

Week 9: What Can Go Wrong and How To Fix It, Diagnostics and Solutions

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Princeton

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¹These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller and Kevin Quinn.

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 - ▶ Wednesday (16):
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- Long Run
 - ▶ regression → diagnostics → causal inference

Questions?

- 1 Assumptions and Violations
- 2 Non-normality
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- 4 Robust Regression Methods
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- 6 Conclusion and Appendix
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Argument for Next Three Classes

Residuals are **important**. Look at them.

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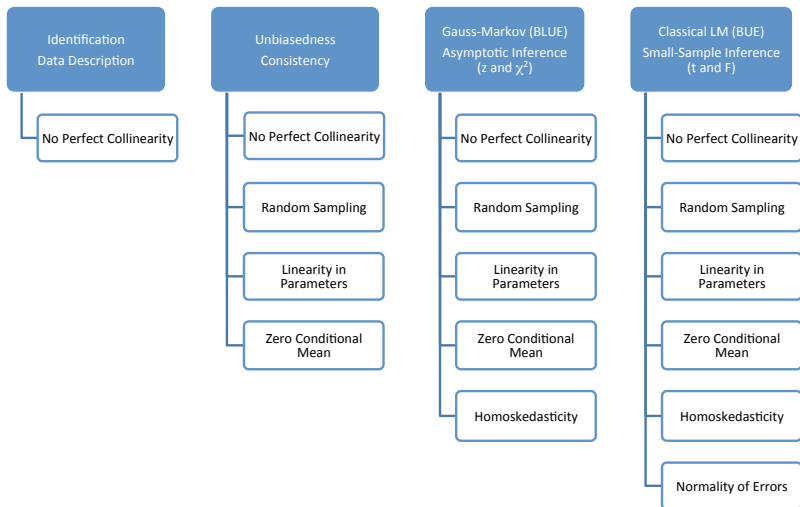
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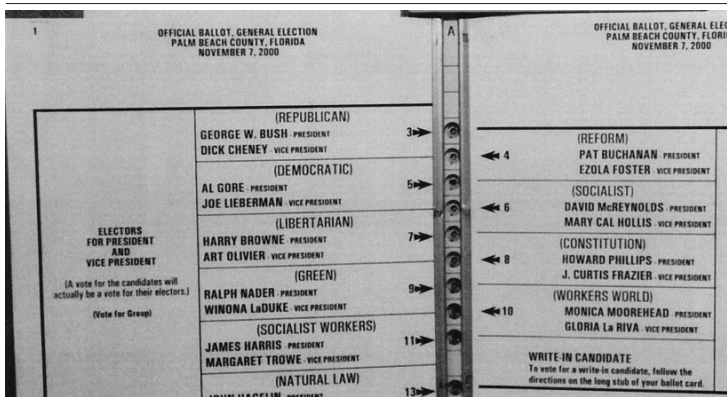
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Example: Buchanan votes in Florida, 2000

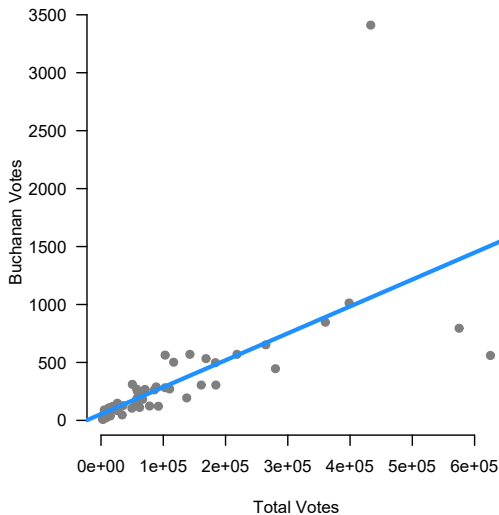
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Wand et al. show that the ballot caused 2,000 Democratic voters to vote by mistake for Buchanan, a number more than enough to have tipped the vote in FL from Bush to Gore, thus giving him FL's 25 electoral votes and the presidency.

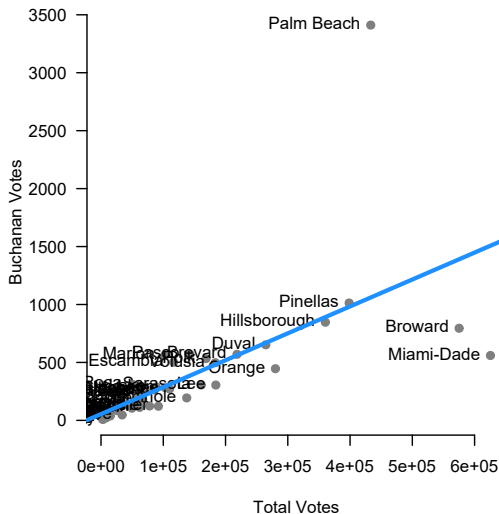
FIGURE 1. The Palm Beach County Butterfly Ballot



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- Fix $\mathbf{x}'_i$ and the distribution of errors should be Normal

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- The sample size (n) needed for approximation to hold depends on how far the errors are from Normal.

Marginal versus conditional

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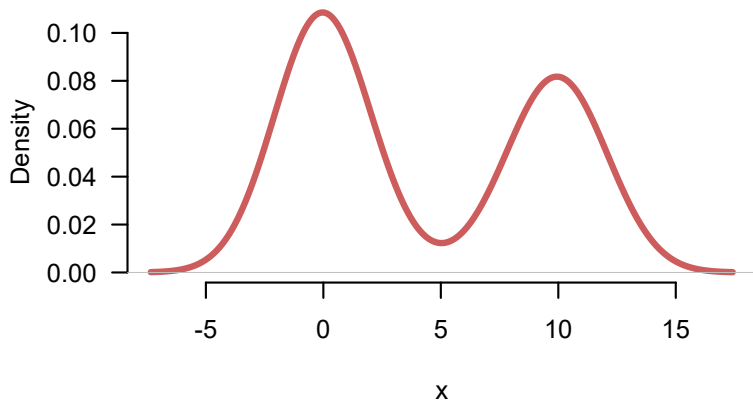
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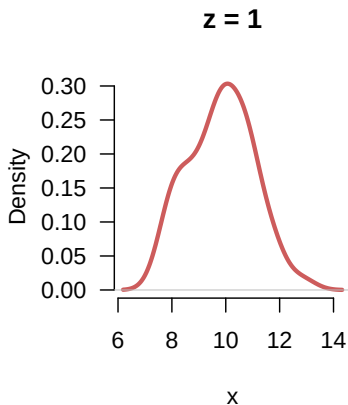
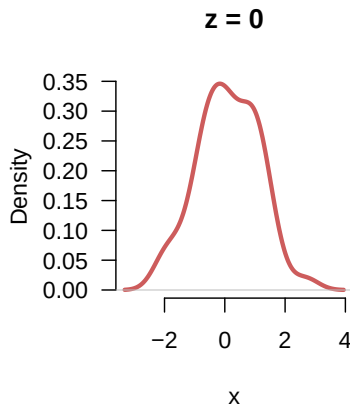
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- The plausibility depends on the X chosen by the researcher.

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To understand the relationship between residuals and errors, we need to derive the distribution of the residuals.

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- Note that the residual is a function of all of the errors

Distribution of the residuals

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The variance of the i th residual $\hat{u}_i$ is $V[\hat{u}_i] = \sigma^2(1 - h_{ii})$, where h_{ii} is the i th diagonal element of the matrix $\mathbf{H}$ (called the **hat value**).

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These properties can obscure the true patterns in the error distribution, and thus are inconvenient for our diagnostics.

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The standardized residuals are still not ideal, since the numerator and denominator of $\hat{u}'_i$ are not independent. This makes the distribution of $\hat{u}'_i$ nonstandard.

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- Deviations from $t \implies$ violation of Normality

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- We examine data from the 2000 presidential election in Florida used in Wand et al. (2001).
- Our analysis takes place at the county level and we will regress the number of Buchanan votes in each county on the total number of votes in each county.

Buchanan Votes and Total Votes

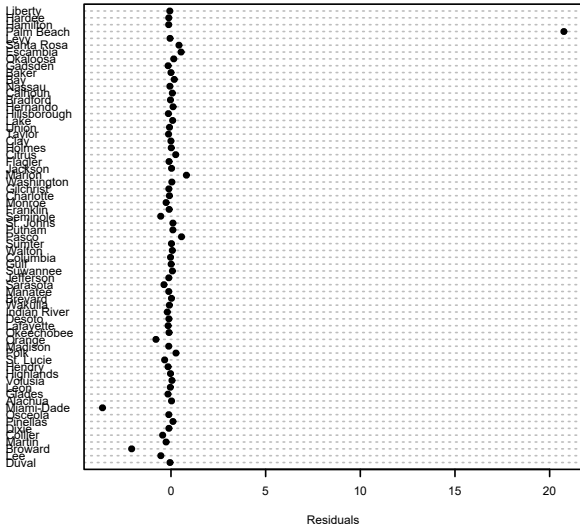
R Code

```
> mod1 <- lm(buchanan00~TotalVotes00,data=dta)
> summary(mod1)
Residuals:
    Min       1Q   Median       3Q      Max
-947.05  -41.74  -19.47   20.20 2350.54

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.423e+01  4.914e+01   1.104   0.274
TotalVotes00 2.323e-03  3.104e-04   7.483 2.42e-10 ***
---
Residual standard error: 332.7 on 65 degrees of freedom
Multiple R-squared:  0.4628,    Adjusted R-squared:  0.4545
F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10

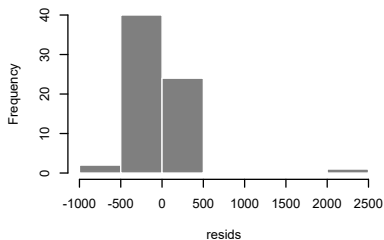
> residuals <- resid(mod1)
> standardized_residuals <- rstandard(mod1)
> studentized_residuals <- rstudent(mod1)
> dotchart(residuals,dta$name,cex=.7,xlab="Residuals")
```

Plotting the residuals

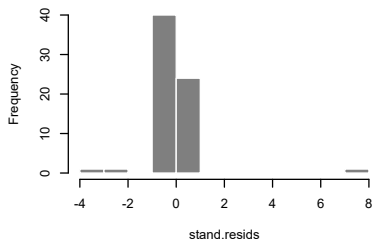


Plotting the residuals

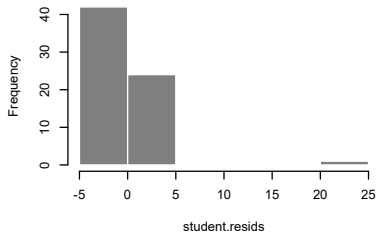
Histogram of resid



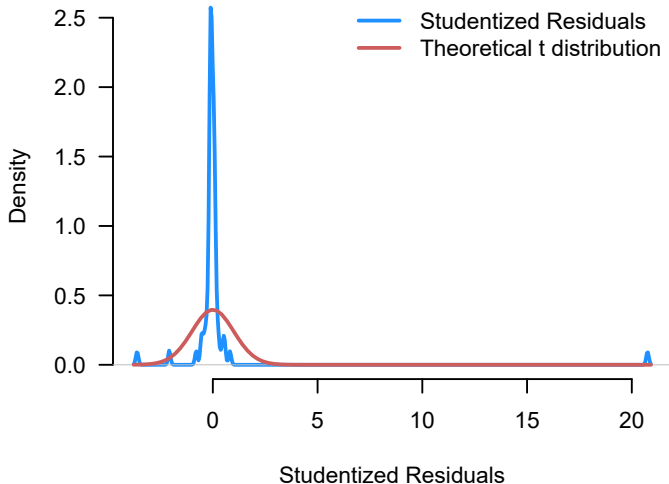
Histogram of stand.resids



Histogram of student.resids



Plotting the residuals



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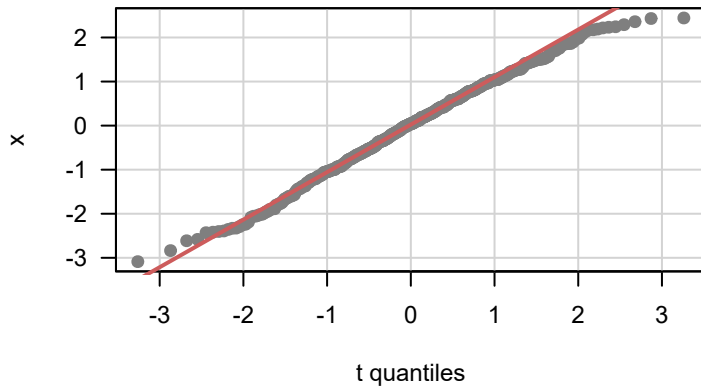
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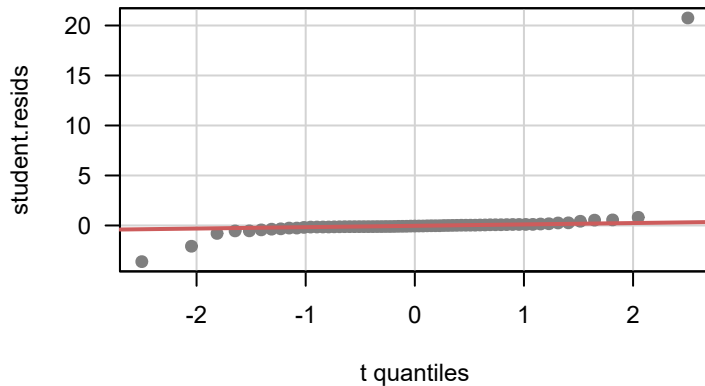
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- If distributions are equal $\implies$ 45 degree line

Good QQ-plot



Buchanan QQ-plot



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- Consider other causes (next two classes)

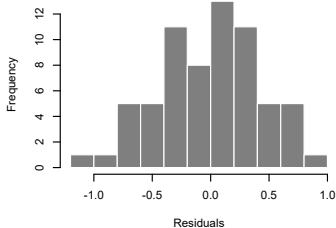
Buchanan revisited

Let's delete Palm Beach and also use log transformations for both variables

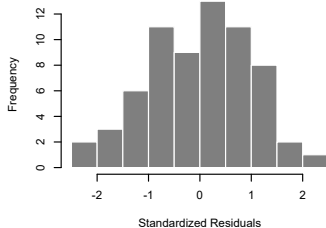
```
##  
## Coefficients:  
##           Estimate Std. Error t value Pr(>|t|)  
## (Intercept)  -2.48597    0.37889  -6.561 1.09e-08 ***  
## log(edaytotal) 0.70311    0.03621  19.417 < 2e-16 ***  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## Residual standard error: 0.4362 on 64 degrees of freedom  
## Multiple R-squared:  0.8549, Adjusted R-squared:  0.8526  
## F-statistic: 377 on 1 and 64 DF, p-value: < 2.2e-16
```

Buchanan revisited

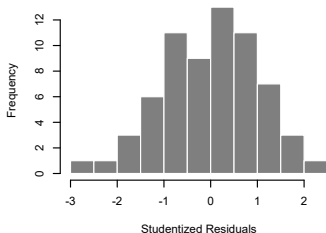
Histogram of resid.nopb



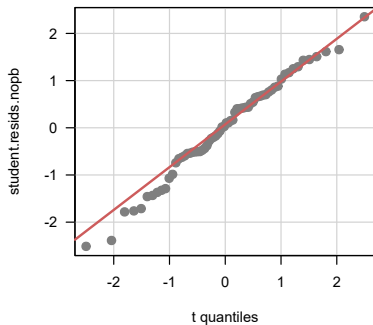
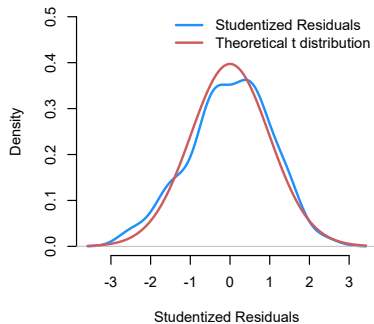
Histogram of stand.resids.nopb



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Buchanan revisited



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- The results will in general be **consistent** which ensures that the bias decreases in sample size.

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- 6 Conclusion and Appendix
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The trouble with Norway

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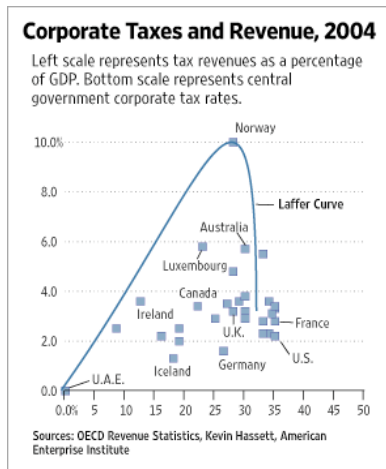
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	Constant	x_1	x_2	$x_1 \cdot x_2$
Norway Obs Included	.814 (4.7)	-.192 (2.0)	-.278 (2.4)	.137 (2.9)
Norway Obs Excluded	.641 (4.8)	-.068 (0.9)	-.138 (1.5)	.054 (1.3)

Creative curve fitting with Norway

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Always Carefully Examine the Data First!!

- 1 Examine summary statistics: `summary(data)`
- 2 Scatterplot matrix for densities and bivariate relationships:
E.g. `scatterplotMatrix(data)` from `car` library.
- 3 Further conditional plots for multivariate data:
E.g. use the `lattice` library or `ggplot2`

Three types of extreme values

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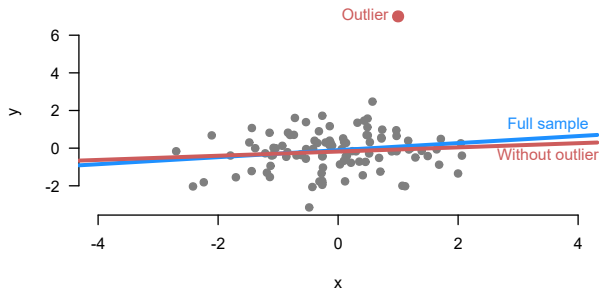
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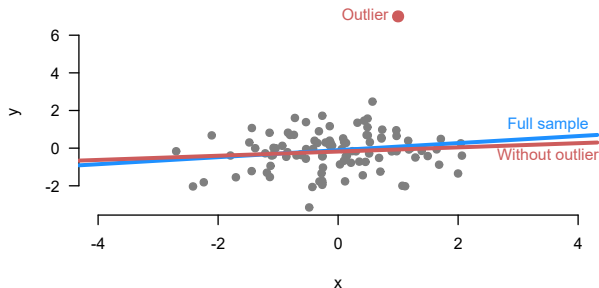
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Outlier definition



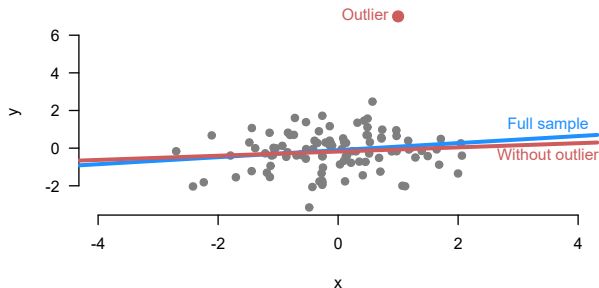
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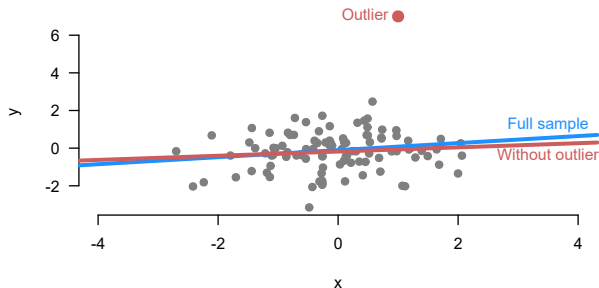
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- $\hat{\sigma} > \hat{\sigma}_{-i}$ because we drop the large residual from the outlier, and so $\hat{u}_i' < \hat{u}_i^*$

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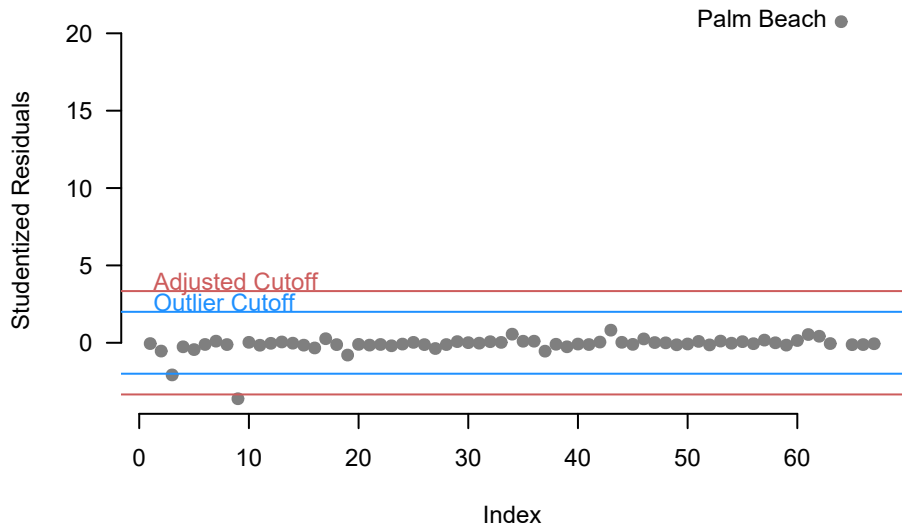
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- People usually adjust cutoff for multiple testing

Buchanan outliers



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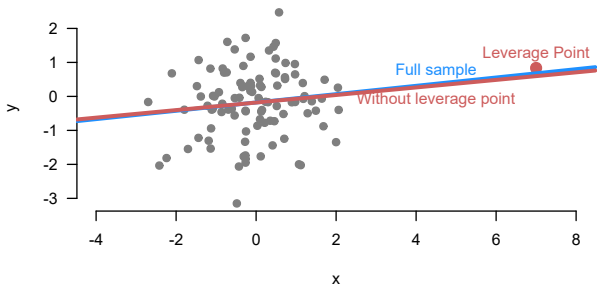
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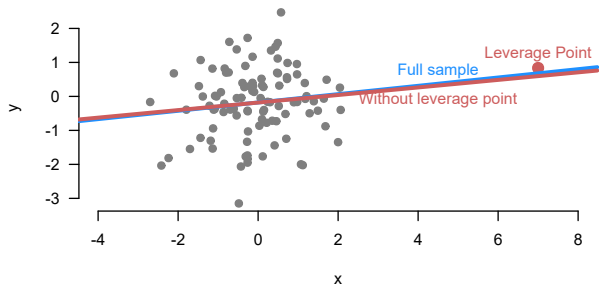
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- The ozone hole was detected in satellite data only when the raw data was reprocessed. When the software was rerun without the pre-processing flags, the ozone hole was seen as far back as 1976.

Leverage point definition



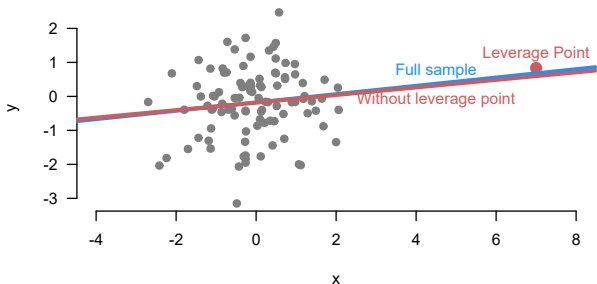
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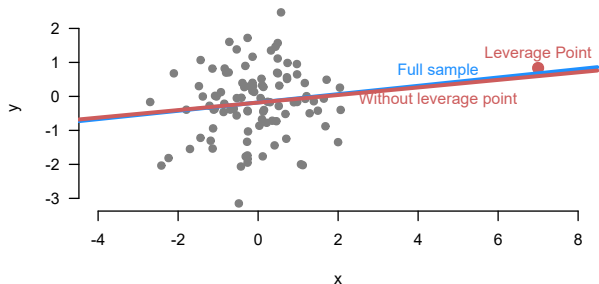
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Leverage Points: Hat values

To measure leverage in multivariate data we will go back to the hat matrix **H**:

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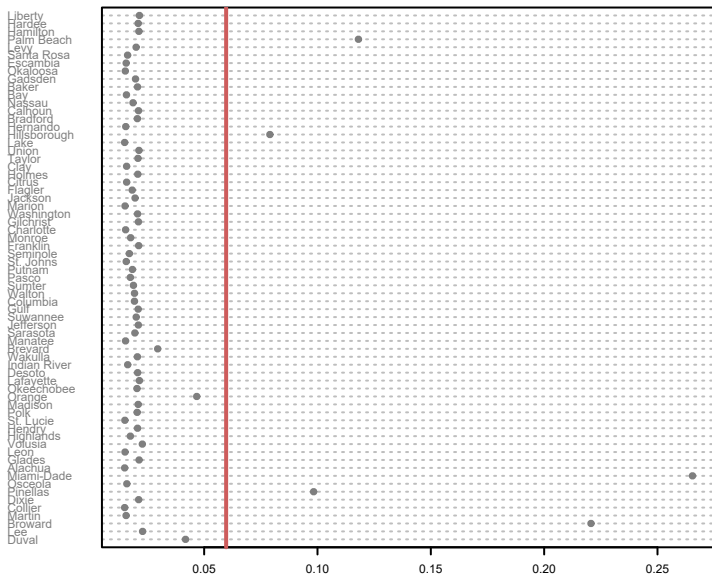
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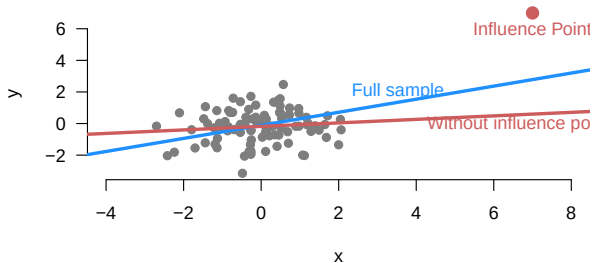
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- **Rule of thumb**: examine hat values greater than $2(k+1)/n$

Buchanan hats

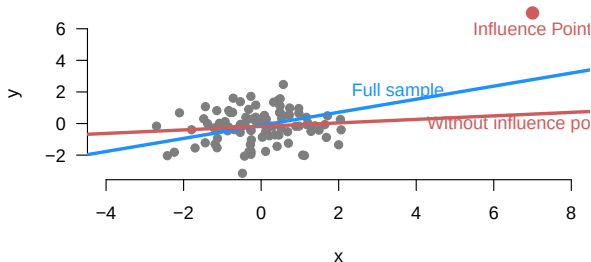


Influence points

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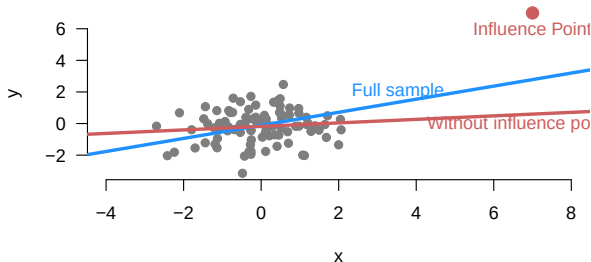


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- D_{ij} is called the **DFbeta**, which measures the **influence** of observation i on the estimated coefficient for the j th explanatory variable.

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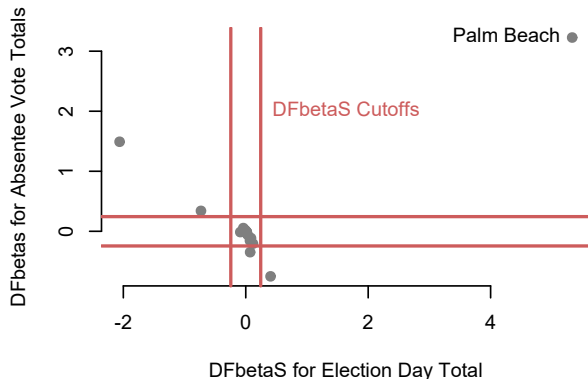
Buchanan influence

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -2.935e+01  5.520e+01  -0.532  0.59686
## edaytotal    1.100e-03  4.797e-04   2.293  0.02529 *
## absnbuchanan  6.895e+00  2.129e+00   3.238  0.00195 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 317.2 on 61 degrees of freedom
## (3 observations deleted due to missingness)
## Multiple R-squared:  0.5361, Adjusted R-squared:  0.5209
## F-statistic: 35.24 on 2 and 61 DF,  p-value: 6.711e-11
```

Buchanan influence

##	(Intercept)	edaytotal	absnbuchanan
## 1	0.3454475146	0.4050504921	-0.7505222758
## 2	-0.0234266617	-0.0241000045	-0.0131672181
## 3	0.0650795039	-0.7319311820	0.3401669862
## 4	-0.0333980968	0.0133802934	-0.0087505576
## 5	-0.0397626659	-0.0073746223	0.0096551713
## 6	-0.0009277798	0.0001505476	0.0002210247

Buchanan influence



- Palm Beach county moves each of the coefficients by more than 3 standard errors!

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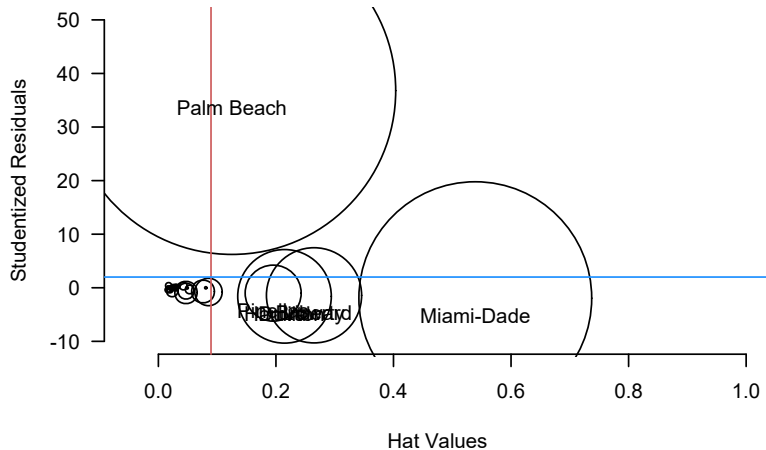
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Influence Plot Buchanan



Code for Influence Plot

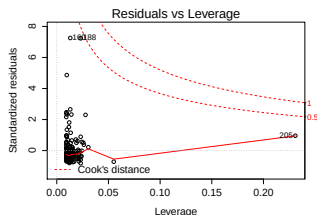
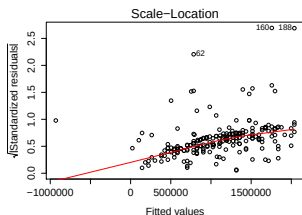
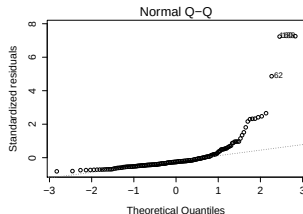
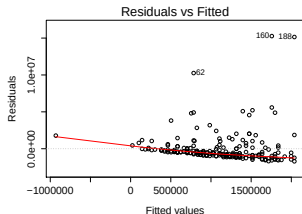
```
mod3 <- lm(edaybuchanan ~ edaytotal + absnbuchanan, data = flvote)
symbols(y = rstudent(mod3), x = hatvalues(mod3),
        circles = sqrt(cooks.distance(mod3)),
        ylab = "Studentized Residuals",
        xlab = "Hat Values", xlim = c(-0.05, 1),
        ylim = c(-10, 50), las = 1, bty = "n")

cutoffstud <- 2
cutoffhat <- 2 * (3)/nrow(flvote)
abline(v = cutoffhat, col = "indianred")
abline(h = cutoffstud, col = "dodgerblue")
filter <- rstudent(mod3) > cutoffstud | hatvalues(mod3) > cutoffhat
text(y = rstudent(mod3)[filter],
     x = hatvalues(mod3)[filter],
     flvote$county[filter], pos = 1)
```

A Quick Function for Standard Diagnostic Plots

R Code

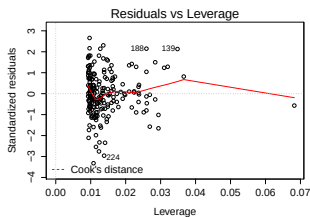
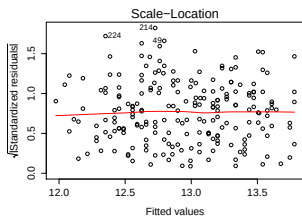
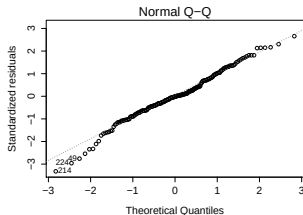
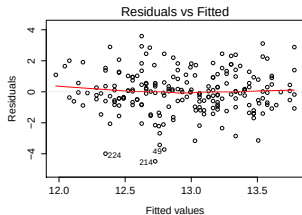
```
> par(mfrow=c(2,2))  
> plot(mod1)
```



The Improved Model

R Code

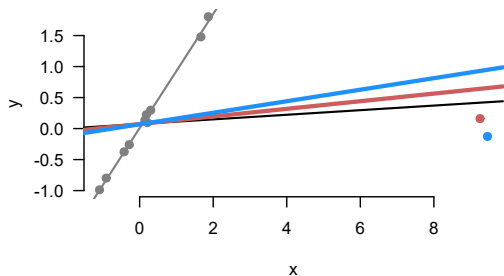
```
> par(mfrow=c(2,2))  
> plot(mod2)
```



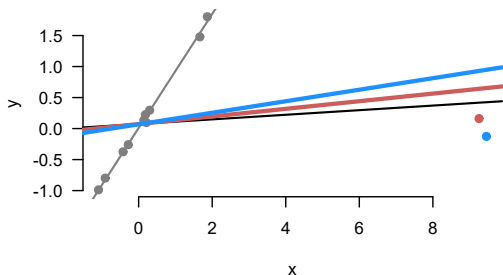
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Limitations of the standard tools

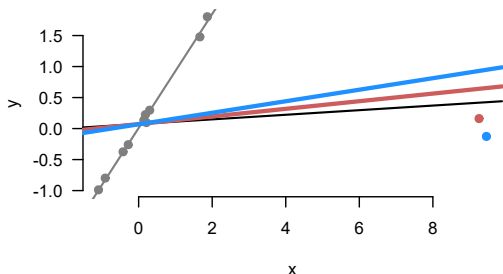


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- Red line drops the red influence point
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- Neither of the “leave-one-out” approaches helps recover the line

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- The **Linear** point is an artificial restriction. It means the estimator has to be of the form $\hat{\beta} = \mathbf{W}y$ but why only use those?
- With normality assumption we get **Best Unbiased Estimator** (BUE) which is quite comforting when $n \gg p$ (number of observations much larger than number of variables).

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"[Even without normally distributed errors] OLS coefficient estimators remain unbiased and efficient." - Berry (1993)

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"[The Gauss-Markov theorem] justifies the use of the OLS method rather than using a variety of competing estimators" - Wooldridge (2013)

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"We need not look for another linear unbiased estimator, for we will not find such an estimator whose variance is smaller than the OLS estimator"
- Gujarati (2004)

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Robustly Estimating a Location

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- We can measure sensitivity with the **influence function** which measures change in estimator based on corruption in one datapoint.

Influence Function

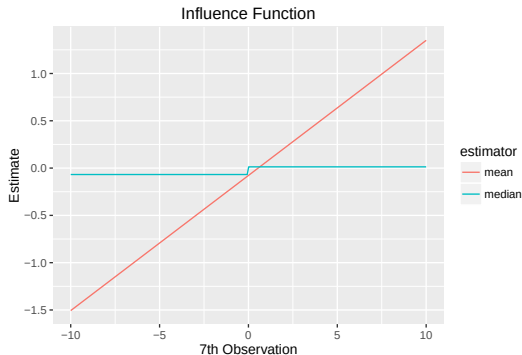
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Example from Fox

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- The breakdown point of the mean is 0 because (as we have seen) a single bad data point can change things a lot.
- The median has a breakdown point of 50% because half the data can be bad without causing the median to become completely unstuck.

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- Calculating robust M estimators often requires an iterative procedure and a careful initialization.

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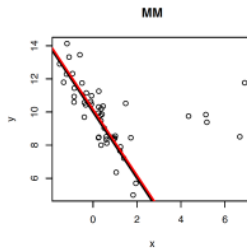
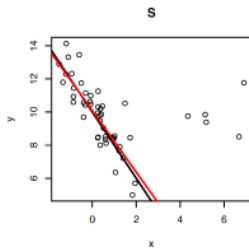
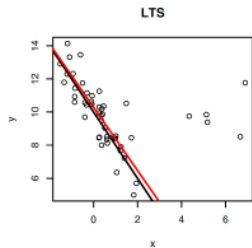
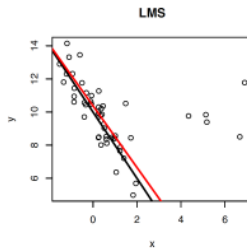
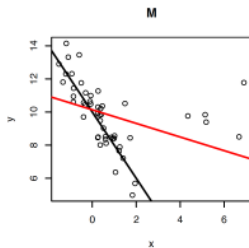
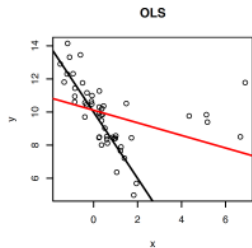
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 - ▶ MM -estimator: with Huber's loss is what I recommend in practice (more in appendix)
- You can find an asymptotic covariance matrix for M -estimators but I would bootstrap it if possible as the asymptotics kick in slowly.


```
library(MASS)
set.seed(588)
n <- 50
x <- rnorm(n)
y <- 10 - 2*x + rnorm(n)
x[1:5] <- rnorm(5, mean=5)
y[1:5] <- 10 + rnorm(5)
ols.out <- lm(y~x)
m.out <- rlm(y~x, method="M")
lms.out <- lqs(y~x, method="lms")
lts.out <- lqs(y~x, method="lts")
s.out <- lqs(y~x, method="S")
mm.out <- rlm(y~x, method="MM")
```

Simulation Results



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- I highly recommend Baissa and Rainey (2016) "When BLUE is Not Best: Non-Normal Errors and the Linear Model" for more on this topic.
- The Fox textbook Chapter 19 is also quite good on this and points out to the key references

Appendix: Characterizing Estimator Robustness (formally)

Definition (Breakdown Point)

The breakdown point of an estimator is the smallest fraction of the data that can be changed an arbitrary amount to produce an arbitrarily large change in the estimate (Seber and Lee 2003, pg 82)

Definition (Influence Function)

Let $F_p = (1 - p)F + p\delta_{\mathbf{z}_0}$ where F is a probability measure, $\delta_{\mathbf{z}_0}$ is the point mass at $\mathbf{z}_0 \in \mathbb{R}^k$, and $p \in (0, 1)$.

Let $T(\cdot)$ be a statistical functional. The influence function of T is

$$IF(\mathbf{z}_0; T, F) = \lim_{p \downarrow 0} \frac{T(F_p) - T(F)}{p}$$

The influence function is a function of $\mathbf{z}_0$ given T and F . It describes how T changes with small amounts of contamination at $\mathbf{z}_0$ (Hampel, Rousseeuw, Ronchetti, and Stahel, (1986), p. 84).

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An *S*-estimator for the regression model is defined as the values of $\hat{\beta}_S$ and s that minimize s subject to the constraint:

$$\frac{1}{n} \sum_{i=1}^n \rho \left(\frac{y_i - \mathbf{x}'_i \hat{\beta}_S}{s} \right) \geq K$$

where K is user-defined constant (typically set to 0.5) and $\rho : \mathbb{R} \rightarrow [0, 1]$ is a function with the following properties (Davies, 1990, p. 1653):

- ① $\rho(0) = 1$
- ② $\rho(u) = \rho(-u)$, $u \in \mathbb{R}$
- ③ $\rho : \mathbb{R}_+ \rightarrow [0, 1]$ is nonincreasing, continuous at 0, and continuous on the left
- ④ for some $c > 0$, $\rho(u) > 0$ if $|u| < c$ and $\rho(u) = 0$ if $|u| > c$

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Good properties, but costly to compute (usually impossible to compute exactly).

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- 16 Fun with Kittens

Measurement Error

“It seems as if measurement error has been pushed into the role of the unwanted child whose existence we would rather deny. Maybe because measurement error is common, insipid, and unsophisticated. Unlike the hidden confounder challenging our intellect, to discover measurement error is a ‘no-brainer’ - it simply lurks everywhere. Our epidemiological fingerprints are contaminated with measurement error. Everything we observe, we observe with error. Since observation is our business, we would probably rather deny that what we observe is imprecise and maybe even inaccurate, but time has come to unveil the secret: measurement error is threatening our profession.”

Karen Michals (2001)

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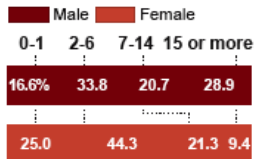
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- Other variables, like gender, number of children, may be measured with less error

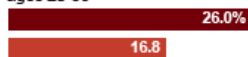
Sex and drugs

Men are more likely to use illegal drugs and have more sexual partners than women, according to a 1999-2002 survey.

Number of sexual partners, ages 20-59



Ever used cocaine or street drugs, ages 20-59



SOURCE: Centers for Disease Control and Prevention AP

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Let's assume that the Gauss-Markov assumptions hold for the model with the true (but unobserved) variables so that OLS would be unbiased and consistent if we observed Y_i . Does the measurement error in Y_i^* cause any problems when fitting OLS to the observed data?

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- If E is correlated with X then OLS is inconsistent and biased.

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- Note: This has nothing to do with assumptions about errors U , we always maintain that $\text{Cov}[X^*, U] = 0$ and $\text{Cov}[X, U] = 0$

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- Can we know the direction of the (asymptotic) bias?

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Notice that given our CEV assumption $\left(\frac{\sigma_X^2}{\sigma_X^2 + \sigma_E^2} \right) < 1$ so that the probability limit of $\hat{\beta}_1$ is always closer to zero than β_1 (**attenuation bias**).

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Bias is small if variance of observed measure σ_X^2 is large relative to variance of error term σ_E^2 (high signal to noise ratio).

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- If we have CEV measurement error in multiple X s then the size and direction of biases are unclear.

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Note: This is true only under fairly strong assumptions including mean zero measurement error.

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- Don't let regression be a magic black box for you- understand why it is giving the answers it gives.

References

- Wand, Jonathan N., Kenneth W. Shotts, Jasjeet S. Sekhon, Walter R. Mebane Jr, Michael C. Herron, and Henry E. Brady. "The butterfly did it: The aberrant vote for Buchanan in Palm Beach County, Florida." *American Political Science Review* (2001): 793-810.
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Where We've Been and Where We're Going...

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- Long Run
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Questions?

Residuals are still **important**. Look at them.

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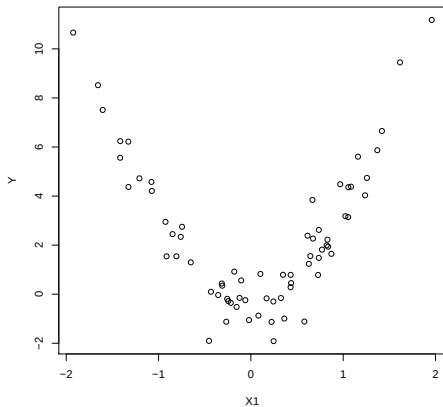
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Nonlinearity

Linearity of the Conditional Expectation Function ($y = \mathbf{X}\beta + \mathbf{u}$) is a key assumption. Why?

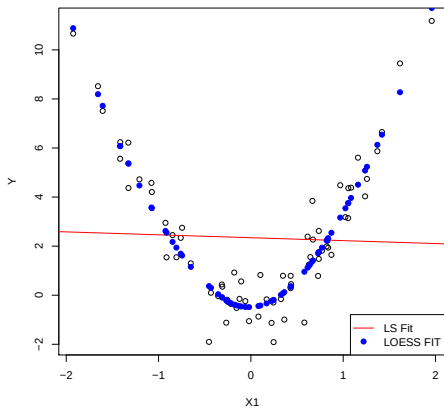
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 - ▶ Possible Exceptions: Returns to scale, constant elasticities, interactive effects, cyclical patterns in time series data, etc.
- Usually we employ “linearity by default” but we should try to make sure this is appropriate: **detect** non-linearities and **model** them accurately

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- For **marginal** relationships Y and X
 - ▶ Scatterplots with loess lines

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 - ▶ Added variables plots and component residual plots
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 - ▶ Non-parametric multiple regression techniques (beyond the scope of this course)

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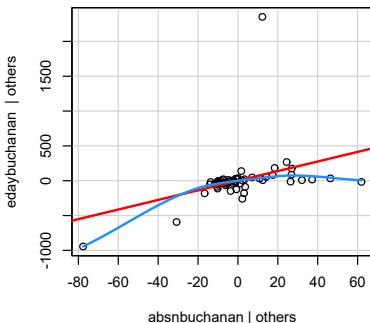
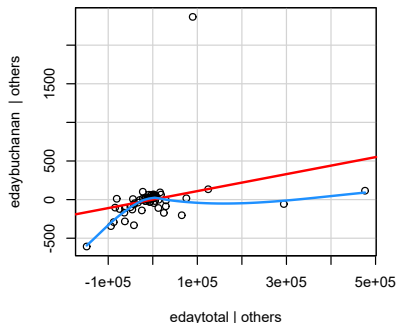
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Buchanan AV plot

R Code

```
par(mfrow = c(1,2))  
out <- avPlots(mod3, "edaytotal")  
lines(loess.smooth(x = out$edaytotal[,1],  
  y= out$edaytotal[,2]), col = "dodgerblue", lwd = 2)  
out2 <- avPlots(mod3, "absnbuchanan")  
lines(loess.smooth(x = out2$absnbuchanan[,1],  
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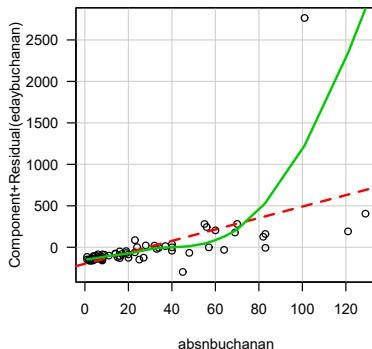
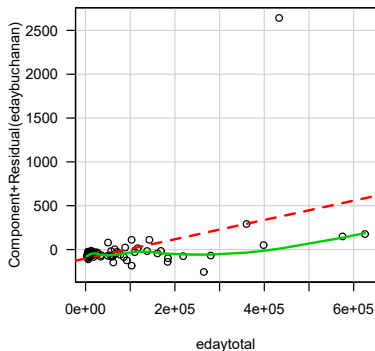
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Buchanan CR plot

R Code

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Component + Residual Plots



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 - ▶ Experience suggests weak non-linearities among X s do not invalidate CR plots

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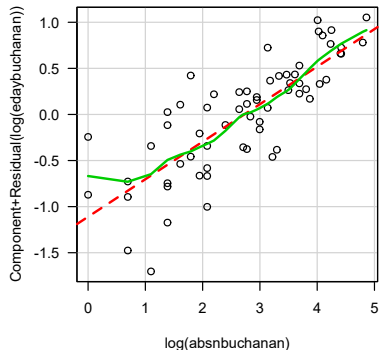
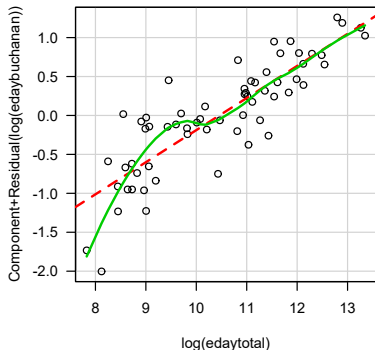
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- Including interactions
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- Transformations such as logs
- Generalized Additive Models (GAM)
- Many more flexible, nonlinear regression models exist beyond the scope of this course.

Transformed Buchanan regression

```
R Code  
mod.nopb2 <- lm(log(edaybuchanan) ~ log(edaytotal) + log(absnbuchanan),  
data = flvvote, subset = county != "Palm Beach")  
crPlots(mod.nopb2, las = 1)
```

Component + Residual Plots

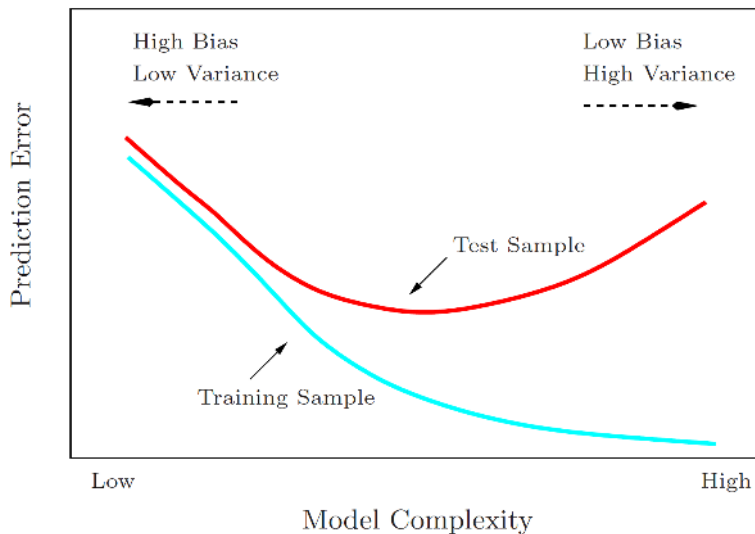


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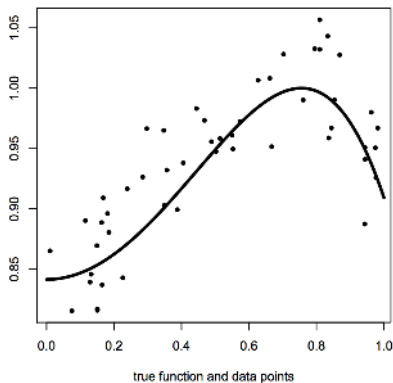
Bias-Variance Tradeoff

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Example Synthetic Problem

$$y = \sin(1 + x^2) + \epsilon$$



This section adapted from slides by Radford Neal.

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- In general the idea is to do a linear regression of y on $\phi_1(x), \phi_2(x), \dots, \phi_{m-1}(x)$ where ϕ_j are **basis functions**.
- The model is now:

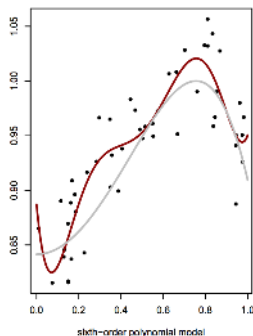
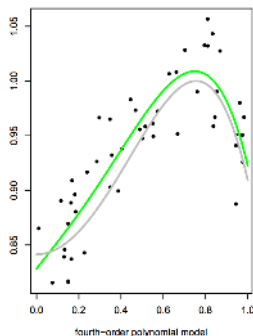
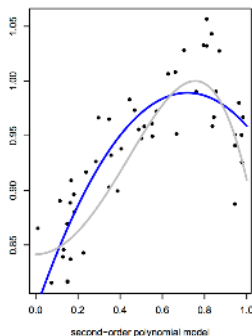
$$y = f(x, \beta) + \epsilon$$
$$f(x, \beta) = \beta_0 + \sum_{j=1}^{m-1} \beta_j \phi_j(x) = \beta^T \phi(x)$$

Polynomial Basis Functions

We can look at OLS fits with polynomial basis functions of increasing order.

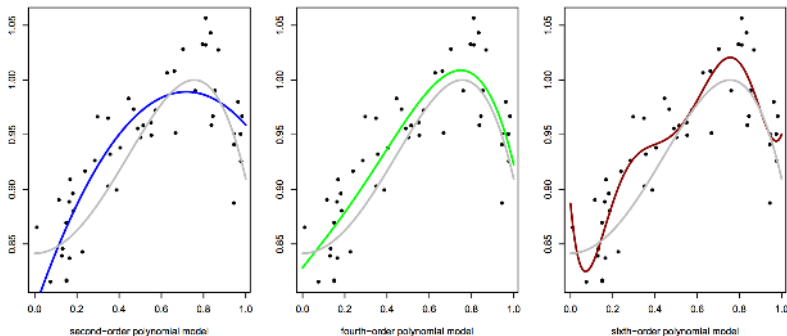
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It appears that the last model is too complex and is overfitting a bit.

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One choice is a Gaussian basis function

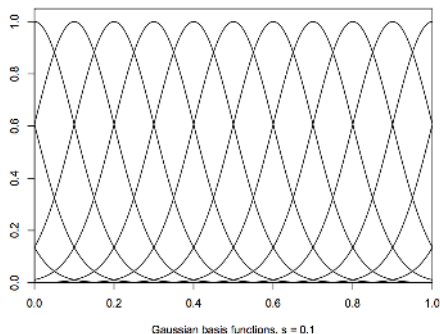
$$\phi_j(x) = \exp(-(x - \mu_j)^2/2s^2)$$

Local Basis Functions

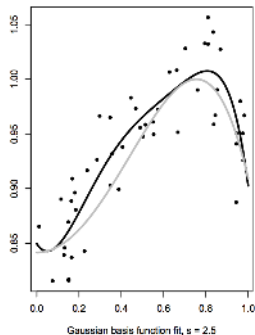
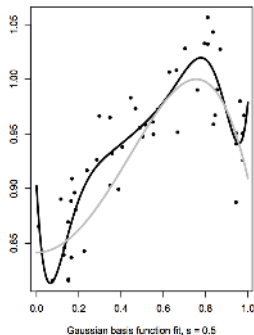
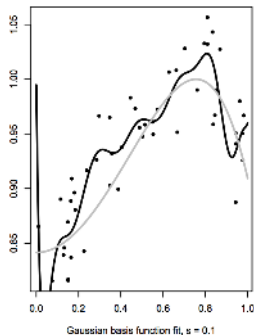
Polynomials are **global** basis functions, each affecting the prediction over the whole input space. Often **local** basis functions are more appropriate.

One choice is a Gaussian basis function

$$\phi_j(x) = \exp(-(x - \mu_j)^2 / 2s^2)$$



Gaussian Basis Fits



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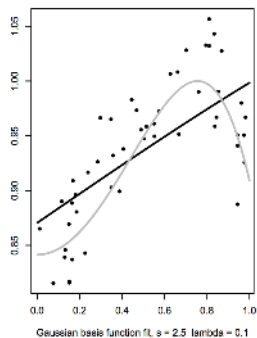
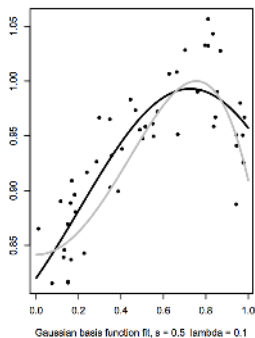
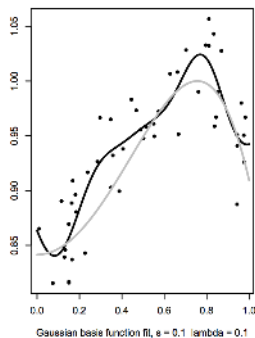
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- The trick in general is how to set λ

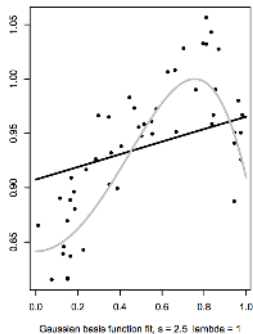
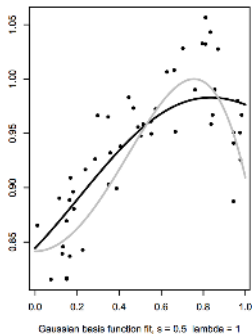
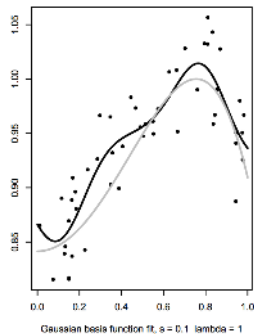
Results

Here are the results with $\lambda = 0.1$:



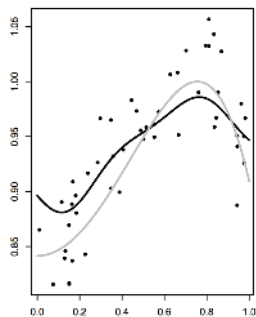
Results

Here are the results with $\lambda = 1$:

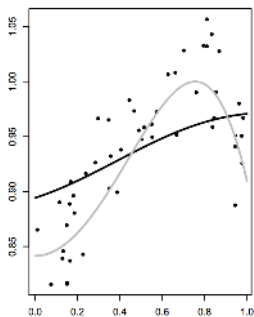


Results

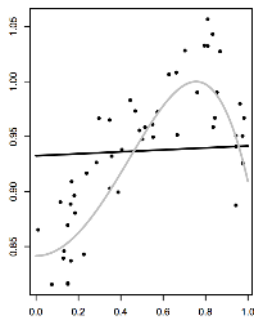
Here are the results with $\lambda = 10$:



Gaussian basis function fit, $\epsilon = 0.1$ $\lambda = 10$



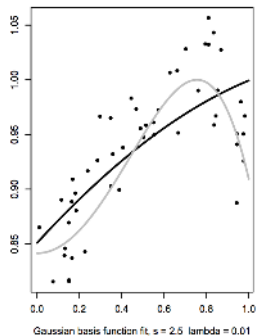
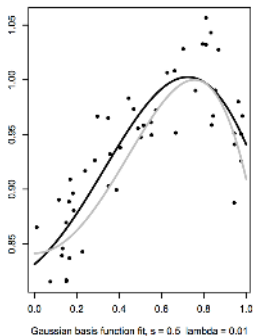
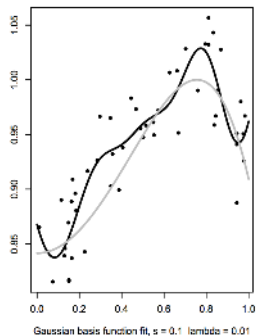
Gaussian basis function fit, $\epsilon = 0.5$ $\lambda = 10$



Gaussian basis function fit, $\epsilon = 2.5$ $\lambda = 10$

Results

Here are the results with $\lambda = 0.01$:



Conclusions from This Example

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- next up, **Generalized Additive Models**

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- Theory and estimation are somewhat involved, but they are easy to use:
 - ▶ `gam.out <- gam(y~s(x1)+s(x2)+x3)`
`plot(gam.out)`
 - ▶ Multiple functions but I recommend `mgcv` package

Generalized Additive Models (GAM)

The GAM approach can be extended to allow **interactions** ($s_{12}(\cdot)$) between explanatory variables, but this eats up degrees of freedom so you need a lot of data.

$$y_i = \beta_0 + s_{12}(x_{1i}, x_{2i}) + s_3(x_{3i}) + u_i$$

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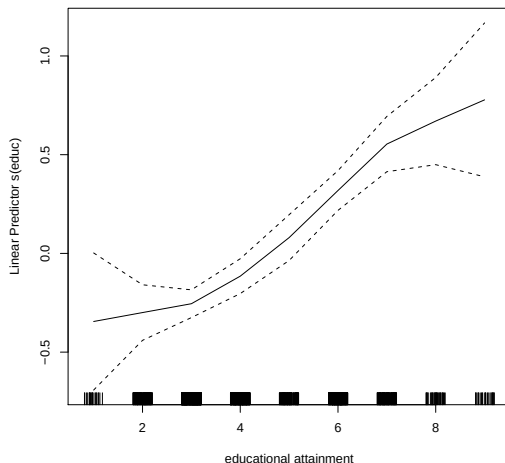
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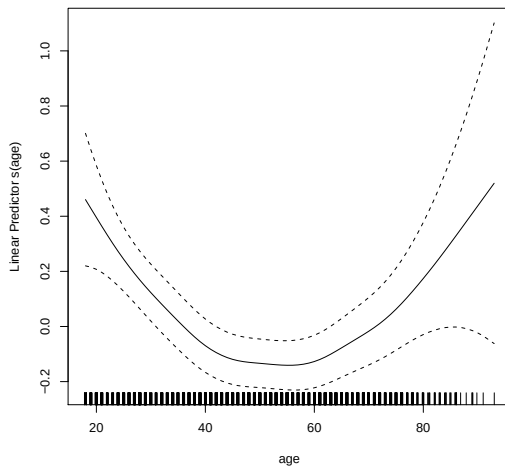
It can also be used for **hybrid models** where we model some variables as parametrically and other with a flexible function:

$$y_i = \beta_0 + \beta_1 x_{1i} + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$

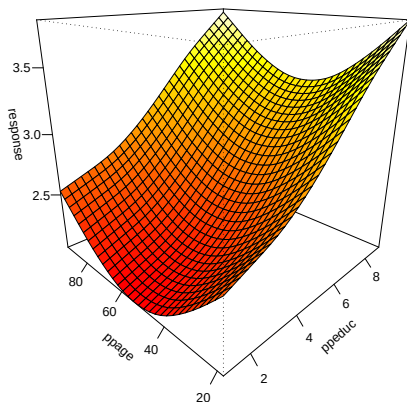
GAM Fit to Attitudes Toward Immigration



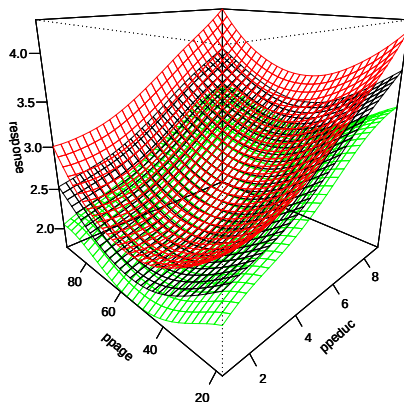
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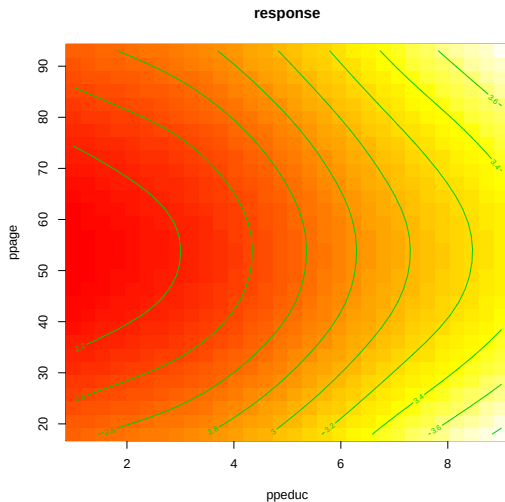


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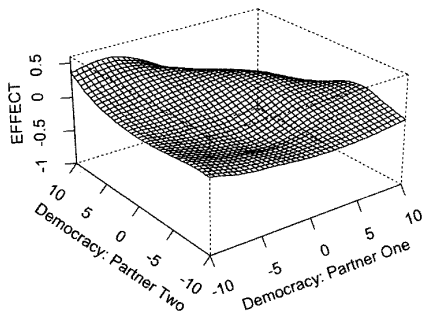
red/green are ± 2 s.e.

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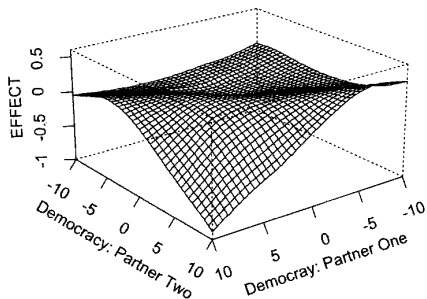


GAM Fit to Dyadic Democracy and Militarized Disputes

(a) Perspective of Non-Democracies



(b) Perspective of Democracies



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- However, be wary of the **global** properties of transformations and polynomials
- Non-linearity concerns are most relevant for continuous covariates with a large range (age)

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Fun With Kernels

Hainmueller and Hazlett (2013). “Kernel Regularized Least Squares: Reducing Misspecification Bias with a Flexible and Interpretable Machine Learning Approach” *Political Analysis*.²

²I thank Chad Hazlett for sharing many of the slides that follow

Motivation: Misspecification Bias

Consider a data generating process such as:

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> # Predictors
> GDP = runif(500)
> Polity = .5*GDP^2 + .2*runif(200)
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> # OLS
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Entirely wrong conclusions!

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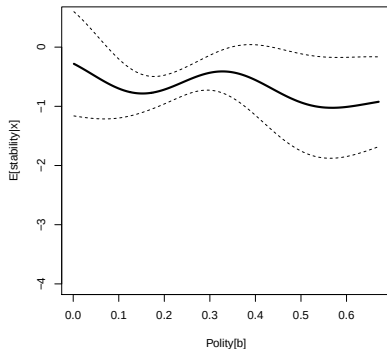
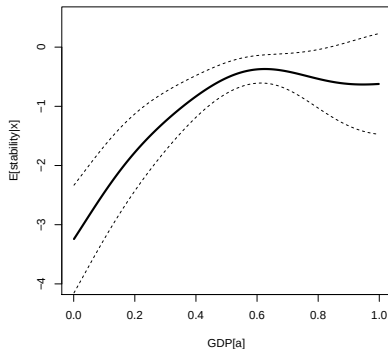
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$E[\text{stability}|\text{gdp, mean}(\text{polity})]$ $E[\text{stability}|\text{mean}(\text{gdp}), \text{polity}]$



Kernel Basics

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For now, a kernel is a function $\mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}$

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Some kernels are naturally interpretable as a distance metric, e.g. the Gaussian:

Gaussian Kernel

$$k(\cdot, \cdot) : \mathbb{R}^D \times \mathbb{R}^P \mapsto \mathbb{R}$$

$$k(x_j, x_i) = e^{-\frac{\|x_j - x_i\|^2}{\sigma^2}}$$

where $\|x_j - x_i\|$ is the Euclidean distance between x_j and x_i

Using the Kernel Trick for Regression

- A feature map, $\phi : \mathbb{R}^P \mapsto \mathbb{R}^{P'}$, such that: $k(X_i, X_j) = \langle \phi(X_i), \phi(X_j) \rangle$

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- Solve the F.O.C.s:

$$R(\theta, \lambda) = \sum_{i=1}^N (Y_i - \phi(X_i)^T \theta)^2 + \lambda \theta^T \theta$$

$$\frac{\partial R(\theta, \lambda)}{\partial \theta} = -2 \sum_{i=1}^N \phi(X_i) (Y_i - \phi(X_i)^T \theta) + 2\lambda \theta = 0$$

How would humans learn this?



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Linear regression?

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Similarity model:

$$E[alt|lat, long] = c_1(\text{similarity to obs1}) + \dots + c_5(\text{similarity to obs5})$$

Intuition: Similarity

Think of this function space as built on similarity:

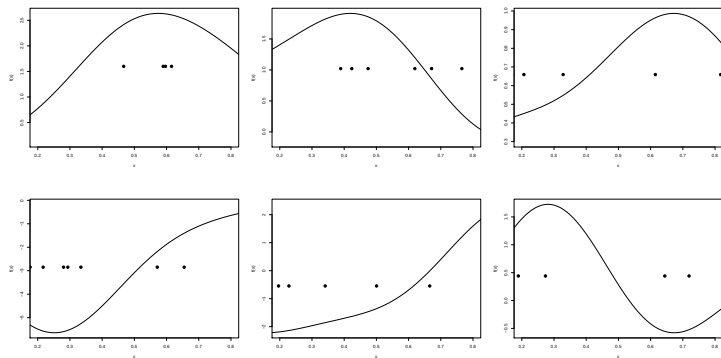
$$\begin{aligned} f(X^*) &= \sum_{i=1}^N c_i k(X^*, X_i) \\ &= c_1(\text{similarity of } X^* \text{ to } X_1) + \dots + c_N(\text{similarity of } X^* \text{ to } X_N) \end{aligned}$$

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Some random functions from this space:



A real example: Harff 2003

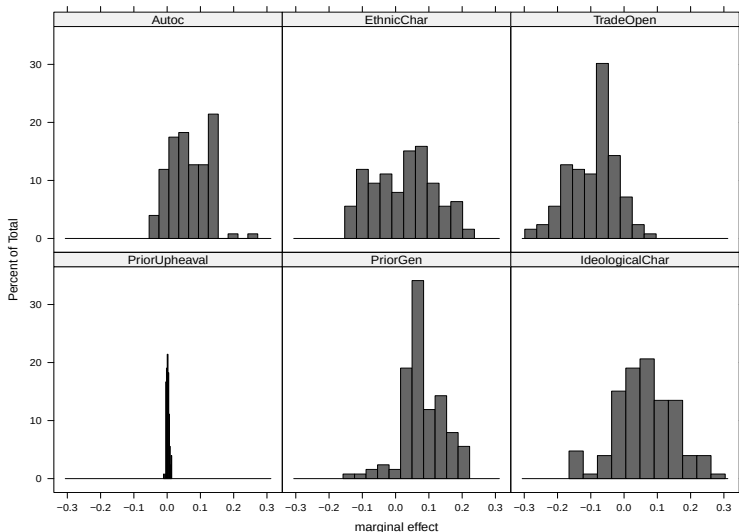
From `summary(krls(y,X))`

DV: Genocide onset		
	β_{OLS}	$E[\frac{\hat{dy}}{dx_i}]$
Prior upheaval	0.009* (0.004)	0.00 0.00
Prior genocide	0.26* (0.12)	0.19* (0.08)
Ideological char. elite	0.15* (0.084)	0.13* (0.08)
Autocracy	0.16* (0.077)	0.12* (0.07)
Ethnic char. elite	0.12 (0.084)	0.05 (0.08)
log(trade openness)	-0.17* (0.057)	-0.09* (0.03)

Behind the averages

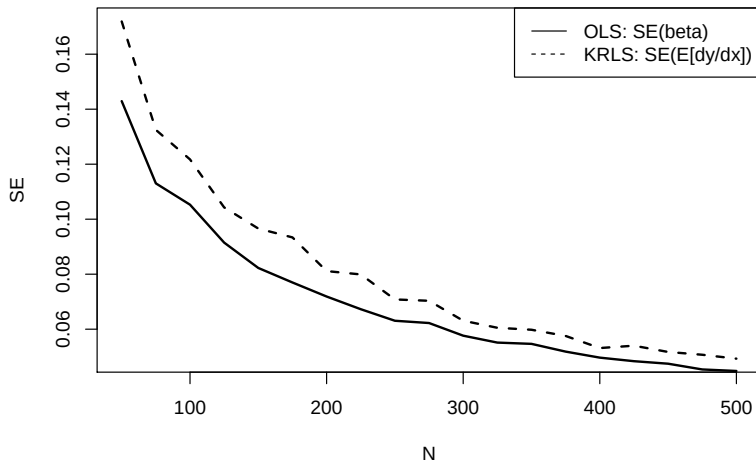
```
plot(krls(X,y))
```

Distributions of pointwise marginal effects

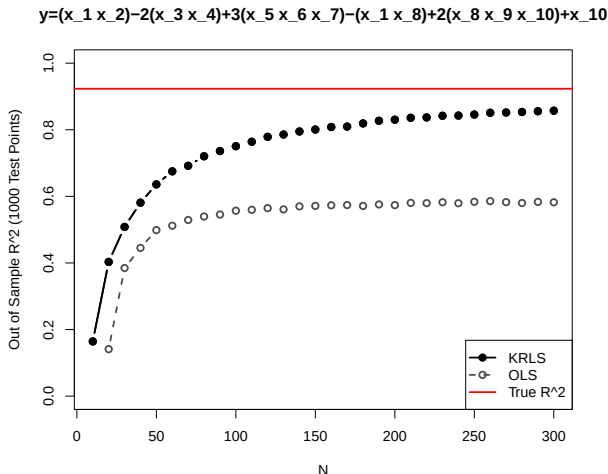


Efficiency Comparison

$$y = 2x + \epsilon, x \sim N(0, 1), \epsilon \sim N(0, 1)$$



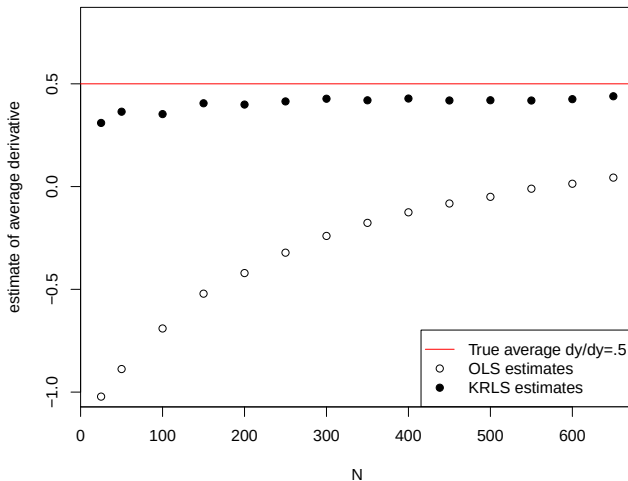
High-dimensional data with non-linearities



$y = (X_1 X_2) - 2(X_3 X_4) + 3(X_5 X_6 X_7) - (X_1 X_8) + 2(X_8 X_9 X_{10}) + X_{10} + \epsilon$ where all X are i.i.d. $Bernoulli(p)$ at varying p , $\epsilon \sim N(0, .5)$. 1,000 test points.

Linear model with bad leverage points

- $y = .5x + \varepsilon$ where $\varepsilon \sim N(0, .3)$
- One bad point, $(y_i = -5, x_i = 5)$.



Interaction or non-linearity?

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 $\varepsilon \sim (0, .44)$. $x_1 \sim \text{Uniform}(0, 2)$

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Estimator	OLS	KRLS			
$\partial y / \partial x_{ij}$	Average	Average	1st Qu.	Median	3rd Qu.
const	-1.50 (0.34)				
x_1	7.51 (0.40)	9.22 (0.52)	5.22 (0.82)	9.38 (0.85)	14.03 (0.79)
x_2	-1.28 (0.21)	0.02 (0.13)	-0.08 (0.19)	0.00 (0.16)	0.10 (0.20)
$(x_1 \cdot x_2)$	1.24 (0.15)				
N	250				

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- ▶ it may model deep interactions but it is still hard to summarize deep interactions

References

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Questions?

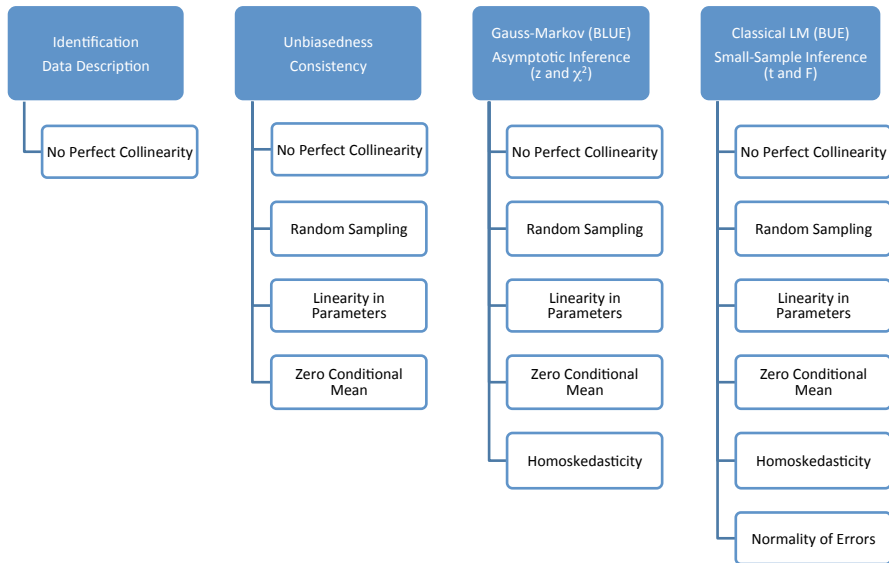
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A Quick Note of Thanks



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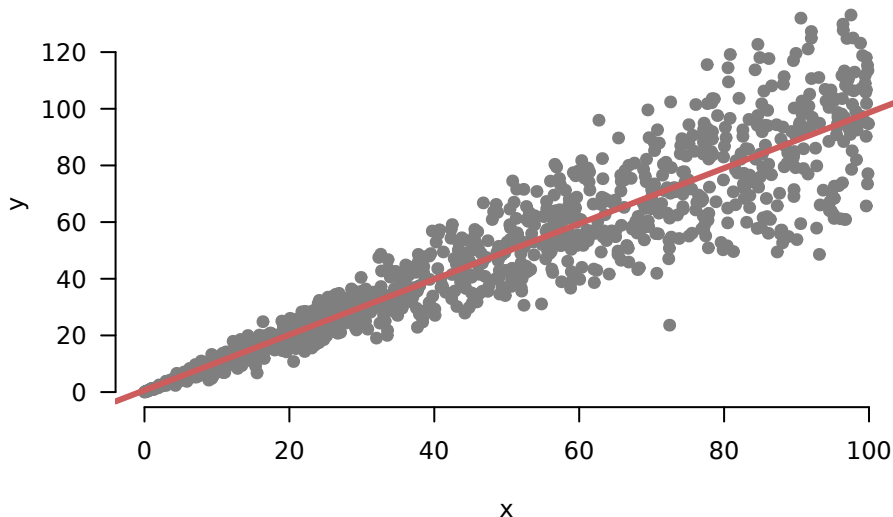
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How Do We Deal With This?



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Then we will discuss a **contrarian** view

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- Replace σ^2 with estimate $\hat{\sigma}^2$ will give us our estimate of the covariance matrix

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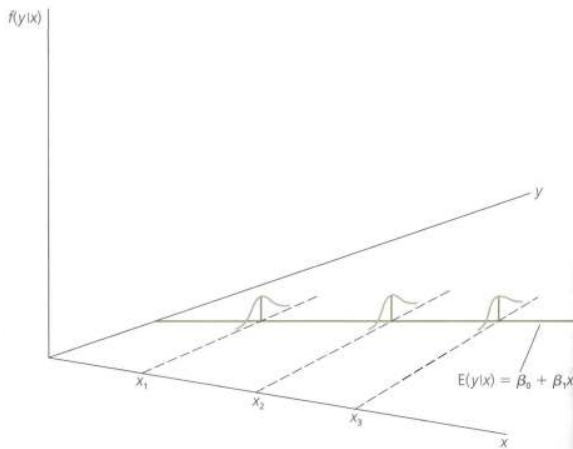
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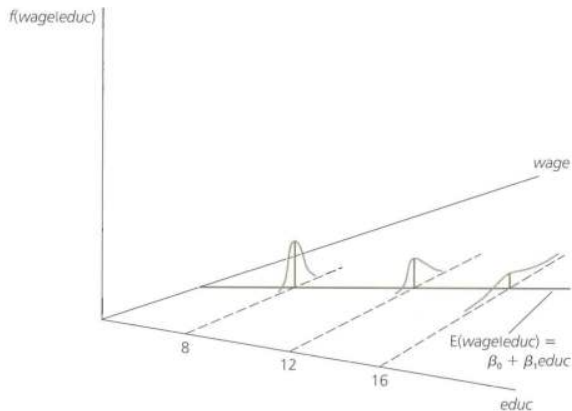
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Visual diagnostics

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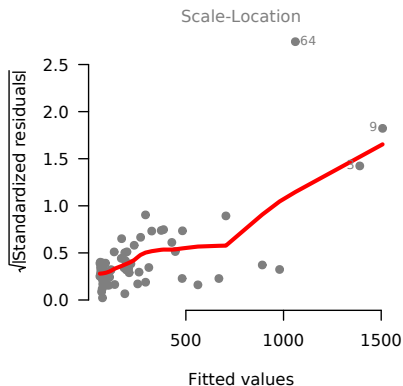
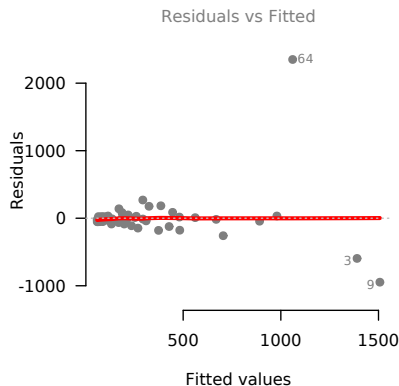
Example: Buchanan votes

```
flvote <- foreign::read.dta("flbuchan.dta")
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)

##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 5.423e+01  4.914e+01   1.104   0.274
## edaytotal    2.323e-03  3.104e-04   7.483 2.42e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 332.7 on 65 degrees of freedom
## Multiple R-squared:  0.4628, Adjusted R-squared:  0.4545
## F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10
```

Diagnostics

```
par(mfrow = c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")  
plot(mod, which = 1, lwd = 3)  
plot(mod, which = 3, lwd = 3)
```



Formal Tests for Non-constant Error Variances

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- The **Breusch-Pagan test**:
 - 1 Regression y_i on $\mathbf{x}_i'$ and store residuals, $\hat{u}_i$
 - 2 Regress $\hat{u}_i^2$ on $\mathbf{x}_i'$
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 - 3 Run F -test against null that all slope coefficients are 0
 - ▶ In R, `bptest` in the `lmtest` package

Breusch-Pagan Example

```
library(lmtest)
bptest(mod)

##
##  studentized Breusch-Pagan test
##
## data:  mod
## BP = 12.59, df = 1, p-value = 0.0003878
```

Dealing with Non-Constant Error Variance

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- 1 **Transform** the dependent variable
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- 4 Admit we have the **wrong model** and use a different approach

Variance Stabilizing Transformations

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Examples:

Transformation	Mean/Variance Relationship
$\sqrt{Y}$	$\sigma_i^2 \propto \mathbf{x}_i\boldsymbol{\beta}$
$\log Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^2$
$1/Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^4$

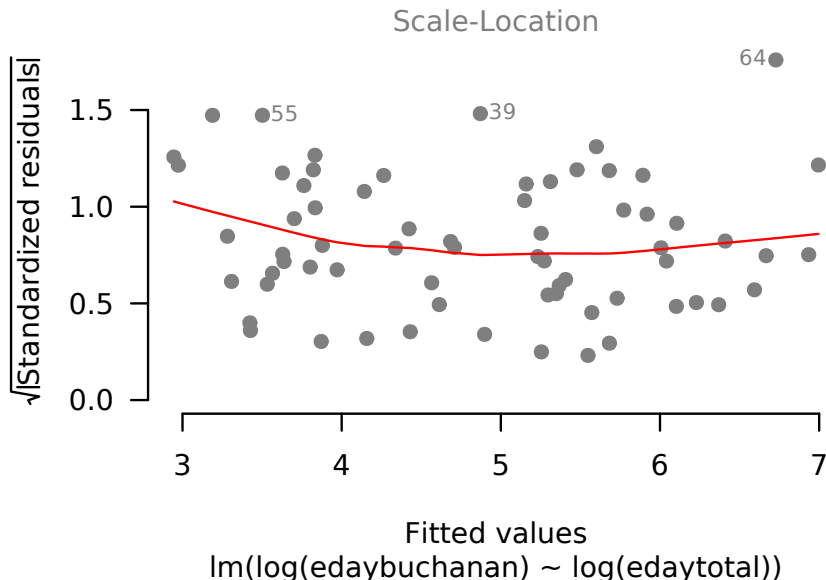
Example: Transforming Buchanan Votes

```
mod2 <- lm(log(edaybuchanan) ~ log(edaytotal), data = flvotef)
summary(mod2)

##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   -2.72789    0.39956  -6.827  3.5e-09 ***
## log(edaytotal)  0.72853    0.03803  19.154 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4688 on 65 degrees of freedom
## Multiple R-squared:  0.8495, Adjusted R-squared:  0.8472
## F-statistic: 366.9 on 1 and 65 DF,  p-value: < 2.2e-16
```

Example: Transformed Scale-Location Plot

```
plot(mod2, which=3)
```



Example: Transformed

```
bptest(mod, studentize=FALSE)
```

```
##  
## Breusch-Pagan test  
##  
## data:  mod  
## BP = 250.07, df = 1, p-value < 2.2e-16
```

```
bptest(mod2, studentize=FALSE)
```

```
##  
## Breusch-Pagan test  
##  
## data:  mod2  
## BP = 0.01105, df = 1, p-value = 0.9163
```

Appendix: Weighted Least Squares

- Suppose that the heteroskedasticity is known up to a multiplicative constant:

$$\text{Var}[u_i|\mathbf{X}] = a_i\sigma^2$$

where $a_i = a_i(\mathbf{x}'_i)$ is a positive and known function of $\mathbf{x}'_i$

- WLS: multiply y_i by $1/\sqrt{a_i}$:

$$y_i/\sqrt{a_i} = \beta_0/\sqrt{a_i} + \beta_1 x_{i1}/\sqrt{a_i} + \cdots + \beta_k x_{ik}/\sqrt{a_i} + u_i/\sqrt{a_i}$$

Appendix: Weighted Least Squares Intuition

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- Rescales errors to $u_i/\sqrt{a_i}$, which maintains zero mean error
- But makes the error variance constant again:

$$\text{Var} \left[\frac{1}{\sqrt{a_i}} u_i | \mathbf{X} \right] = \frac{1}{a_i} \text{Var} [u_i | \mathbf{X}]$$

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- But makes the error variance constant again:

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- But makes the error variance constant again:

$$\begin{aligned}\text{Var} \left[\frac{1}{\sqrt{a_i}} u_i | \mathbf{X} \right] &= \frac{1}{a_i} \text{Var} [u_i | \mathbf{X}] \\ &= \frac{1}{a_i} a_i \sigma^2 \\ &= \sigma^2\end{aligned}$$

- If you know a_i , then you can use this approach to makes the model homoskedastic and, thus, BLUE again
- When do we know a_i ?

Appendix: Weighted Least Squares procedure

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- Define the weighting matrix:

$$\mathbf{W} = \begin{bmatrix} 1/\sqrt{a_1} & 0 & 0 & 0 \\ 0 & 1/\sqrt{a_2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1/\sqrt{a_n} \end{bmatrix}$$

- Run the following regression:

$$\mathbf{W}\mathbf{y} = \mathbf{W}\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{u}$$

$$\mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \mathbf{u}^*$$

- Run regression of $\mathbf{y}^* = \mathbf{W}\mathbf{y}$ on $\mathbf{X}^* = \mathbf{W}\mathbf{X}$ and all Gauss-Markov assumptions are satisfied
- Plugging into the usual formula for $\hat{\boldsymbol{\beta}}$:

$$\hat{\boldsymbol{\beta}}_W = (\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{y}$$

Appendix: WLS Example

- In R, use `weights = argument to lm` and give the weights squared: $1/a_i$
- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

Appendix: WLS Example

- In R, use `weights =` argument to `lm` and give the weights squared: $1/a_i$
- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

```
mod.wls <- lm(edaybuchanan ~ edaytotal, weights = 1/edaytotal,  
              data = flvot)
```

```
summary(mod.wls)
```

```
##
```

```
## Coefficients:
```

```
##           Estimate Std. Error t value Pr(>|t|)
```

```
## (Intercept) 2.707e+01  8.507e+00   3.182  0.00225 **
```

```
## edaytotal    2.628e-03  2.502e-04  10.503 1.22e-15 ***
```

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
##
```

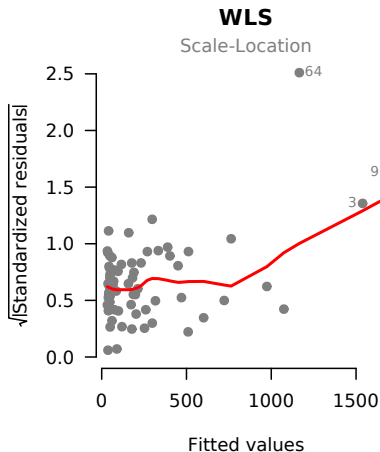
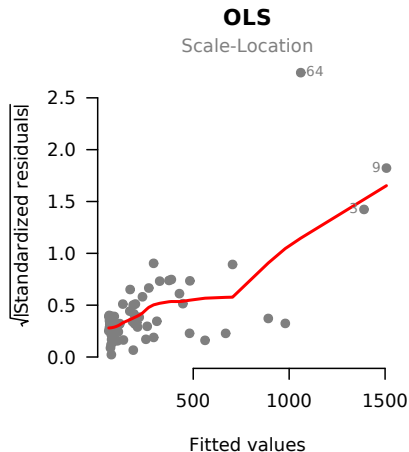
```
## Residual standard error: 0.5645 on 65 degrees of freedom
```

```
## Multiple R-squared:  0.6292, Adjusted R-squared:  0.6235
```

```
## F-statistic: 110.3 on 1 and 65 DF,  p-value: 1.22e-15
```

Appendix: Comparing WLS to OLS

```
par(mfrow=c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")  
plot(mod, which = 3, main = "OLS", lwd = 2)  
plot(mod.wls, which = 3, main = "WLS", lwd = 2)
```



Heteroskedasticity Consistent Estimator

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- Under non-constant error variance:

$$\text{Var}[\mathbf{u}] = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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- When $\Sigma \neq \sigma^2 \mathbf{I}$, we are stuck with this expression:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\Sigma\mathbf{X} (\mathbf{X}'\mathbf{X})^{-1}$$

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$$\text{Var}[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\Sigma\mathbf{X} (\mathbf{X}'\mathbf{X})^{-1}$$

- Idea: If we can consistently estimate the components of Σ , we could directly use this expression by replacing Σ with its estimate, $\hat{\Sigma}$.

White's Heteroskedasticity Consistent Estimator

Suppose we have heteroskedasticity of unknown form:

$$V[\mathbf{u}] = \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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then $V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\Sigma\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$ and White (1980) shows that

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$$\widehat{V[\hat{\beta}|\mathbf{X}]} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{u}_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \hat{u}_n^2 \end{bmatrix} \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

is a consistent estimator of $V[\hat{\beta}|\mathbf{X}]$ **under any form of heteroskedasticity** consistent with $V[\mathbf{u}]$ above.

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is a consistent estimator of $V[\hat{\beta}|\mathbf{X}]$ **under any form of heteroskedasticity** consistent with $V[\mathbf{u}]$ above.

The estimate based on the above is called the **heteroskedasticity consistent (HC)** or **robust standard errors**.

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- 1 Fit the regression and obtain the residuals $\hat{\mathbf{u}}$
- 2 Construct the “meat” matrix $\hat{\Sigma}$ with squared residuals in diagonal:

$$\hat{\Sigma} = \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{u}_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \hat{u}_n^2 \end{bmatrix}$$

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- 3 Plug $\hat{\Sigma}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix

$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\hat{\Sigma}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

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$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\hat{\Sigma}\mathbf{X} (\mathbf{X}'\mathbf{X})^{-1}$$

- There are various **small sample corrections** to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|\mathbf{X}] = \frac{n}{n-k-1} \cdot (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\hat{\Sigma}\mathbf{X} (\mathbf{X}'\mathbf{X})^{-1}$$

Regular & Robust Standard Errors in Florida Example

R Code

```
> library(sandwich)
> library(lmtest)
> coeftest(mod1) # homoskedasticity
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.9141e+01  1.1036   0.2738
TotalVotes00 2.3229e-03 3.1041e-04  7.4831 2.417e-10 ***

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC0")) # classic White
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.0612e+01  1.3353  0.18642
TotalVotes00 2.3229e-03 8.7047e-04  2.6685  0.00961 **

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC1")) # small sample correction
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.1232e+01  1.3153  0.19304
TotalVotes00 2.3229e-03 8.8376e-04  2.6284  0.01069 *
```

WLS vs. White's Estimator

- WLS:

WLS vs. White's Estimator

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 - ▶ With known weights, WLS is efficient

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 - ▶ but **weights usually aren't known**
- White's Estimator:

WLS vs. White's Estimator

- WLS:
 - ▶ With known weights, WLS is efficient
 - ▶ and $\widehat{SE}[\widehat{\beta}_{WLS}]$ is unbiased and consistent
 - ▶ but **weights usually aren't known**
- White's Estimator:
 - ▶ **Doesn't change estimate $\widehat{\beta}$**

WLS vs. White's Estimator

- WLS:

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- ▶ but **weights usually aren't known**

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- ▶ Because it relies on consistency, it is a **large sample result**, best with large n
- ▶ For small n , performance might be poor (correction factors exist but are often insufficient)

- 1 Assumptions and Violations
- 2 Non-normality
- 3 Outliers
- 4 Robust Regression Methods
- 5 Optional: Measurement Error
- 6 Conclusion and Appendix
- 7 Detecting Nonlinearity
- 8 Linear Basis Function Models
- 9 Generalized Additive Models
- 10 Fun With Kernels
- 11 Heteroskedasticity
- 12 Clustering
- 13 Optional: Serial Correlation
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- Called clustering or clustered dependence

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- Ignoring clustering is “cheating”: units not independent

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- $\rho \in (0, 1)$ is called the within-cluster correlation.
- What if we ignore this structure and just use ε_{ij} as the error?
- Variance of the composite error is σ^2 :

$$\begin{aligned}\text{Var}[\varepsilon_{ij}] &= \text{Var}[v_j + u_{ij}] \\ &= \text{Var}[v_j] + \text{Var}[u_{ij}] \\ &= \rho\sigma^2 + (1 - \rho)\sigma^2 = \sigma^2\end{aligned}$$

Lack of Independence

- Covariance between two units i and s in the same cluster is $\rho\sigma^2$:

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sj}] = \rho\sigma^2$$

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- Zero covariance of two units i and s in different clusters j and k :

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sk}] = 0$$

Example Covariance Matrix

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{bmatrix}'$$

$$\text{Var}[\boldsymbol{\varepsilon}] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 \end{bmatrix}$$

Appendix: Example 6 Units, 2 Clusters

$$\varepsilon = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{bmatrix}'$$

$$V[\varepsilon] = \Sigma = \begin{bmatrix} V[\varepsilon_{1,1}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{1,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{1,1}] & \cdot & \cdot & \cdot \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{2,1}] & V[\varepsilon_{2,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{2,1}] & \cdot & \cdot & \cdot \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{3,1}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{3,1}] & V[\varepsilon_{3,1}] & \cdot & \cdot & \cdot \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{4,2}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{4,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{4,2}] & V[\varepsilon_{4,2}] & \cdot & \cdot \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{4,2}, \varepsilon_{5,2}] & V[\varepsilon_{5,2}] & \cdot \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{4,2}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{5,2}, \varepsilon_{6,2}] & V[\varepsilon_{6,2}] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 \end{bmatrix}$$

which can be verified as follows:

- $V[\varepsilon_{ij}] = V[v_j + u_{ij}] = V[v_j] + V[u_{ij}] = \rho\sigma^2 + (1 - \rho)\sigma^2 = \sigma^2$
- $\text{Cov}[\varepsilon_{ij}, \varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] - E[\varepsilon_{ij}]E[\varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] = E[(v_j + u_{ij})(v_j + u_{lj})]$
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 $= E[v_j v_k] + E[v_j u_{lk}] + E[v_k u_{ij}] + E[u_{ij} u_{lk}]$
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 - ▶ If n_j varies by cluster, then cluster-level errors will have heteroskedasticity
 - ▶ Can use WLS with cluster size as the weights

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- Under this clustered dependence, we can write this as:

$$\text{Var}[\hat{\boldsymbol{\beta}}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{j=1}^m \mathbf{x}'_j \boldsymbol{\Sigma}_j \mathbf{x}_j \right) (\mathbf{X}'\mathbf{X})^{-1}$$

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- 3 The within cluster correlation is then estimated as: $\hat{\rho} = \frac{\hat{\sigma}^2 - \hat{\tilde{\sigma}}^2}{\hat{\sigma}^2}$

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- 2 Plug $\widehat{\Sigma}$ into the sandwich estimator to obtain the cluster “corrected” estimator of the variance-covariance matrix

$$V[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\widehat{\Sigma}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

Estimating Cluster Robust Standard Errors

We can now compute the CRSEs using our sandwich formula:

- 1 Take your estimates of $\widehat{\sigma}^2$ and $\widehat{\rho}$ and construct the block diagonal variance-covariance matrix $\widehat{\Sigma}$:

$$\widehat{\Sigma} = \begin{bmatrix} \widehat{\Sigma}_1 & 0 & \dots & 0 \\ 0 & \widehat{\Sigma}_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \widehat{\Sigma}_M \end{bmatrix} \quad \text{with } \widehat{\Sigma}_j = \begin{bmatrix} \widehat{\sigma}^2 & \widehat{\sigma}^2 \cdot \widehat{\rho} & \dots & \widehat{\sigma}^2 \cdot \widehat{\rho} \\ \widehat{\sigma}^2 \cdot \widehat{\rho} & \widehat{\sigma}^2 & \dots & \widehat{\sigma}^2 \cdot \widehat{\rho} \\ & & \ddots & \\ \widehat{\sigma}^2 \cdot \widehat{\rho} & \widehat{\sigma}^2 \cdot \widehat{\rho} & \dots & \widehat{\sigma}^2 \end{bmatrix}$$

- 2 Plug $\widehat{\Sigma}$ into the sandwich estimator to obtain the cluster “corrected” estimator of the variance-covariance matrix

$$V[\widehat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\widehat{\Sigma}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

- No canned function for CRSE in R; use our custom function posted on the course website

```
> source("vcovCluster.r")  
> coeftest(model, vcov = vcovCluster(model, cluster = clusterID))
```

Example: Gerber, Green, Larimer

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY — VOTE!

		Aug 04	Nov 04	Aug 06
MAPLE DR				
9995 JOSEPH JAMES SMITH		Voted	Voted	_____
9995 JENNIFER KAY SMITH			Voted	_____
9997 RICHARD B JACKSON			Voted	_____
9999 KATHY MARIE JACKSON			Voted	_____

Social Pressure Model

```
load("gerber_green_larimer.RData")
social$voted <- 1 * (social$voted == "Yes")
social$treatment <- factor(social$treatment,
  levels = c("Control", "Hawthorne", "Civic Duty",
             "Neighbors", "Self"))
mod1 <- lm(voted ~ treatment, data = social)
coeftest(mod1)

##
## t test of coefficients:
##
##              Estimate Std. Error  t value  Pr(>|t|)
## (Intercept)    0.2966383   0.0010612  279.5250 < 2.2e-16 ***
## treatmentHawthorne 0.0257363   0.0026007   9.8958 < 2.2e-16 ***
## treatmentCivic Duty 0.0178993   0.0026003   6.8835 5.849e-12 ***
## treatmentNeighbors 0.0813099   0.0026008  31.2634 < 2.2e-16 ***
## treatmentSelf     0.0485132   0.0026003  18.6566 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Social Pressure Model, CRSEs

Again no canned CRSE in R, so we use our own.

```
source("vcovCluster.R")
coeftest(mod1, vcov = vcovCluster(mod1, "hh_id"))

##
## t test of coefficients:
##
##              Estimate Std. Error  t value  Pr(>|t|)
## (Intercept)    0.2966383   0.0013096 226.5172 < 2.2e-16 ***
## treatmentHawthorne 0.0257363   0.0032579   7.8997 2.804e-15 ***
## treatmentCivic Duty 0.0178993   0.0032366   5.5302 3.200e-08 ***
## treatmentNeighbors 0.0813099   0.0033696  24.1308 < 2.2e-16 ***
## treatmentSelf    0.0485132   0.0033000  14.7009 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

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 - ▶ Block bootstrap can be a useful alternative (key idea: bootstrap by resampling the clusters)

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- 3 Outliers
- 4 Robust Regression Methods
- 5 Optional: Measurement Error
- 6 Conclusion and Appendix
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- 8 Linear Basis Function Models
- 9 Generalized Additive Models
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- Often have **serially correlated**: errors in one time period are correlated with errors in other time periods
- Many different ways for this to happen, but we often assume a very limited type of dependence called AR(1).

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$$y_t = \beta_0 + \beta_1 x_t + u_t$$

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- Typically assume stationarity meaning that $V[u_t]$ and $\text{Cov}[u_t, u_{t+h}]$ are independent of t
- Generalizes to higher order serial correlation (e.g. an AR(2) model is given by $u_t = \rho u_{t-1} + \delta u_{t-2} + e_t$).

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That is, the covariance between errors in $t = 1$ and $t = 2$ is $\frac{\sigma^2}{(1 - \rho^2)}\rho$, between errors in $t = 1$ and $t = 3$ is $\frac{\sigma^2}{(1 - \rho^2)}\rho^2$, etc.

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ρ is usually positive, which implies that we underestimate the variance if we ignore serial correlation.

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- Plot residuals over time (or more fancy “autocorrelation” plots)

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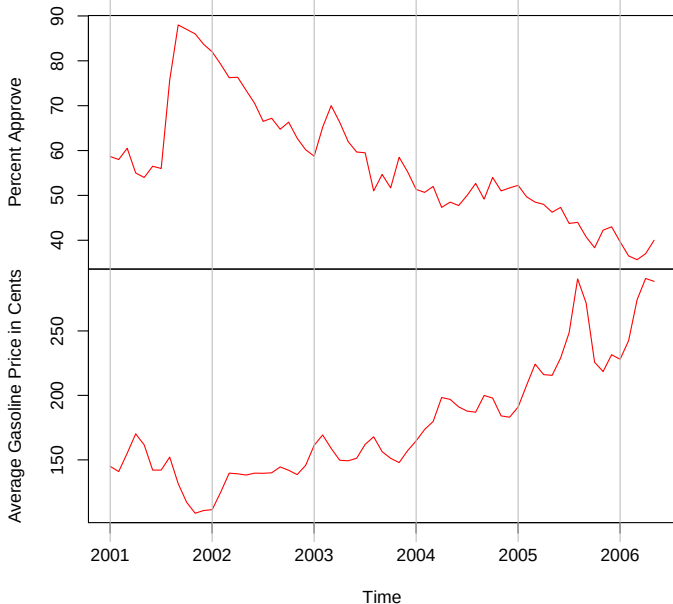
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- First-differencing the data

Monthly Presidential Approval Ratings and Gas Prices



Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(Zelig)
> data(approval)
> mod1 <- lm(approve ~ avg.price, data=approval)
> coeftest(mod1)
```

t test of coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	100.472076	3.567277	28.165	< 2.2e-16 ***
avg.price	-0.243885	0.019465	-12.529	< 2.2e-16 ***

Tests for Serial Correlation: Durbin-Watson

Recall our AR(1) model is:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

where $u_t = \rho u_{t-1} + e_t$, $e_t \sim N(0, \sigma^2)$, and ρ is our unknown **autoregressive coefficient** (with $|\rho| < 1$).

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One common test for serial correlation is the **Durbin-Watson statistic**:

$$DW = \frac{\sum_{t=2}^n \hat{u}_t - \hat{u}_{t-1}}{\sum_{t=1}^n \hat{u}_t^2} \quad \text{where} \quad DW \approx 2(1 - \hat{\rho})$$

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- If $DW \approx 2$ then $\hat{\rho} \approx 0$ (Note that $0 \leq DW \leq 4$)
- If $DW < 1$ we have serious positive serial correlation
- If $DW > 3$ we have serious negative serial correlation

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(lmtest)
> dwtest(approve ~ avg.price, data=approval)

Durbin-Watson test

data:  approve ~ avg.price
DW = 0.4863, p-value = 1.326e-14
alternative hypothesis: true autocorrelation is greater than 0
```

The test suggests strong positive serial correlation. Standard errors are severely downward biased.

Corrections: HAC Standard Errors

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 - ▶ HAC are consistent estimator for $V[\hat{\beta}]$ in the presence of heteroskedasticity and or autocorrelation
 - ▶ The `sandwich` package in R implements a variety of HAC estimators
 - ▶ A common option is `NeweyWest`

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> mod1 <- lm(approve~avg.price,data=approval)
> coeftest(mod1) # homoskedastic errors
t test of coefficients:

              Estimate Std. Error t value Pr(>|t|)
(Intercept) 100.472076   3.567277  28.165 < 2.2e-16 ***
avg.price    -0.243885   0.019465 -12.529 < 2.2e-16 ***

> coeftest(mod1, vcov = NeweyWest) # HAC errors
t test of coefficients:

              Estimate Std. Error t value Pr(>|t|)
(Intercept) 100.472076  14.499337   6.9294 2.652e-09 ***
avg.price    -0.243885   0.071733  -3.3999 0.001174 **
```

Once we correct for autocorrelation, standard errors increase dramatically.

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- “Robust SEs” of various forms are consistent even when these problems are present

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Appendix: Derivation of Error Structure for the AR(1) Model

We have

$$V[u_t] = V[\rho u_{t-1} + e_t] = \rho^2 V[u_{t-1}] + \sigma^2$$

with stationarity, $V[u_t] = V[u_{t-1}]$, and so

$$V[u_t](1 - \rho^2) = \sigma^2 \Rightarrow V[u_t] = \frac{\sigma^2}{(1 - \rho^2)}$$

also

$$\text{Cov}[u_t, u_{t-1}] = E[u_t u_{t-1}] = E[(\rho u_{t-1} + e_t) e_{t-1}] = \rho V[e_{t-1}] = \rho \frac{\sigma^2}{(1 - \rho^2)}$$

or generally

$$\text{Cov}[u_t, u_{t-h}] = \rho^h \frac{\sigma^2}{(1 - \rho^2)}$$

- 1 Assumptions and Violations
- 2 Non-normality
- 3 Outliers
- 4 Robust Regression Methods
- 5 Optional: Measurement Error
- 6 Conclusion and Appendix
- 7 Detecting Nonlinearity
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A Contrarian View of Robust Standard Errors

King, Gary and Margaret E. Roberts. “How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It” *Political Analysis* (2015) 23: 159-179.³

³I thank Gary and Molly for the slides that follow.

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RSEs and SEs are the same

- Consistent with a correctly specified model
- RSEs are not useful, as a “fix”

Their Alternative Procedure

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Robust standard errors:

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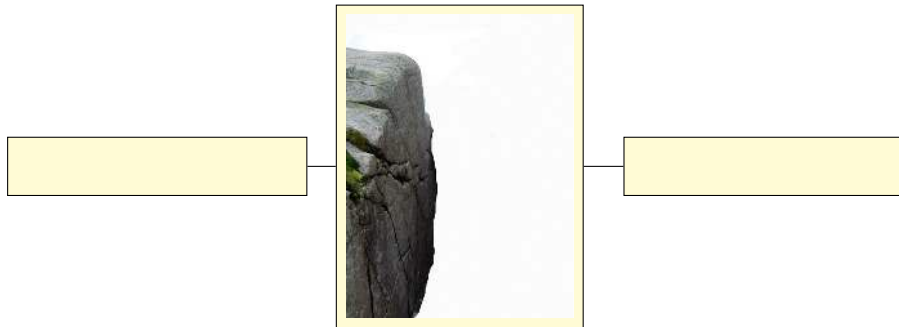
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 - ▶ Use model diagnostics (e.g. residual plots, qq-plots, misspecification tests)
 - ▶ Evaluate misspecification
 - ▶ Respecify the model
- 3 Keeping going, until they don't differ.

For RSEs to help: Everything has to be Juuuussttt Right

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For RSEs to help: Everything has to be Juuussttt Right

Model Correct



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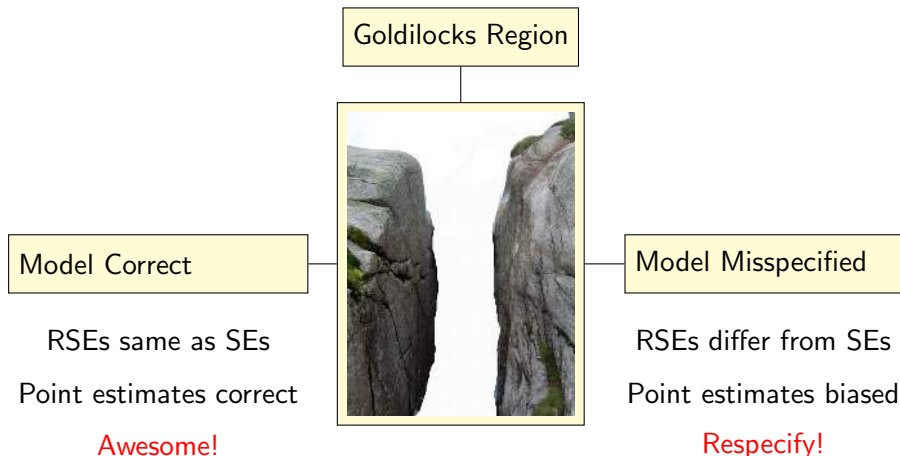
Model Misspecified

RSEs differ from SEs

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Respecify!

For RSEs to help: Everything has to be Juuussttt Right



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Goldilocks Region



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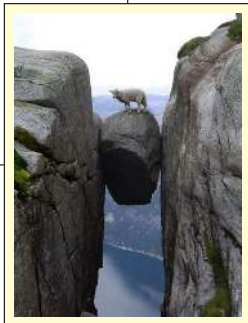
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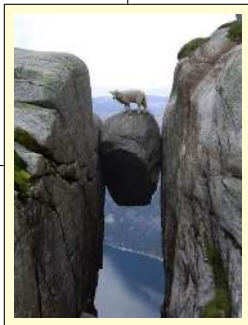
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but not so much as to
bias everything else

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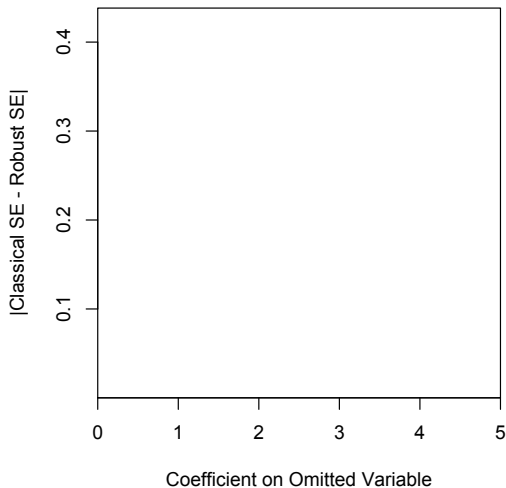
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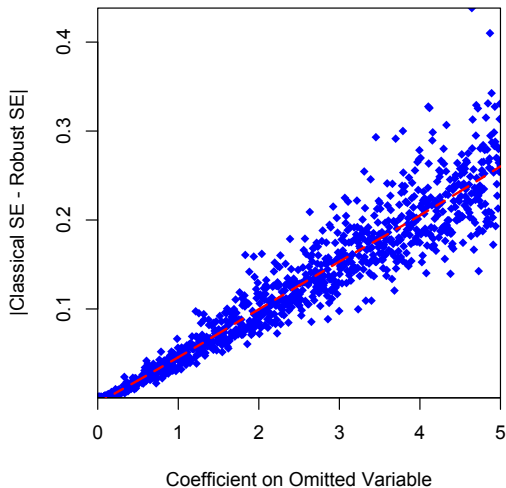
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- Parts of the model are wrong; **why do we think the rest are right?**

Difference Between SE and RSE Exposes Misspecification

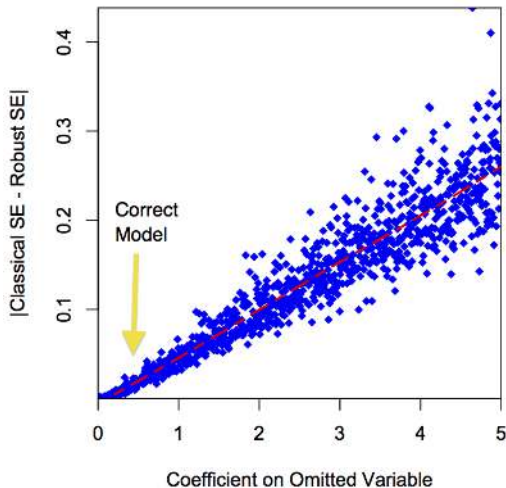
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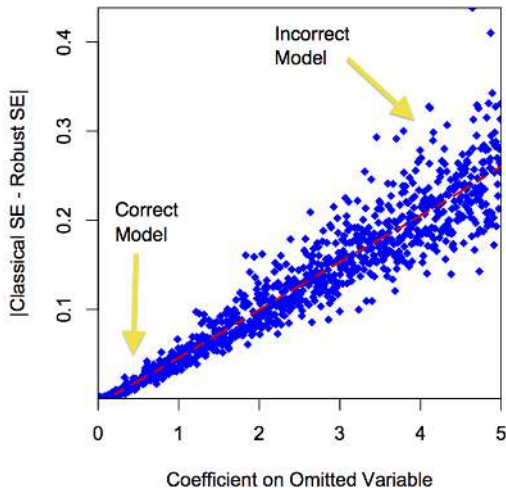
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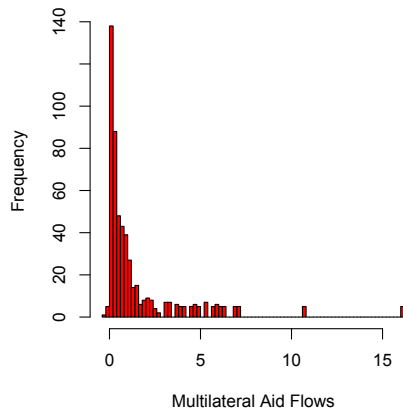
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 - ▶ $\Rightarrow$ indicates model misspecification

Problem: Highly Skewed Dependent Variable

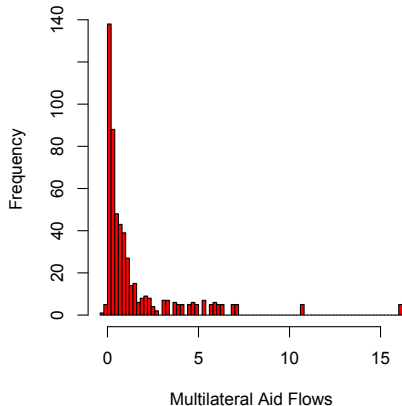
Problem: Highly Skewed Dependent Variable

Original

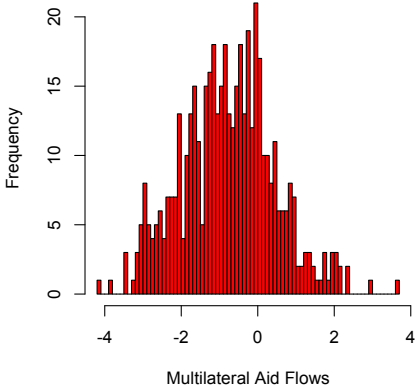


Problem: Highly Skewed Dependent Variable

Original



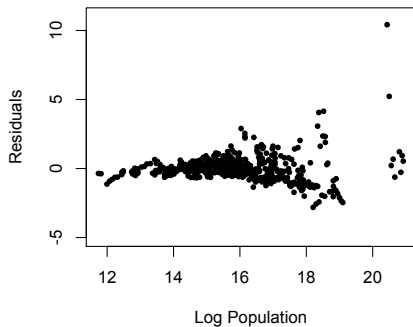
Transformed



Diagnostics: Reveal Misspecification

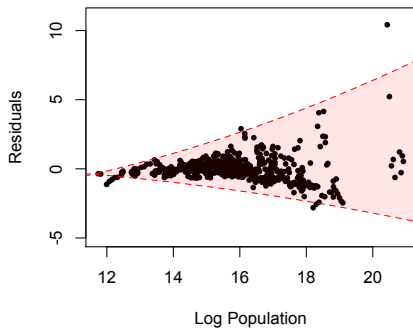
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Population vs Residuals, Author's Model



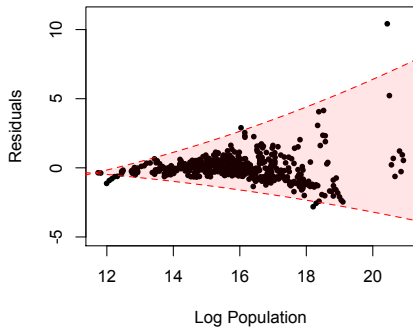
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Population vs Residuals, Author's Model



Diagnostics: Reveal Misspecification

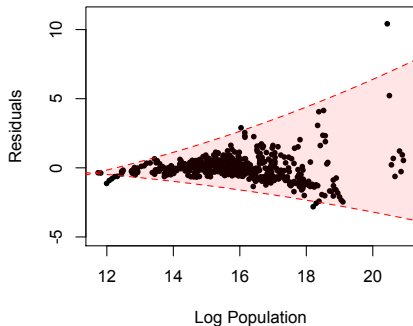
Population vs Residuals, Author's Model



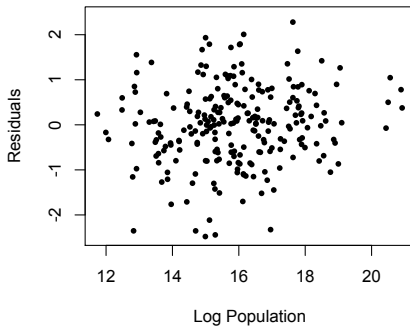
Textbook case of heteroskedasticity

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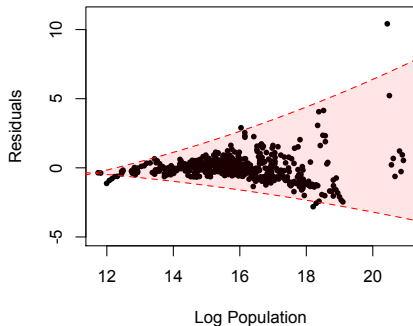
Population vs Residuals, Altered Model



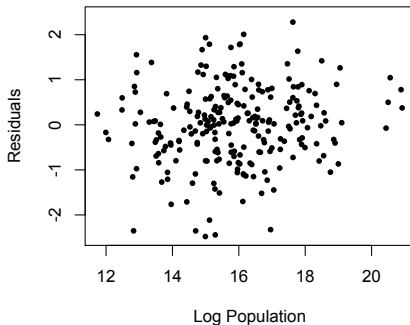
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Diagnostics: Reveal Misspecification

Population vs Residuals, Author's Model



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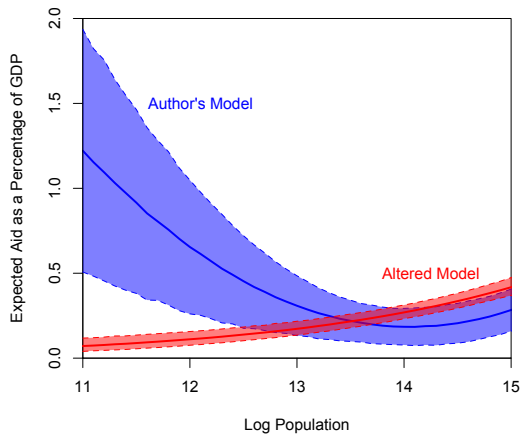


Textbook case of heteroskedasticity

Textbook case of homoskedasticity

After Fix: Different Conclusion

After Fix: Different Conclusion



Concluding Contrarian Thoughts

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Their advice:

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Their advice:

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Concluding Contrarian Thoughts

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- Instead: a sensitive **detector** of misspecification.
- Evaluate misspecification,

Concluding Contrarian Thoughts

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- Evaluate misspecification, by conducting diagnostic tests.

Concluding Contrarian Thoughts

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- Respecify the model,

Concluding Contrarian Thoughts

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Their Examples:

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- Robust SEs indicate fundamental modelling problems

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Their Examples:

- Robust SEs indicate fundamental modelling problems
- Easily identified with diagnostics
- Fixing these problems $\Rightarrow$ **hugely** different substantive conclusions

Concluding Thoughts on Diagnostics

Residuals are **important**. Look at them.

Next Week

- Causality with Measured Confounding
- Reading:
 - ▶ Angrist and Pischke Chapter 2 (The Experimental Ideal) Chapter 3.2 (Regression and Causality)
 - ▶ Morgan and Winship Chapters 3-4 (Causal Graphs and Conditioning Estimators)
 - ▶ Optional: Elwert and Winship (2014) "Endogenous selection bias: The problem of conditioning on a collider variable" *Annual Review of Sociology*
 - ▶ Optional: Morgan and Winship Chapter 11 Repeated Observations and the Estimation of Causal Effects
- As a side note: if you want to read the argument against the contrarian response: Aronow (2016) "A Note on 'How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It.'" It is an interesting piece- feel free to come talk to me about this debate!

Appendix: Derivation of Variance under Homoskedasticity

$$\begin{aligned}\hat{\beta} &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y} \\ &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'(\mathbf{X}\beta + \mathbf{u}) \\ &= \beta + (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}\end{aligned}$$

$$\begin{aligned}V[\hat{\beta}|\mathbf{X}] &= V[\beta|\mathbf{X}] + V[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}|\mathbf{X}] \\ &= V[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{u}|\mathbf{X}] \\ &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' V[\mathbf{u}|\mathbf{X}] ((\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}')' \text{ (note: } \mathbf{X} \text{ nonrandom } |\mathbf{X}) \\ &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' V[\mathbf{u}|\mathbf{X}] \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \sigma^2 \mathbf{I} \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \text{ (by homoskedasticity)} \\ &= \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}\end{aligned}$$

Replacing σ^2 with our estimator $\hat{\sigma}^2$ gives us our estimator for the $(k+1) \times (k+1)$ variance-covariance matrix for the vector of regression coefficients:

$$\widehat{V[\hat{\beta}|\mathbf{X}]} = \hat{\sigma}^2 (\mathbf{X}'\mathbf{X})^{-1}$$

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Fun With Neighbors

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Fun With Neighbors

- We talked about error dependence induced by **time** and by **cluster**.
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Zhukov, Yuri M. and Brandon M. Stewart. "Choosing Your Neighbors: Networks of Diffusion in International Relations" *International Studies Quarterly* 2013; 57: 271-287.

Our Main Questions

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How Do We Generally Choose Neighbors?

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How Do We Generally Choose Neighbors?

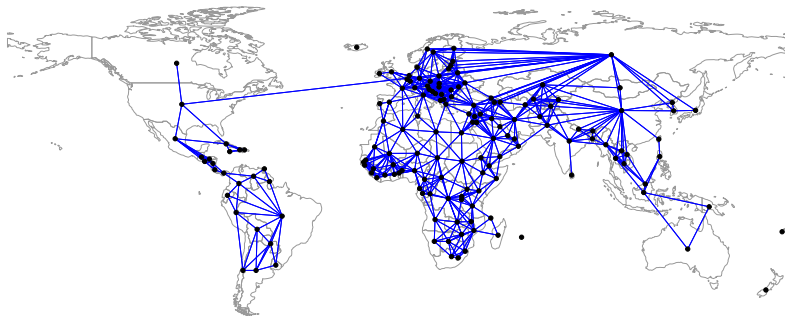
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How Do We Generally Choose Neighbors?

- ① Contiguity is the most common variable
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- ③ Atheoretical choices, rarely justified
- ④ Different types of neighbors tell different stories

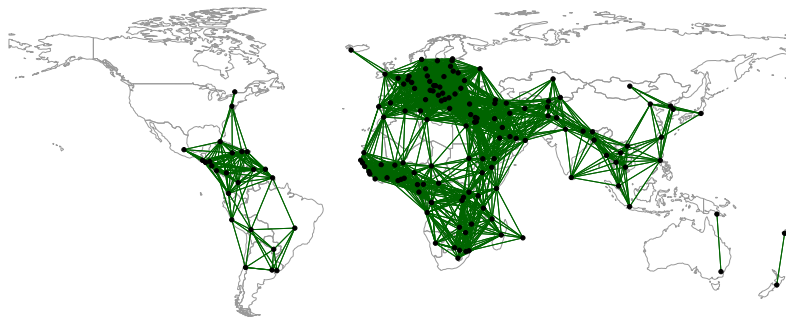
Visualization of Connections: Contiguity

Figure: Contiguity neighbors with 500 km snap distance



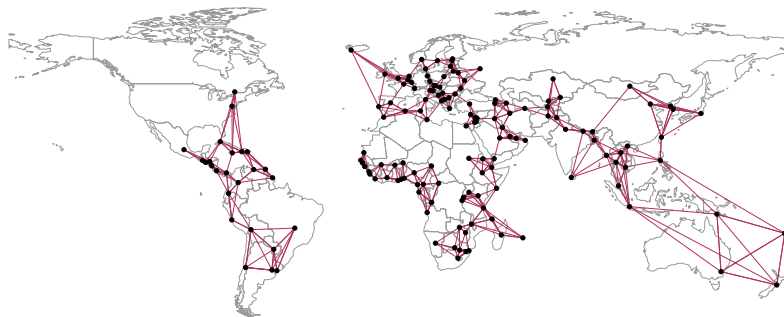
Visualization of Connections: Minimum Distance

Figure: Minimum distance neighbors (capital cities)



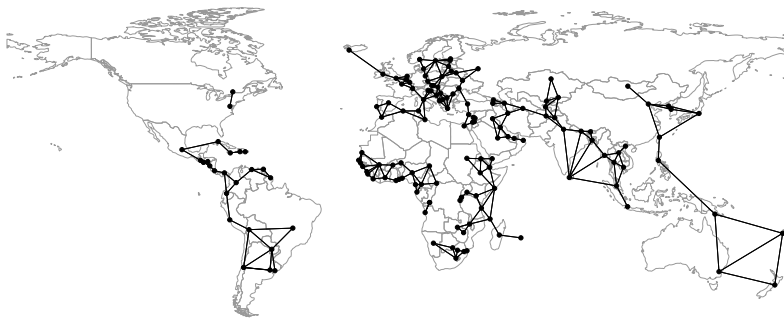
Visualization of Connections: K-Nearest Neighbors

Figure: $k = 4$ Nearest Neighbors (capital cities)



Visualization of Connections: Graph-based Neighbors

Figure: Sphere of Influence Neighbors (capital cities)



Application: Democratic Diffusion

Gleditsch and Ward (2006)

Changes of political regime modeled as a first-order Markov chain process with the transition matrix

$$\mathbf{K} = \begin{bmatrix} Pr(y_{i,t} = 0 | y_{i,t-1} = 0) & Pr(y_{i,t} = 1 | y_{i,t-1} = 0) \\ Pr(y_{i,t} = 0 | y_{i,t-1} = 1) & Pr(y_{i,t} = 1 | y_{i,t-1} = 1) \end{bmatrix}$$

where $y_{i,t} = 1$ if an (A)utocratic regime exists in country i at time t , and $y_{i,t} = 0$ if the regime is (D)emocratic.

... in other words:

$$\mathbf{K} = \begin{bmatrix} Pr(D \rightarrow D) & Pr(D \rightarrow A) \\ Pr(A \rightarrow D) & Pr(A \rightarrow A) \end{bmatrix}$$

Equilibrium Effects of Democratic Transition

If a regime transition takes place in country i , what is the change in predicted probability of a regime transition in country j (country i 's neighbor)?

$$QI = Pr(y_{j,t} | y_{i,t} = y_{i,t-1}) - Pr(y_{j,t} | y_{i,t} \neq y_{i,t-1})$$

where $y_{i,t} = 0$ if country i is a democracy at time t and $y_{i,t} = 1$ if it is an autocracy. All other covariates are held constant.

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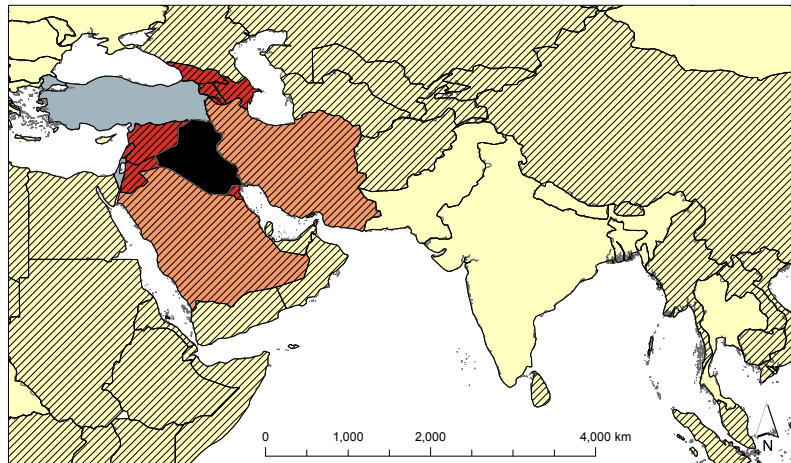
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Illustrative cases

- Iraq transitions from autocracy to democracy.
- Russia transitions from democracy to autocracy.

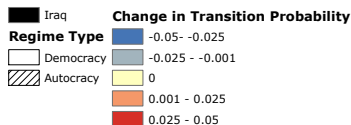
Iraq's democratization and regional regime stability



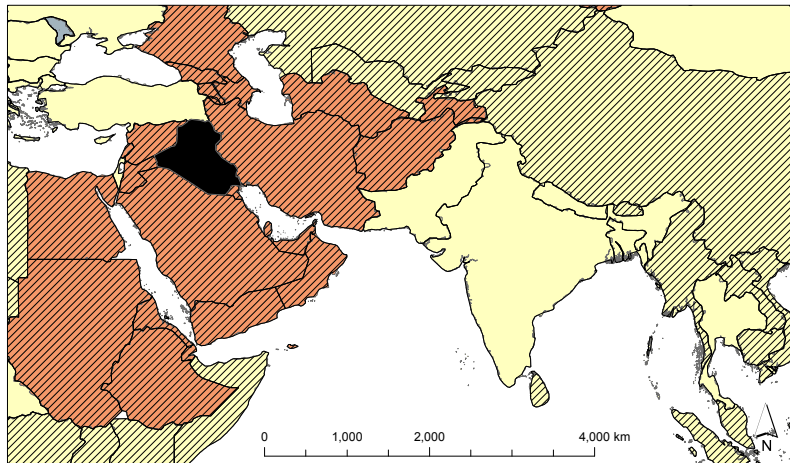
Contiguity + 500 km

Iraq transitions from autocracy to democracy
(1998 data)

Monte Carlo simulation (1,000 runs)



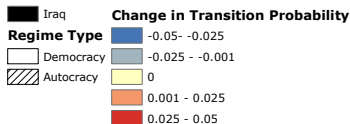
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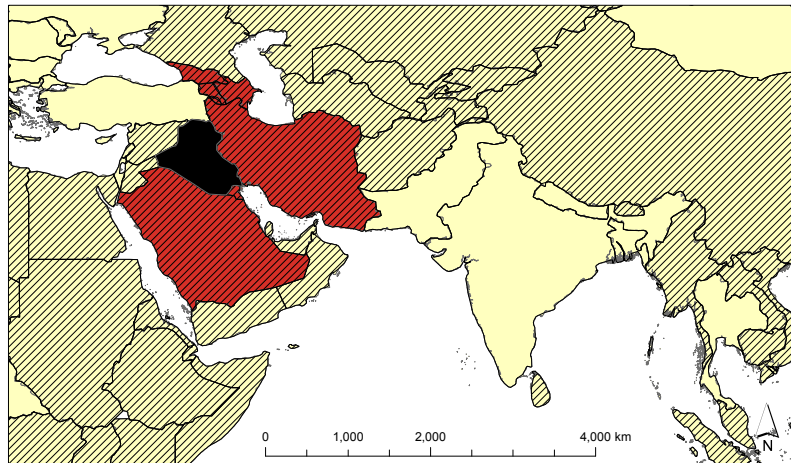
Minimum Distance

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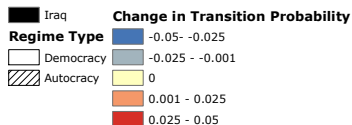
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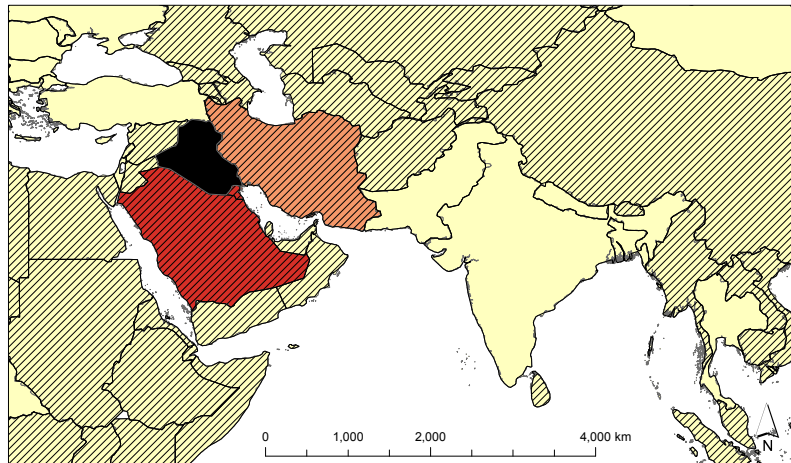
k = 4 Nearest Neighbors

Iraq transitions from autocracy to democracy
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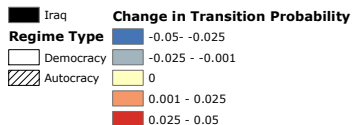
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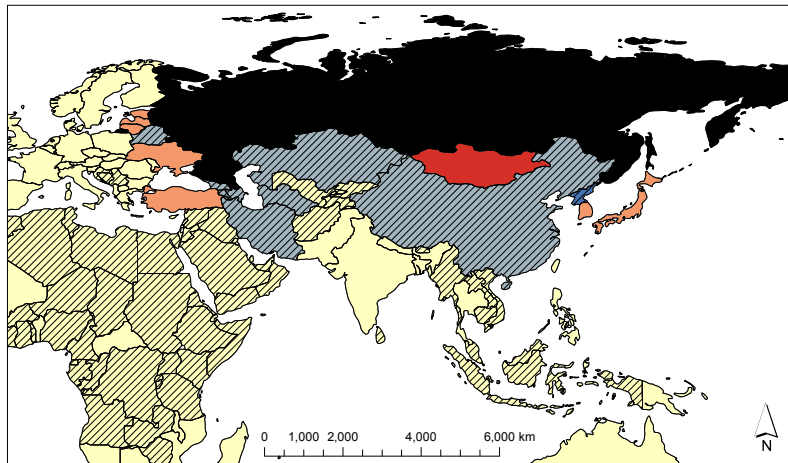
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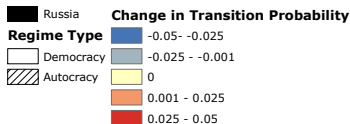
Russia's autocratization and regional regime stability



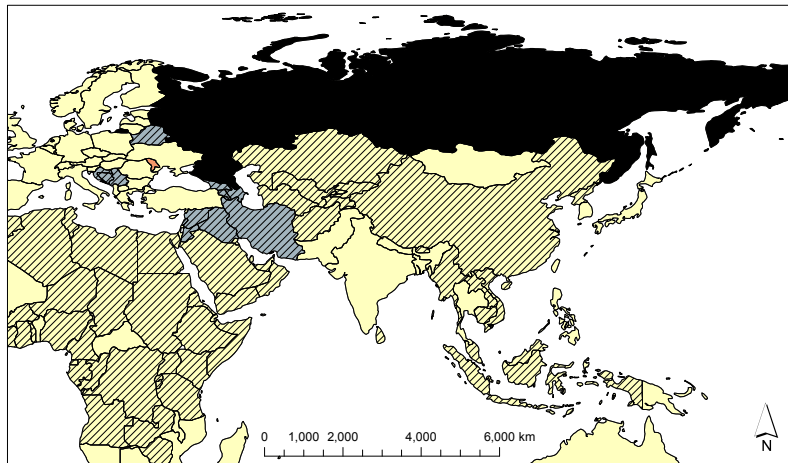
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Russia transitions from democracy to autocracy (1998 data)

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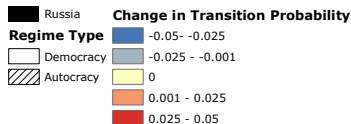
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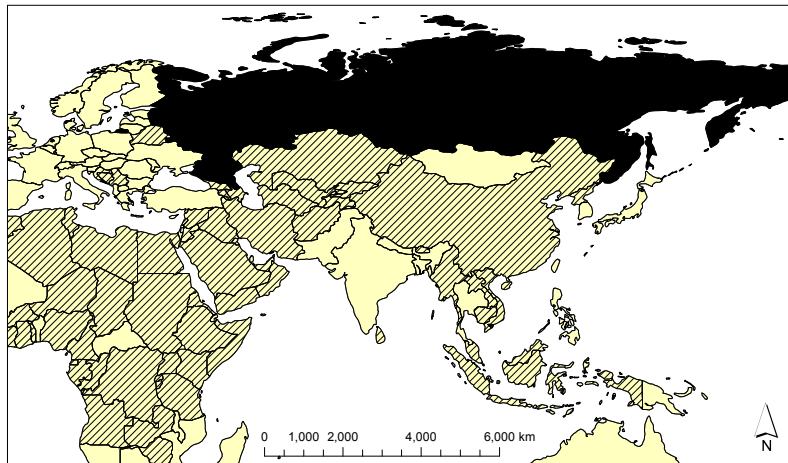
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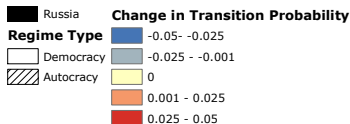
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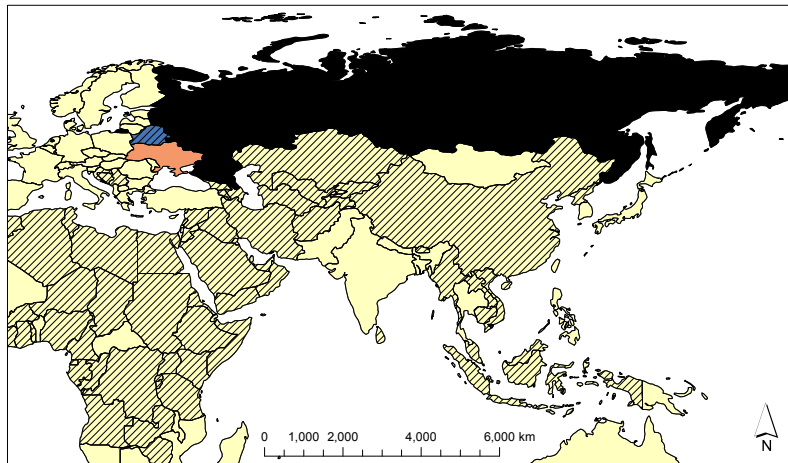
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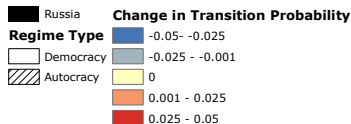
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Kitten Wars

Kitten Wars



kittenwar

Warm, Flaky, Delicious
Crowd Pleaser,
Pillsbury™ Crescent Rolls [Save \\$1.00](#)



Alfredo vs. Amy



Click on the cutest to decide the winner!!!
Can't decide? Refresh the page for a draw.

Home

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kitten search:

Kitten Wars

Winningest Kittens!

[more](#)



1.

Bitley
has won 76% of 8642 battles.



2.

Freddie
has won 76% of 3768 battles.

Kitten Wars

Winningest Kittens!

[more](#)



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Losingest Kittens!

[more](#)



1.

Scary Cat
has lost 79% of 11211 battles.



2.

Benjam
has lost 79% of 1919 battles.

Kitten Wars for Ideas (Salganik and Levy)

Kitten Wars for Ideas (Salganik and Levy)

RESEARCH ARTICLE

Wiki Surveys: Open and Quantifiable Social Data Collection

Matthew J. Salganik, Karen E. C. Levy

Published: May 20, 2015 • DOI: 10.1371/journal.pone.0123483

Article	Authors	Metrics	Comments	Related Content
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Abstract

Introduction

Wiki surveys

Pairwise Wiki Surveys

Case studies

Discussion

Ethics Statement

Supporting Information

Acknowledgments

Author Contributions

References

Reader Comments (0)

Media Coverage

Figures

Abstract

In the social sciences, there is a longstanding tension between data collection methods that facilitate quantification and those that are open to unanticipated information. Advances in technology now enable new, hybrid methods that combine some of the benefits of both approaches. Drawing inspiration from online information aggregation systems like Wikipedia and from traditional survey research, we propose a new class of research instruments called *wiki surveys*. Just as Wikipedia evolves over time based on contributions from participants, we envision an evolving survey driven by contributions from respondents. We develop three general principles that underlie wiki surveys: they should be greedy, collaborative, and adaptive. Building on these principles, we develop methods for data collection and data analysis for one type of wiki survey, a pairwise wiki survey. Using two proof-of-concept case studies involving our free and open-source website www.allcoundeas.org, we show that pairwise wiki surveys can yield insights that would be difficult to obtain with other methods.

Figures



Kitten Wars for Ideas (Salganik and Levy)



Bringing survey research into the digital age.

Mix core ideas from survey research with new insights from crowdsourcing.
Add a heavy dose of statistics. Stir in a bit of fresh thinking. Enjoy.

Try a Wiki Survey

Create a Wiki Survey

HOW A WIKI SURVEY WORKS



Create

Start with a question and some seed ideas, and you can create a wiki survey in moments.



Participate

The participants you invite will enjoy our simple process of voting and adding new ideas.



Discover

The best ideas will bubble to the top using our system that is open, transparent, and powerful.

Kitten Wars for Ideas (Salganik and Levy)



[Cast Votes](#) [View Results](#) [About this page](#)

This is a copy of a wild survey that was used by New York City Mayor's Office. You can read more about the project here: <http://bit.ly/planyc>

Which do you think is better for creating a greener, greater New York City?

Install and maintain drinking-water fountains on busy city streets.

Create or enhance public plazas in every community

Users can vote by clicking one of these options

I can't decide

54578 votes on 288 ideas

Users can vote by clicking one of these options

Add your own idea here...

Users can add their own ideas here

powered by
ALL OUR IDEAS

[About Privacy and Consent Policy](#)

Kitten Wars for Ideas (Salganik and Levy)

Which do you think is better for creating a greener, greater New York City?

Ideas	Score (0 - 100) 
Require all big buildings to make certain energy efficiency upgrades	67
Promote cycling by installing safe bike lanes	65
Promote the use of solar energy using the latest technology on all high-rise buildings.	65
Invest in multiple modes of transportation and provide both improved infrastructure and improved safety	65
Continue enhancing bike lane network, to finally connect separated bike lane systems to each other across all five boroughs.	65
Replace sodium vapor street lights with LED or other energy-saving lights.	64
Utilize NYC Rooftops to install Solar PV panels	63
Plant more trees	62
Create a network of protected bike paths throughout the entire city	62
Add improvements to the bike lanes in the inner city. This will encourage exercise and reduce city's carbon footprint.	62

The Power of Releasing Software

The Power of Releasing Software

The Governor asks ... again



Governor Tarso Genro of the state of Rio Grande do Sul, Brazil has done it again. The Governor and his team completed a second round of their amazing open government project called Governador Pergunta (The Governor Asks), which collects public feedback on important policy challenges using a customized version of allourideas.org.

The Power of Releasing Software

wiki surveys to assess risks of state-led mass killings



As part of their work with the [Holocaust Museum's Center for the Prevention of Genocide](#), [Jay Ulfelder](#) and [Ben Valentino](#) launched a wiki survey to help assess the risks of state-led mass killing onsets in 2014. You can read about their results on this [interesting blog post](#).

The Power of Releasing Software

UN Global Sustainability Report 2013



We are happy to announce that the United Nations Division for Sustainable Development is using allourideas.org to solicit ideas from scientists around the world for the 2013 UN Global Sustainability Report. The report will

Powered By Research (You Can Do It Too!)

Powered By Research (You Can Do It Too!)

Backed by research

All Our Ideas is a research project based at Princeton University that is dedicated to creating new ways of collecting social data. You can learn more about the theory and methods behind our project by [reading our paper](#) or [watching our talk](#). Thanks to Google, the National Science Foundation, and Princeton for supporting this research.

$$z_i \sim \begin{cases} N(\dot{\mathbf{x}}_i^T \boldsymbol{\theta}_{\mathbf{v}}, 1) I(z_i^* > 0) & \text{if } y_i = 1 \\ N(\dot{\mathbf{x}}_i^T \boldsymbol{\theta}_{\mathbf{v}}, 1) I(z_i^* < 0) & \text{if } y_i = 0 \end{cases}$$

Powered By Research (You Can Do It Too!)

Bradley–Terry model

From Wikipedia, the free encyclopedia

The **Bradley–Terry model** is a [probability model](#) that can predict the outcome of a comparison. Given a pair of individuals i and j drawn from some [population](#), it estimates the probability that the [pairwise comparison](#) $i > j$ turns out true, as

$$P(i > j) = \frac{p_i}{p_i + p_j}$$

where p_i is a positive [real-valued](#) score assigned to individual i . The comparison $i > j$ can be read as " i is preferred to j ", " i ranks higher than j ", or " i beats j ", depending on the application.

For example, p_i may represent the skill of a team in a sports tournament, estimated from the number of times i has won a match. $P(i > j)$ then represents the probability that i will win a match against j .^{[1][2]} Another example used to explain the model's purpose is that of scoring products in a certain category by quality. While it's hard for a person to draft a direct ranking of (many) brands of wine, it may be feasible to compare a sample of pairs of wines and say, for each pair, which one is better. The Bradley–Terry model can then be used to derive a full ranking.^[2]

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