reason about the bias.

DAGs: Colliders

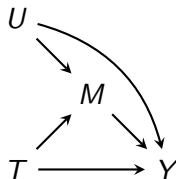
Theory (example next slide):



- Which of these variables is a collider? On what path?

DAGs: Colliders

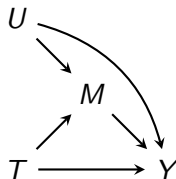
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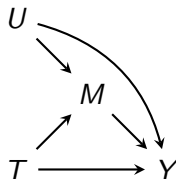
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- Which of these variables is a collider? On what path?
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- What happens if we ignore M ?

DAGs: Colliders

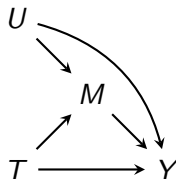
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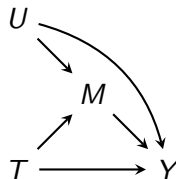
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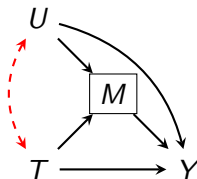
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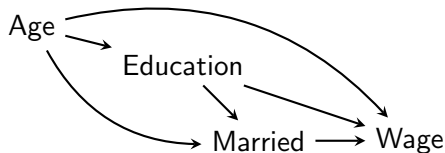
- Which of these variables is a collider? On what path?
- M is a collider on the path $T \rightarrow M \leftarrow U \rightarrow Y$
- What happens if we ignore M ? (Nothing: $T \rightarrow Y$ is identified!)
- What happens if we condition on M ? Bad things: Induced association!)

DAGs: Induced association from conditioning on a collider



Back to our example

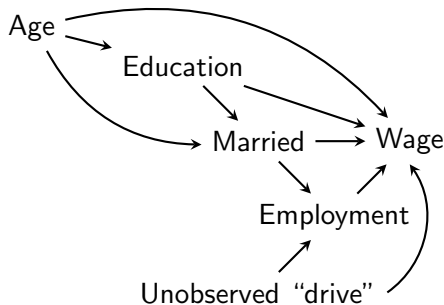
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DAGs: Collider example

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But, our sample is only among those who are employed, since others don't have wages.

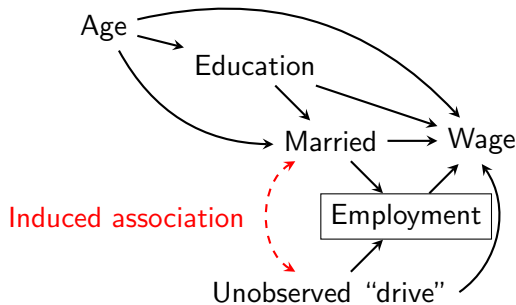


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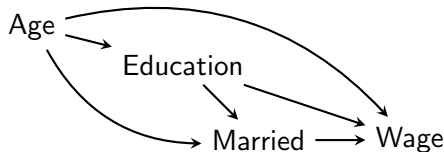
But, our sample is only among those who are employed, since others don't have wages.

This effectively conditions on employment, causing problems!



Estimation

Given the assumptions (selection on observables) in the DAG below, we have identified the effect of marriage on men's wages.



To **estimate** this effect, we will assume an additive linear regression model.

$$Y = \beta_0 + \beta_1(Age) + \beta_2(Education) + \beta_3(Married) + u$$

Causal inference with observed variables: Big ideas

- Define a causal quantity of interest
- Consider the experimental ideal
- Identification
- Estimation

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For R errors, also check that you didn't forget to load an R package (they need to be loaded within the RMarkdown file, not by hand in the console).