

Week 8: What Can Go Wrong and How To Fix It, Diagnostics and Solutions

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¹These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller, Erin Hartman and Kevin Quinn.

Where We've Been and Where We're Going...

- Last Week
 - ▶ multiple regression
- This “Week”
 - ▶ Monday (5):
 - ★ unusual and influential data → robust estimation
 - ▶ Wednesday (7):
 - ★ non-linearity → generalized additive models
 - ▶ Monday (12):
 - ★ unusual errors → sandwich SEs
- Next Week
 - ▶ regression in social science
- Long Run
 - ▶ probability → inference → regression → causal inference

Questions?

- 1 Assumptions and Violations
- 2 Non-normality
- 3 Outliers
- 4 Robust Regression Methods
- 5 Fun with Outliers
- 6 Appendix: Robustness
- 7 Detecting Nonlinearity
- 8 Linear Basis Function Models
- 9 Generalized Additive Models
- 10 Fun With Kernels
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- 13 A Contrarian View of Robust Standard Errors
- 14 Fun with Neighbors
- 15 Appendix: WLS and Serial Correlation

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Argument for Next Three Classes

Residuals are **important**. Look at them.

Review of the OLS assumptions

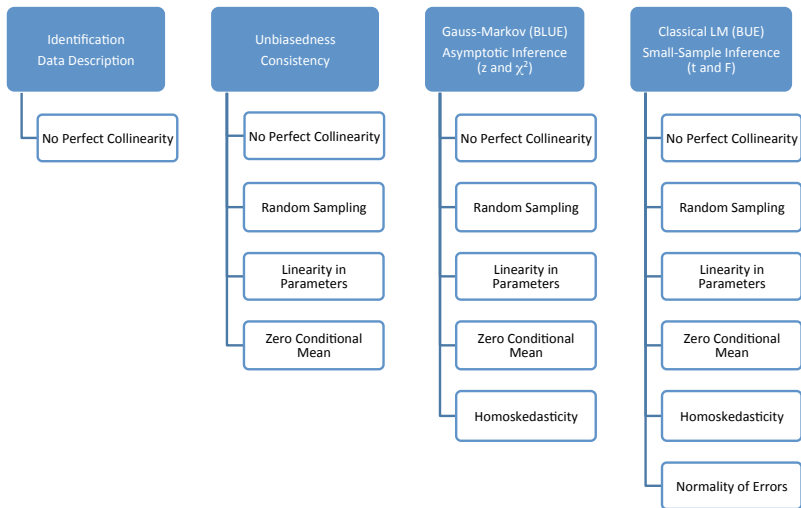
- ① Linearity: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$
 - ② Random/iid sample: $(y_i, \mathbf{x}_i')$ are a iid sample from the population.
 - ③ No perfect collinearity: $\mathbf{X}$ is an $n \times (K + 1)$ matrix with rank $K + 1$
 - ④ Zero conditional mean: $\mathbb{E}[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
 - ⑤ Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
 - ⑥ Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I}_n)$
- 1-4 give us unbiasedness/consistency
 - 1-5 are the Gauss-Markov, allow for large-sample inference
 - 1-6 allow for small-sample inference

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Let's talk about **what's at stake** in diagnostics under different views of what regression is doing.

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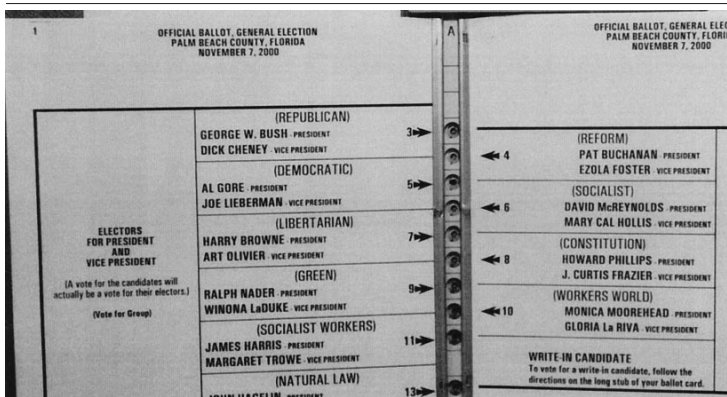


Example: Buchanan votes in Florida, 2000

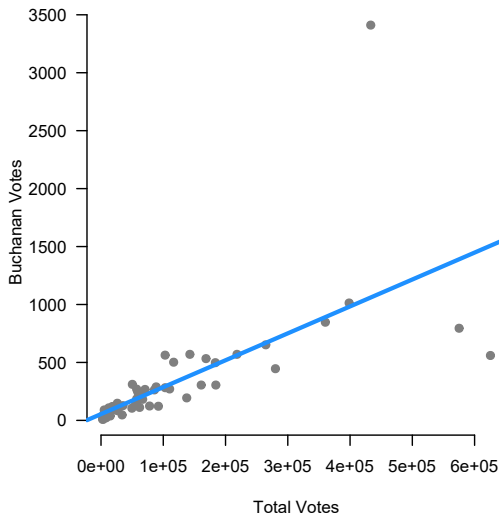
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Wand et al. show that the ballot caused 2,000 Democratic voters to vote by mistake for Buchanan, a number more than enough to have tipped the vote in FL from Bush to Gore, thus giving him FL's 25 electoral votes and the presidency.

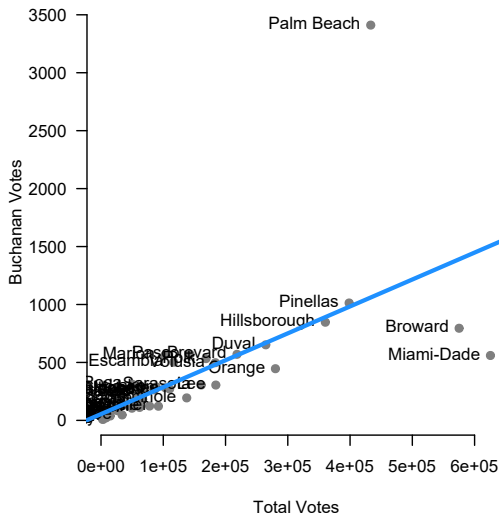
FIGURE 1. The Palm Beach County Butterfly Ballot



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- Fix $\mathbf{x}'_i$ and the distribution of errors should be Normal

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- The sample size (n) needed for approximation to hold depends on how far the errors are from Normal.

Marginal versus conditional

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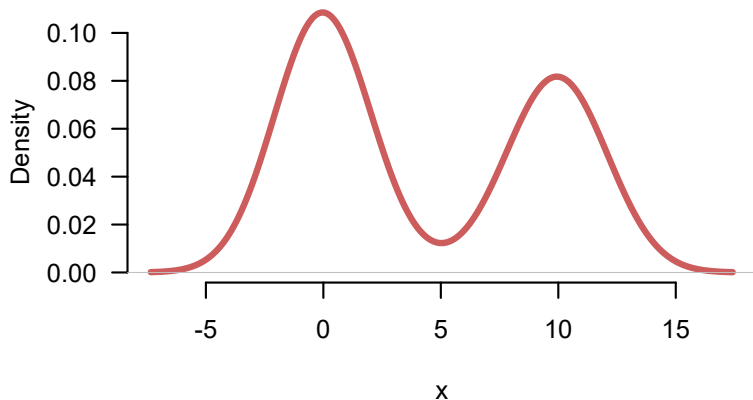
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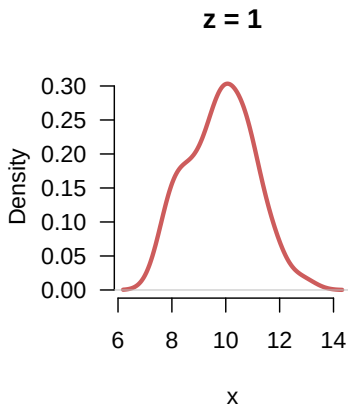
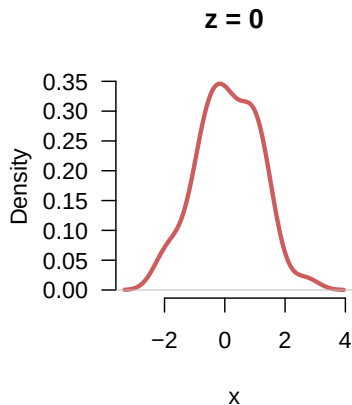
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- The plausibility depends on the X chosen by the researcher.

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To understand the relationship between residuals and errors, we need to **derive the distribution of the residuals** (which we will do over the next few slides).

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- ▶ $\mathbf{H}$ is **idempotent**: $\mathbf{H}\mathbf{H} = \mathbf{H}$

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- Note that each residual is a function of **all** of the errors.

Characterizing the distribution of the residuals

What can we say about the distribution of the residuals now that we have the expression: $\hat{\mathbf{u}} = (\mathbf{I} - \mathbf{H})\mathbf{u}$.

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The variance of the i th residual $\hat{u}_i$ is $V[\hat{u}_i] = \sigma^2(1 - h_{ii})$, where h_{ii} is the i th diagonal element of the matrix $\mathbf{H}$ (called the **hat value**).

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What if we could **transform** the residuals to address the two issues above?

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This produces **standardized** (or “internally studentized”) **residuals**:

$$\hat{u}'_i = \frac{\hat{u}_i}{\hat{\sigma}\sqrt{1 - h_{ii}}}$$

where $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ and $\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$ is our usual estimate of the error variance.

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The standardized residuals are still not ideal, since the numerator and denominator of $\hat{u}'_i$ are not independent. This makes the distribution of $\hat{u}'_i$ nonstandard. If the distribution is non-standard, we can't easily check for violations.

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- Deviations from this t distribution of the **residuals** imply violation of Normality in the **errors**.

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- We examine data from the 2000 presidential election in Florida used in Wand et al. (2001).
- Our analysis takes place at the county level and we will regress the number of Buchanan votes in each county on the total number of votes in each county.

Buchanan Votes and Total Votes

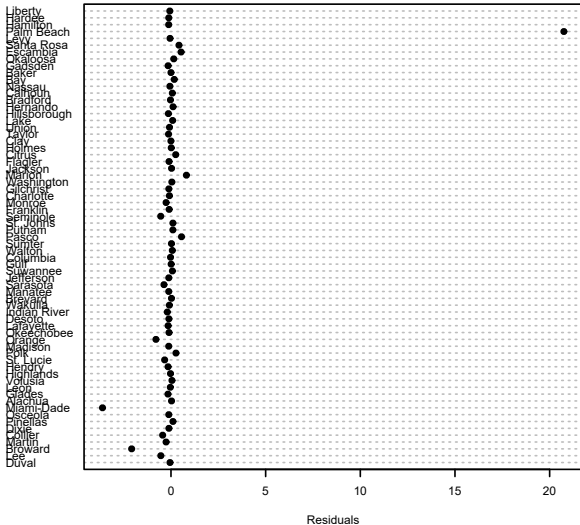
R Code

```
> mod1 <- lm(buchanan00~TotalVotes00,data=dta)
> summary(mod1)
Residuals:
    Min       1Q   Median       3Q      Max
-947.05  -41.74  -19.47   20.20 2350.54

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.423e+01  4.914e+01   1.104   0.274
TotalVotes00 2.323e-03  3.104e-04   7.483 2.42e-10 ***
---
Residual standard error: 332.7 on 65 degrees of freedom
Multiple R-squared:  0.4628,    Adjusted R-squared:  0.4545
F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10

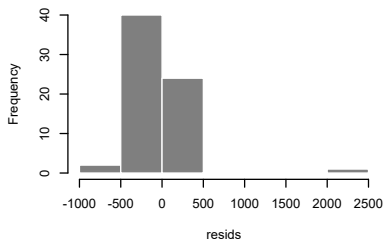
> residuals <- resid(mod1)
> standardized_residuals <- rstandard(mod1)
> studentized_residuals <- rstudent(mod1)
> dotchart(residuals,dta$name,cex=.7,xlab="Residuals")
```

Plotting the residuals

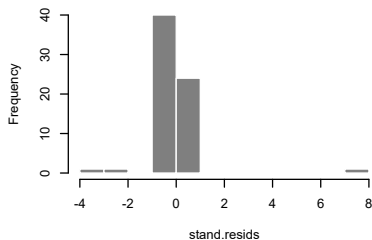


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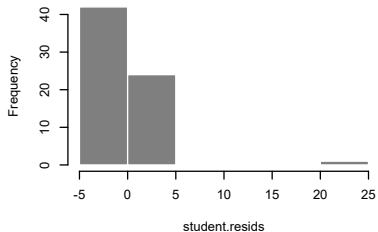
Histogram of resid



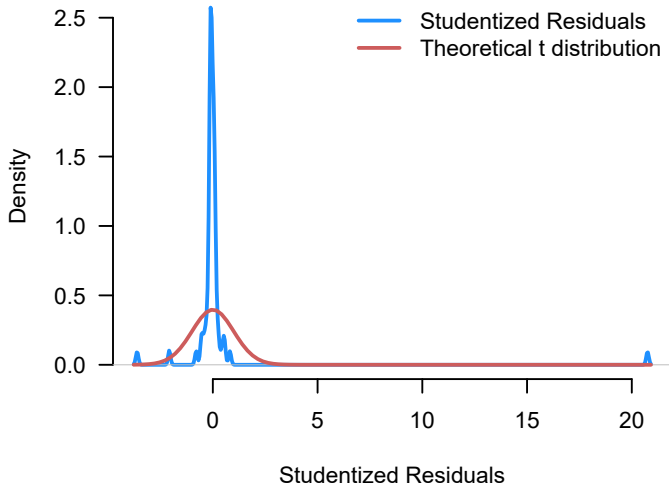
Histogram of stand.resids



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Plotting the residuals



Quantile-Quantile plots

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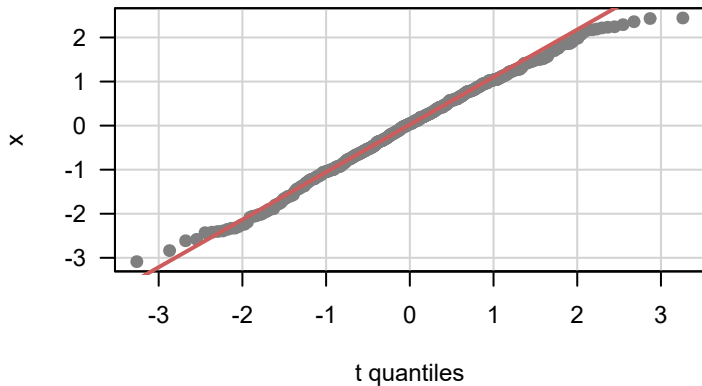
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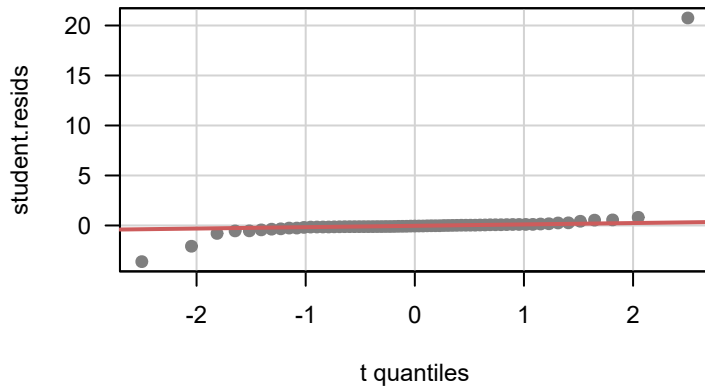
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- If distributions are equal $\implies$ 45 degree line

Good QQ-plot



Buchanan QQ-plot



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- Consider other causes (next two classes)

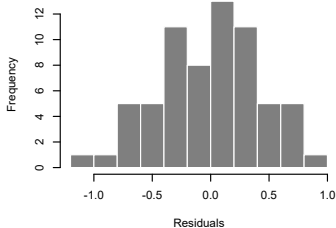
Buchanan revisited

Let's delete Palm Beach and also use log transformations for both variables

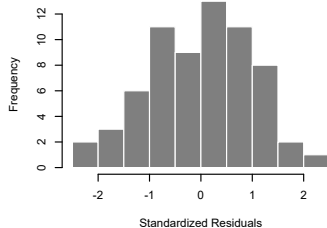
```
##  
## Coefficients:  
##           Estimate Std. Error t value Pr(>|t|)  
## (Intercept)  -2.48597    0.37889  -6.561 1.09e-08 ***  
## log(edaytotal) 0.70311    0.03621  19.417 < 2e-16 ***  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## Residual standard error: 0.4362 on 64 degrees of freedom  
## Multiple R-squared:  0.8549, Adjusted R-squared:  0.8526  
## F-statistic: 377 on 1 and 64 DF, p-value: < 2.2e-16
```

Buchanan revisited

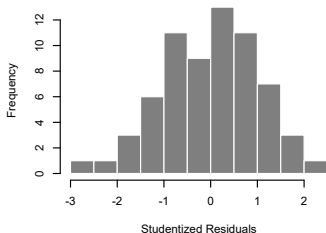
Histogram of resid.nopb



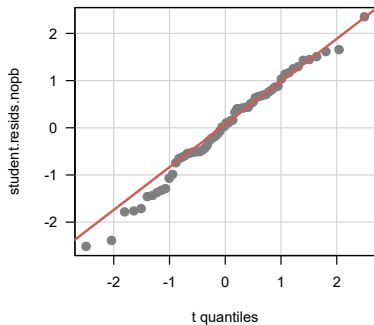
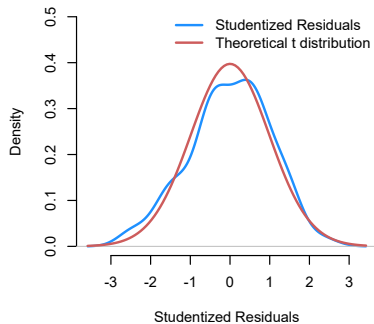
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- The results will in general be **consistent** which ensures that the bias decreases in sample size.

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- 4 Robust Regression Methods
- 5 Fun with Outliers
- 6 Appendix: Robustness
- 7 Detecting Nonlinearity
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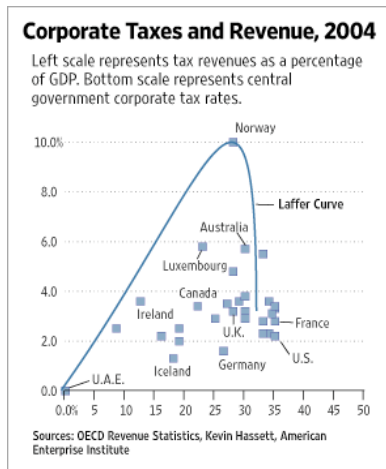
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	Constant	x_1	x_2	$x_1 \cdot x_2$
Norway Obs Included	.814 (4.7)	-.192 (2.0)	-.278 (2.4)	.137 (2.9)
Norway Obs Excluded	.641 (4.8)	-.068 (0.9)	-.138 (1.5)	.054 (1.3)

Creative curve fitting with Norway

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The Most Important Lesson: Check Your Data

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Always Carefully Examine the Data First!!

- 1 Examine summary statistics: `summary(data)`
- 2 Scatterplot matrix for densities and bivariate relationships:
E.g. `scatterplotMatrix(data)` from `car` library.
- 3 Further conditional plots for multivariate data:
E.g. `ggplot2`

Three types of extreme values

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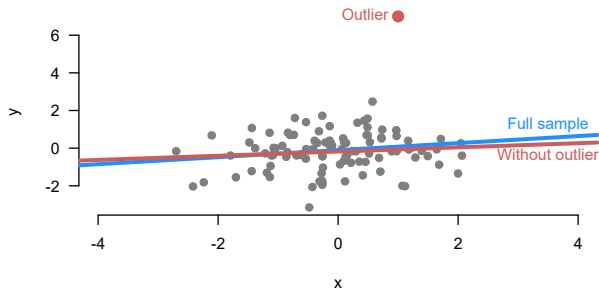
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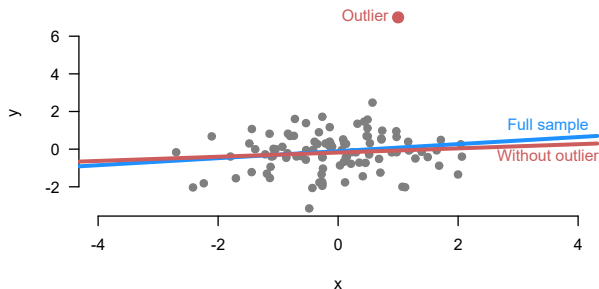
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 - Can be a violation of iid (not identically distributed)

Outlier definition



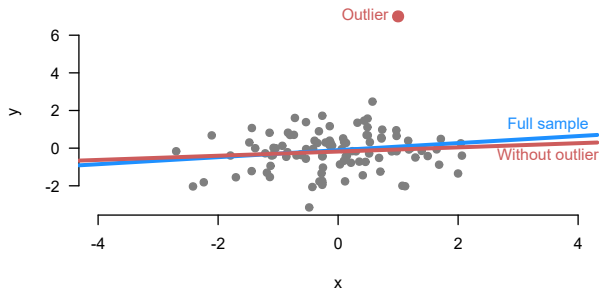
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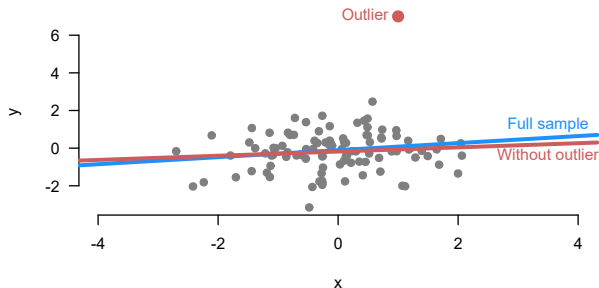
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- No bias if typical in the x 's

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- $\hat{\sigma} > \hat{\sigma}_{-i}$ because we drop the large residual from the outlier, and so $\hat{u}_i' < \hat{u}_i^*$

Cutoff rules for outliers

- The studentized residuals follow a t distribution, $u_i^* \sim t_{n-k-2}$, when $u_i \sim N(0, \sigma^2)$

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- Rule of thumb: $|\hat{u}_i^*| > 2$ will be relatively rare

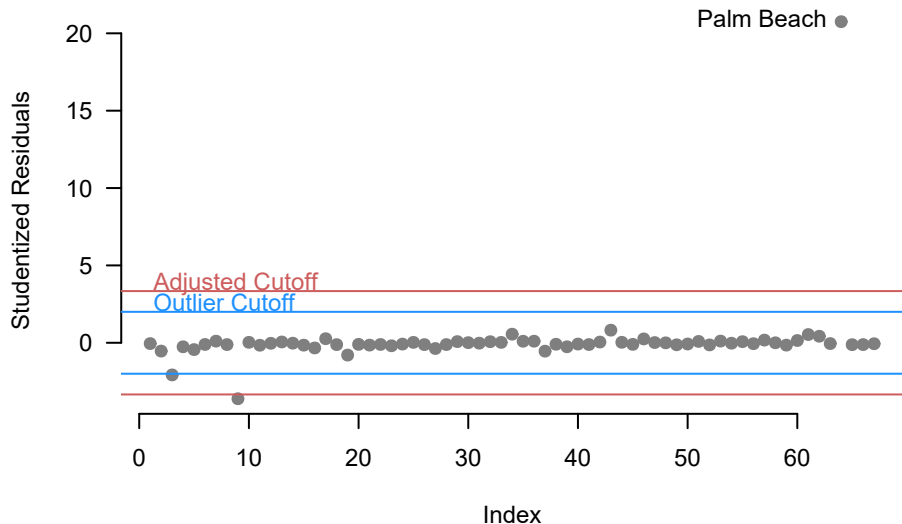
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- People usually adjust cutoff for multiple testing

Buchanan outliers



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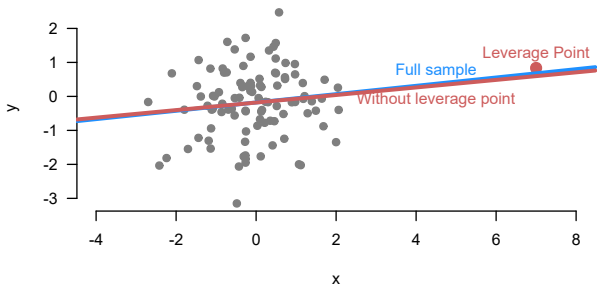
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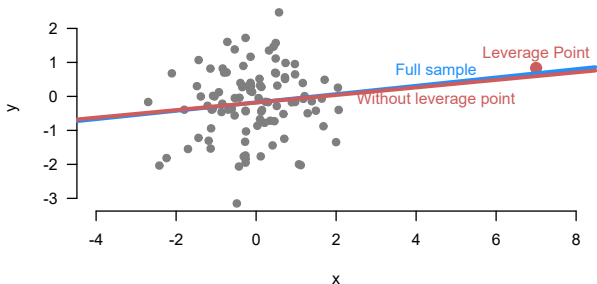
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- The ozone hole was detected in satellite data only when the raw data was reprocessed. When the software was rerun without the pre-processing flags, the ozone hole was seen as far back as 1976.

Leverage point definition



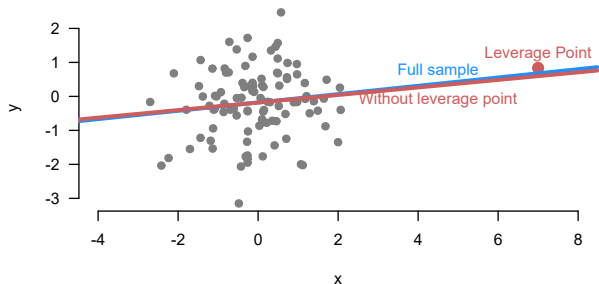
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Leverage point definition



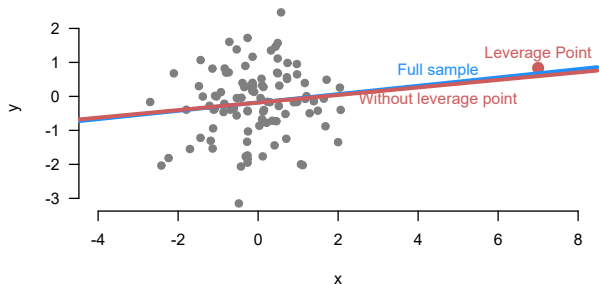
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Leverage point definition



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- That is, values far from the center of the covariate distribution
- Decrease SEs (more X variation)
- No bias if typical in y dimension

Leverage Points: Hat values

To measure leverage in multivariate data we will go back to the hat matrix **H**:

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{H}\mathbf{y}$$

H is $n \times n$, symmetric, and idempotent. It generates fitted values as follows:

$$\hat{y}_i = \mathbf{h}_i' \mathbf{y} = \begin{bmatrix} h_{i,1} & h_{i,2} & \cdots & h_{i,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{j=1}^n h_{i,j} y_j$$

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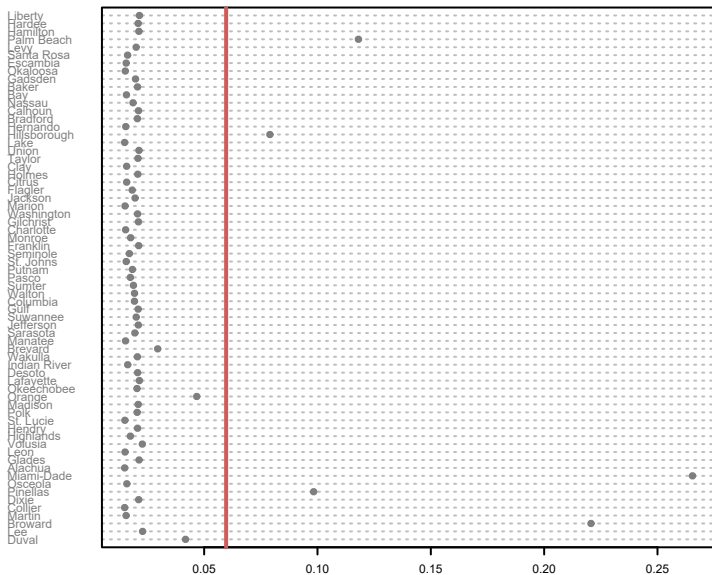
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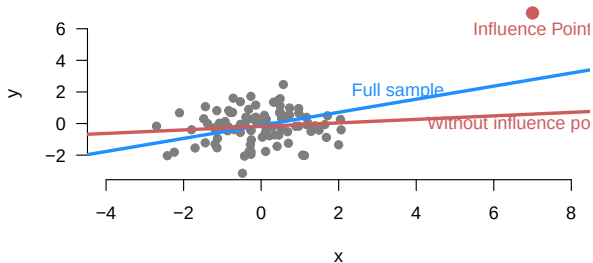
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- Intuitively, the hat values measure how far a unit's vector of characteristics $\mathbf{x}_i$ is from the vector of means of **X**
- **Rule of thumb**: examine hat values greater than $2(k+1)/n$

Buchanan hats

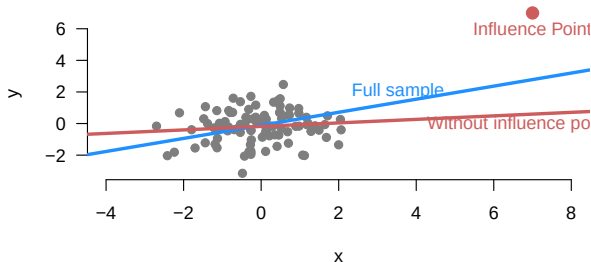


Influence points

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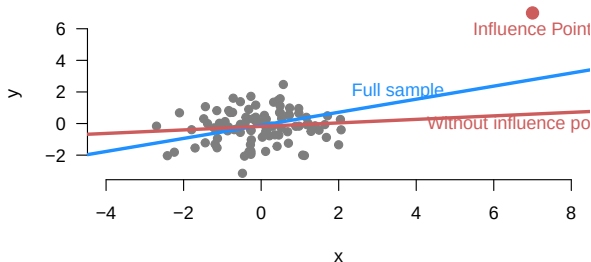


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- Causes the regression line to move toward it (bias?)

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- D_{ij} is called the **DFbeta**, which measures the **influence** of observation i on the estimated coefficient for the j th explanatory variable.

Standardized Influence

To make comparisons across coefficients, it is helpful to scale D_{ij} by the estimated standard error of the coefficients:

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- In R: `dfbetas(model)`

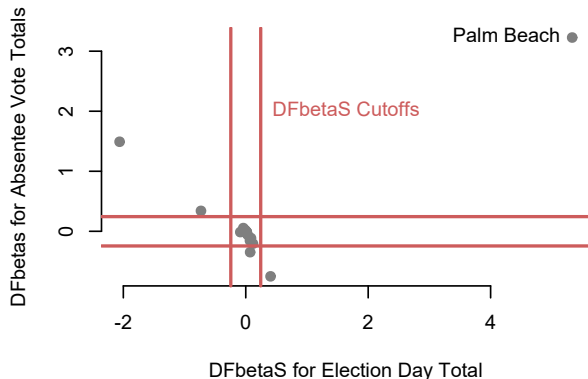
Buchanan influence

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -2.935e+01  5.520e+01  -0.532  0.59686
## edaytotal    1.100e-03  4.797e-04   2.293  0.02529 *
## absnbuchanan  6.895e+00  2.129e+00   3.238  0.00195 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 317.2 on 61 degrees of freedom
## (3 observations deleted due to missingness)
## Multiple R-squared:  0.5361, Adjusted R-squared:  0.5209
## F-statistic: 35.24 on 2 and 61 DF,  p-value: 6.711e-11
```

Buchanan influence

##	(Intercept)	edaytotal	absnbuchanan
## 1	0.3454475146	0.4050504921	-0.7505222758
## 2	-0.0234266617	-0.0241000045	-0.0131672181
## 3	0.0650795039	-0.7319311820	0.3401669862
## 4	-0.0333980968	0.0133802934	-0.0087505576
## 5	-0.0397626659	-0.0073746223	0.0096551713
## 6	-0.0009277798	0.0001505476	0.0002210247

Buchanan influence



- Palm Beach county moves each of the coefficients by more than 3 standard errors!

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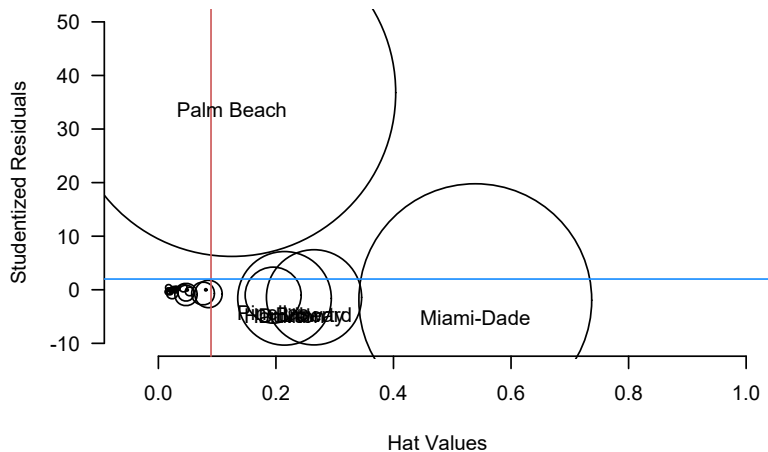
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Influence Plot Buchanan



Code for Influence Plot

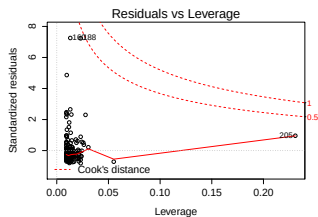
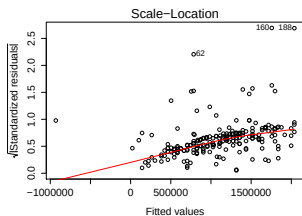
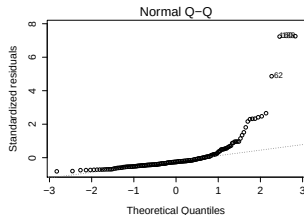
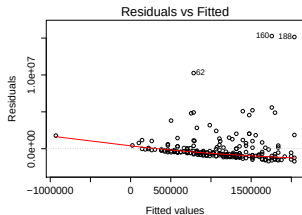
```
mod3 <- lm(edaybuchanan ~ edaytotal + absnbuchanan, data = flvote)
symbols(y = rstudent(mod3), x = hatvalues(mod3),
        circles = sqrt(cooks.distance(mod3)),
        ylab = "Studentized Residuals",
        xlab = "Hat Values", xlim = c(-0.05, 1),
        ylim = c(-10, 50), las = 1, bty = "n")

cutoffstud <- 2
cutoffhat <- 2 * (3)/nrow(flvote)
abline(v = cutoffhat, col = "indianred")
abline(h = cutoffstud, col = "dodgerblue")
filter <- rstudent(mod3) > cutoffstud | hatvalues(mod3) > cutoffhat
text(y = rstudent(mod3)[filter],
     x = hatvalues(mod3)[filter],
     flvote$county[filter], pos = 1)
```

A Quick Function for Standard Diagnostic Plots

R Code

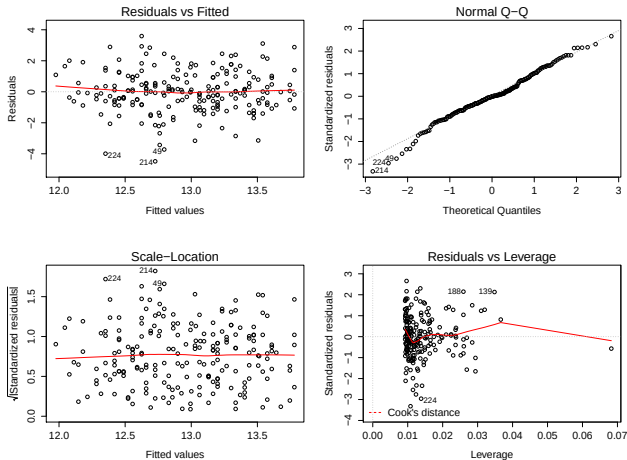
```
> par(mfrow=c(2,2))  
> plot(mod1)
```



The Improved Model

R Code

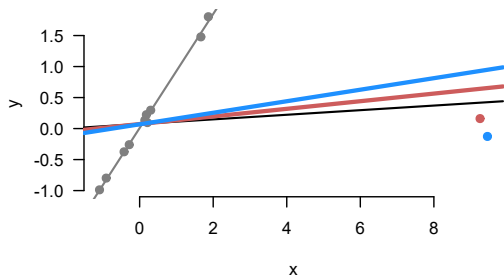
```
> par(mfrow=c(2,2))  
> plot(mod2)
```



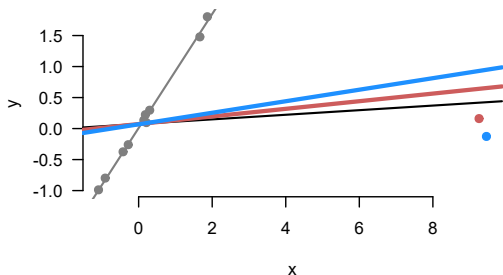
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Limitations of the standard tools

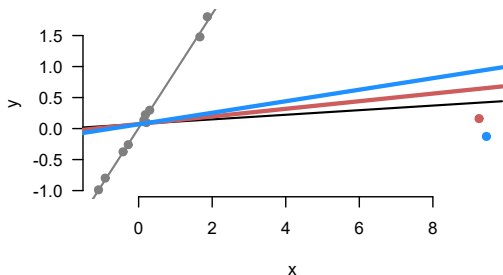


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- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point

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- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point
- Neither of the “leave-one-out” approaches helps recover the line

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- We will cover a few ideas in **robust** statistics over the next few days (much of which is due directly or indirectly to Peter Huber)

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- Robust methods are procedures that are designed to continue to provide 'reasonable' answers in the presence of violation of some assumptions.

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- The **Linear** point is an artificial restriction. It means the estimator has to be of the form $\hat{\beta} = \mathbf{W}y$ but why only use those?
- With normality assumption we get **Best Unbiased Estimator** (BUE) which is quite comforting when $n \gg p$ (number of observations much larger than number of variables).

This Point is Not Obvious

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"[Even without normally distributed errors] OLS coefficient estimators remain unbiased and efficient." - Berry (1993)

Quotes from Rainey and Baissa (2015) presentation

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"[The Gauss-Markov theorem] justifies the use of the OLS method rather than using a variety of competing estimators" - Wooldridge (2013)

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This flies in the face of most conventional wisdom in textbooks.

"We need not look for another linear unbiased estimator, for we will not find such an estimator whose variance is smaller than the OLS estimator"
- Gujarati (2004)

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"The Gauss-Markov theorem allows us to have considerable confidence in the least squares estimators." - Berry and Feldman (1993)

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Robustly Estimating a Location

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- For normal data $y_i \sim \mathcal{N}(\mu, \sigma^2)$, median is less efficient:
 - ▶ $V(\hat{\mu}_{\text{mean}}) = \frac{\sigma^2}{n}$
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- We can measure sensitivity with the **influence function** which measures change in estimator based on corruption in one datapoint.

Influence Function

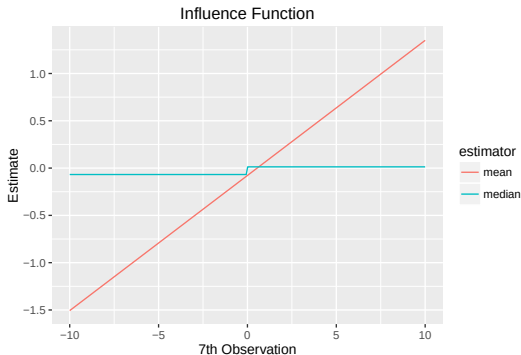
- Imagine that we had a sample Y from a standard normal: -0.068, -1.282, 0.013, 0.141, -0.980, 1.63. $\bar{Y} = -1.52$

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Example from Fox

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- The breakdown point of the mean is 0 because (as we have seen) a single bad data point can change things a lot.
- The median has a breakdown point of 50% because half the data can be bad without causing the median to become completely unstuck.

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- Other objectives include the Huber objective and Tukey's biweight objective which have different properties.
- Calculating robust M estimators often requires an iterative procedure and a careful initialization.

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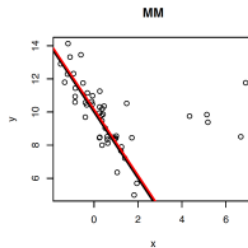
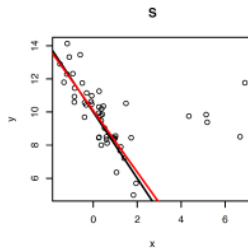
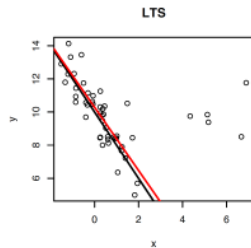
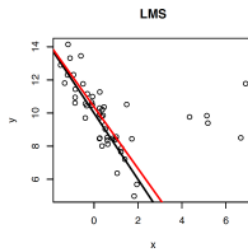
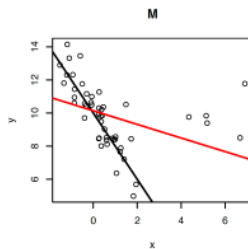
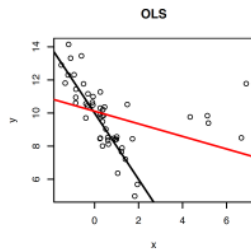
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 - ▶ MM -estimator: with Huber's loss is what I recommend in practice (more in appendix)
- You can find an asymptotic covariance matrix for M -estimators but I would bootstrap it if possible as the asymptotics kick in slowly.

```
library(MASS)
set.seed(588)
n <- 50
x <- rnorm(n)
y <- 10 - 2*x + rnorm(n)
x[1:5] <- rnorm(5, mean=5)
y[1:5] <- 10 + rnorm(5)
ols.out <- lm(y~x)
m.out <- rlm(y~x, method="M")
lms.out <- lqs(y~x, method="lms")
lts.out <- lqs(y~x, method="lts")
s.out <- lqs(y~x, method="S")
mm.out <- rlm(y~x, method="MM")
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Simulation Results



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- In most cases I personally would start with OLS, do diagnostics and then consider a robust alternative. If I don't have time for diagnostics, maybe robust is better from the outset.
- I highly recommend Baissa and Rainey (2016) "When BLUE is Not Best: Non-Normal Errors and the Linear Model" for more on this topic.
- The Fox textbook Chapter 19 is also quite good on this and points out to the key references

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- Easy to test some of these and hard to test others.
- Always **check your data!**
- Don't let regression be a magic black box for you- understand what is in your data that is leading to the findings.

Fun With Outliers



Prostitution and the sex discrepancy in reported number of sexual partners

Devon D. Brewer^{*†}, John J. Potterat[‡], Sharon B. Garrett^{*}, Stephen Q. Muth[†], John M. Roberts, Jr.[§], Danuta Kasprzyk[¶], Daniel E. Montano[¶], and William W. Darrow[¶]

^{*}Alcohol and Drug Abuse Institute, University of Washington, 3937 15th Avenue NE, Seattle, WA 98105; [†]El Paso County Department of Health and Environment, 301 South Union Boulevard, Colorado Springs, CO 80910; [‡]Department of Sociology, University of New Mexico, Albuquerque, NM 87131; [§]Centers for Public Health Research and Evaluation, Batelle Memorial Institute, 4000 NE 41st Street, P.O. Box 5395, Seattle, WA 98105-5395; [¶]Department of Public Health, Florida International University, 3000 NE 145th Street, ACI-394F, North Miami, FL 33181

Communicated by A. Kimball Romney, University of California, Irvine, CA, August 16, 2000 (received for review June 21, 2000)

One of the most reliable and perplexing findings from surveys of sexual behavior is that men report substantially more sexual partners than women do. We use data from national sex surveys and studies of prostitutes and their clients in the United States to examine sampling bias as an explanation for this disparity. We find that prostitute women are underrepresented in the national surveys. Once their undersampling and very high numbers of sexual partners are factored in, the discrepancy disappears. Prostitution's role in the discrepancy is not readily apparent because men are reluctant to acknowledge that their reported partners include prostitutes.

Our analyses of the GSS are based on data from 1988 to 1991 combined. Overall sample sizes are 5,907 for the GSS and 3,159 for the NHSLS.

Analysis and Results. Because of slight differences in the numbers of men and women in the population at large and differences in the proportions of men and women who are heterosexual, estimates must be obtained at the United States population level rather than simply by relying on the surveys' sample means for men's and women's numbers of partners. (Detailed calculations

Thanks to Matt Salganik for pointing me to this example

Does Diversity Pay? A Replication of Herring (2009)

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© American Sociological
Association 2017
DOI: 10.1177/0003122417714422
journals.sagepub.com/home/asr



**Dragana Stojmenovska,^a Thijs Bol,^a
and Thomas Leopold^a**

Abstract

In an influential article published in the *American Sociological Review* in 2009, Herring finds that diverse workforces are beneficial for business. His analysis supports seven out of eight hypotheses on the positive effects of gender and racial diversity on sales revenue, number of customers, perceived relative market share, and perceived relative profitability. This comment points out that Herring's analysis contains two errors. First, missing codes on the outcome variables are treated as substantive codes. Second, two control variables—company size and establishment size—are highly skewed, and this skew obscures their positive associations with the predictor and outcome variables. We replicate Herring's analysis correcting for both errors. The findings support only one of the original eight hypotheses, suggesting that diversity is nonconsequential, rather than beneficial, to business success.

In our correspondence with Herring, he did not offer a definitive explanation for these discrepancies, but indicated that he may have treated all codes other than “not applicable” (–999) as substantive codes. Given (1) the large difference between his sample size and the number of valid observations in the NOS, and (2) the large number of missing values due to reasons other than “not applicable”—in particular for sales revenue and number of customers—this coding error appears likely to account for much of the discrepancies. This means, for example, that 206 business organizations in which the sales revenue was unknown were treated as if they had sales of 88,888,888,888 US Dollars. Yet, even when we replicated this error (i.e., keeping all organizations with missing values other than –999 in our sample), we were unable to recover Herring’s sample sizes, although the differences were smaller.

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- 7 Detecting Nonlinearity
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Appendix: Characterizing Estimator Robustness (formally)

Definition (Breakdown Point)

The breakdown point of an estimator is the smallest fraction of the data that can be changed an arbitrary amount to produce an arbitrarily large change in the estimate (Seber and Lee 2003, pg 82)

Definition (Influence Function)

Let $F_p = (1 - p)F + p\delta_{\mathbf{z}_0}$ where F is a probability measure, $\delta_{\mathbf{z}_0}$ is the point mass at $\mathbf{z}_0 \in \mathbb{R}^k$, and $p \in (0, 1)$.

Let $T(\cdot)$ be a statistical functional. The influence function of T is

$$IF(\mathbf{z}_0; T, F) = \lim_{p \downarrow 0} \frac{T(F_p) - T(F)}{p}$$

The influence function is a function of $\mathbf{z}_0$ given T and F . It describes how T changes with small amounts of contamination at $\mathbf{z}_0$ (Hampel, Rousseeuw, Ronchetti, and Stahel, (1986), p. 84).

Appendix: S Estimators

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An *S*-estimator for the regression model is defined as the values of $\hat{\beta}_S$ and s that minimize s subject to the constraint:

$$\frac{1}{n} \sum_{i=1}^n \rho \left(\frac{y_i - \mathbf{x}'_i \hat{\beta}_S}{s} \right) \geq K$$

where K is user-defined constant (typically set to 0.5) and $\rho : \mathbb{R} \rightarrow [0, 1]$ is a function with the following properties (Davies, 1990, p. 1653):

- ① $\rho(0) = 1$
- ② $\rho(u) = \rho(-u)$, $u \in \mathbb{R}$
- ③ $\rho : \mathbb{R}_+ \rightarrow [0, 1]$ is nonincreasing, continuous at 0, and continuous on the left
- ④ for some $c > 0$, $\rho(u) > 0$ if $|u| < c$ and $\rho(u) = 0$ if $|u| > c$

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The work by first calculating S-estimates of the scale and coefficients and then using these as starting values for a particular M-estimator.

Good properties, but costly to compute (usually impossible to compute exactly).

References

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Where We've Been and Where We're Going...

- Last Week
 - ▶ multiple regression
- This “Week”
 - ▶ Monday (5):
 - ★ unusual and influential data → robust estimation
 - ▶ Wednesday (7):
 - ★ non-linearity → generalized additive models
 - ▶ Monday (12):
 - ★ unusual errors → sandwich SEs
- Next Week
 - ▶ regression in social science
- Long Run
 - ▶ probability → inference → regression → causal inference

Questions?

Residuals are still **important**. Look at them.

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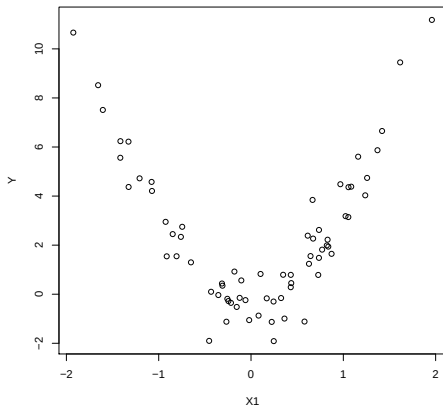
- 1 Assumptions and Violations
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Nonlinearity

Linearity of the Conditional Expectation Function ($y = \mathbf{X}\beta + \mathbf{u}$) is a key assumption. Why?

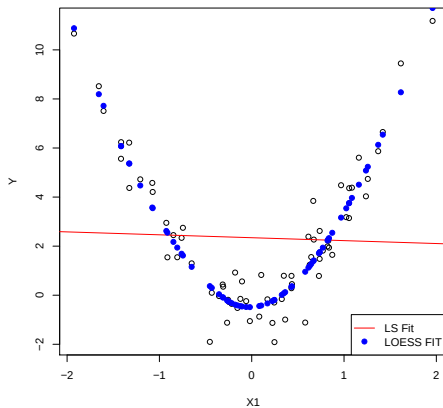
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 - ▶ Statements like “y increases with x” (monotonicity) are as specific as most social theories get.
 - ▶ Possible Exceptions: Returns to scale, constant elasticities, interactive effects, cyclical patterns in time series data, etc.
- Usually we employ “linearity by default” but we should try to make sure this is appropriate: **detect** non-linearities and **model** them accurately

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- For **marginal** relationships Y and X
 - ▶ Scatterplots with loess lines

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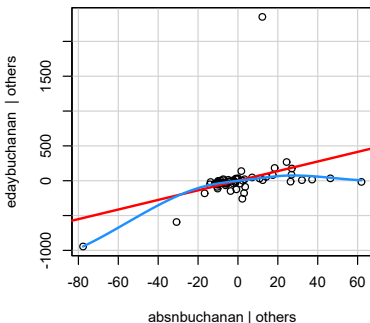
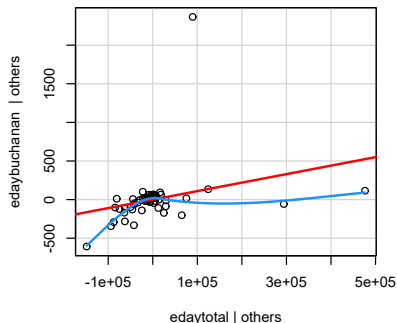
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Buchanan AV plot

R Code

```
par(mfrow = c(1,2))
out <- avPlots(mod3, "edaytotal")
lines(loess.smooth(x = out$edaytotal[,1],
  y= out$edaytotal[,2]), col = "dodgerblue", lwd = 2)
out2 <- avPlots(mod3, "absnbuchanan")
lines(loess.smooth(x = out2$absnbuchanan[,1],
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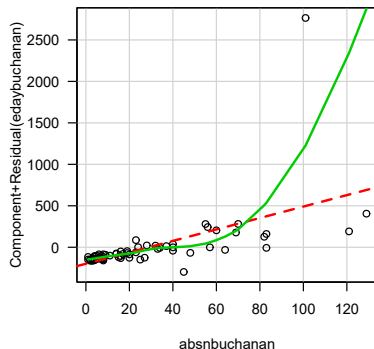
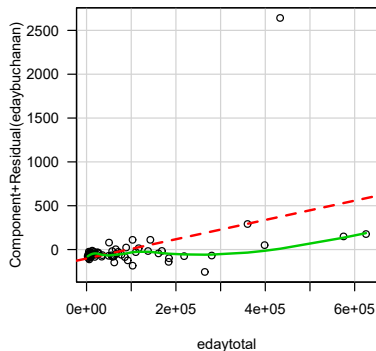
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Buchanan CR plot

R Code

```
crPlots(mod3, las = 1)
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Component + Residual Plots



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 - ▶ This suggests that linearizing the relationship between the X s through transformations can be helpful
 - ▶ Experience suggests weak non-linearities among X s do not invalidate CR plots

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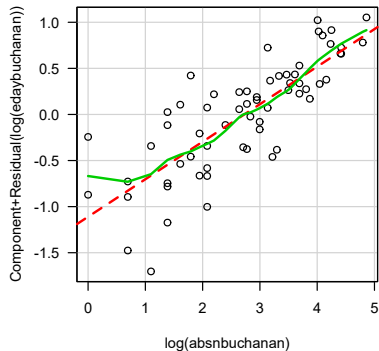
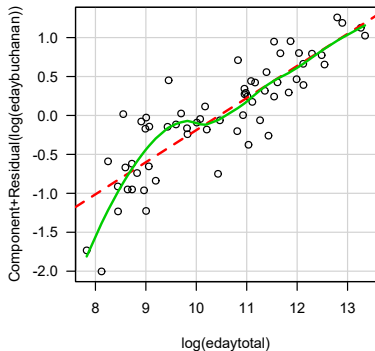
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I will teach you some, but many options.

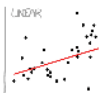
Transformed Buchanan regression

```
R Code  
mod.nopb2 <- lm(log(edaybuchanan) ~ log(edaytotal) + log(absnbuchanan),  
data = flvvote, subset = county != "Palm Beach")  
crPlots(mod.nopb2, las = 1)
```

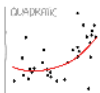
Component + Residual Plots



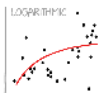
CURVE-FITTING METHODS AND THE MESSAGES THEY SEND



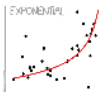
"HEY, I DID A
REGRESSION!"



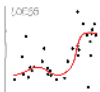
"I WANTED A CURVED
LINE, SO I MADE ONE
WITH MATH!"



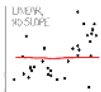
"LOOK, IT'S
TAPERING OFF!"



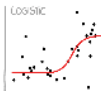
"LOOK, IT'S GROWING
UNCONTROLABLY!"



"I'M SOPHISTICATED, NOT
LIKE THOSE BUMBLING
POLYNOMIAL PEOPLE!"



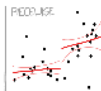
"I'M MAKING A
SCATTER PLOT, BUT
I DON'T WANT TO!"



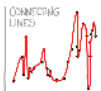
"I NEED TO CONNECT THESE
TWO LINES, BUT MY FIRST IDEAS
DON'T HAVE ENOUGH MATH!"



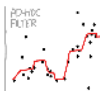
"LISTEN, SCIENCE IS HARD,
BUT I'M A SERIOUS
PERSON DOING MY BEST."



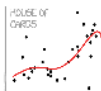
"I HAVE A THEORY,
AND THIS IS THE ONLY
DATA I COULD FIND!"



"I CONNECTED 'SMOOTH
LINES' IN EXCEL!"



"I HAD AN IDEA FOR HOW
TO CLEAN UP THE DATA.
WHAT DO YOU THINK?"



"YES YOU CAN SEE, THIS
MODEL SMOOTHLY FITS
THE-- WAIT NO NO DON'T
EXTEND IT ANAAAA!"

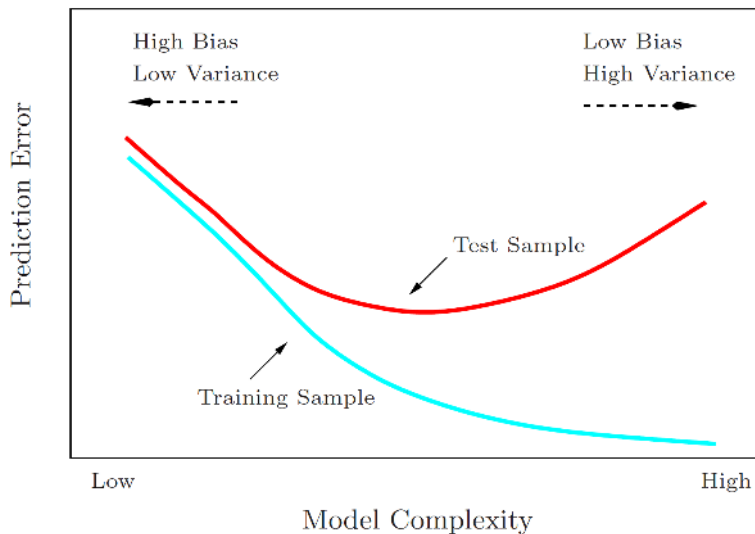
Thanks XKCD for having a comic for everything!

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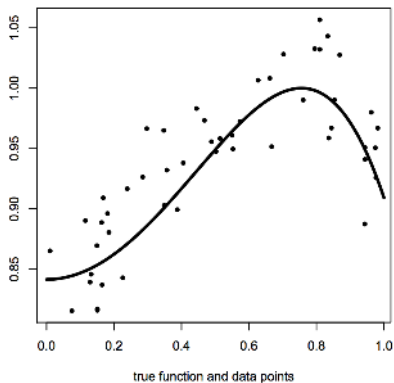
Bias-Variance Tradeoff

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Example Synthetic Problem

$$y = \sin(1 + x^2) + \epsilon$$



This section adapted from slides by Radford Neal.

Linear Basis Function Models

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- In general the idea is to do a linear regression of y on $\phi_1(x), \phi_2(x), \dots, \phi_{m-1}(x)$ where ϕ_j are **basis functions**.
- The model is now:

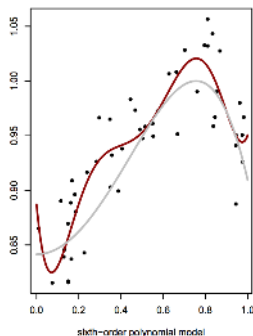
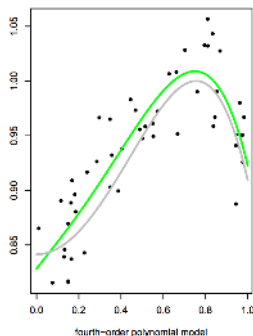
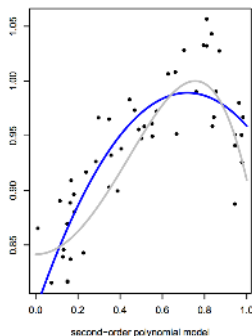
$$y = f(x, \beta) + \epsilon$$
$$f(x, \beta) = \beta_0 + \sum_{j=1}^{m-1} \beta_j \phi_j(x) = \beta^T \phi(x)$$

Polynomial Basis Functions

We can look at OLS fits with polynomial basis functions of increasing order.

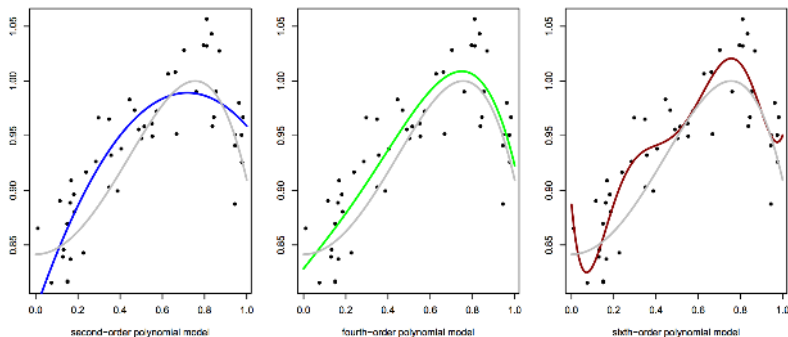
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It appears that the last model is too complex and is overfitting a bit.

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One choice is a Gaussian basis function

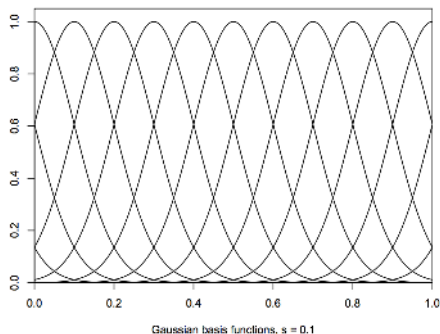
$$\phi_j(x) = \exp(-(x - \mu_j)^2/2s^2)$$

Local Basis Functions

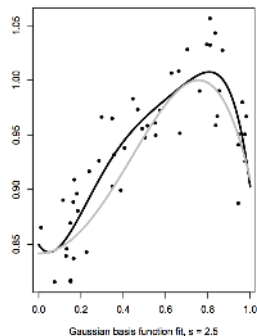
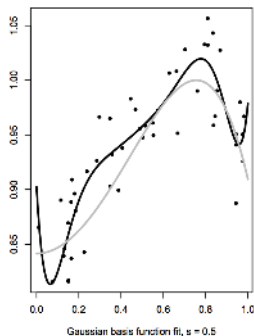
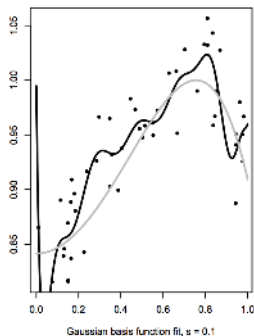
Polynomials are **global** basis functions, each affecting the prediction over the whole input space. Often **local** basis functions are more appropriate.

One choice is a Gaussian basis function

$$\phi_j(x) = \exp(-(x - \mu_j)^2 / 2s^2)$$



Gaussian Basis Fits



Regularization

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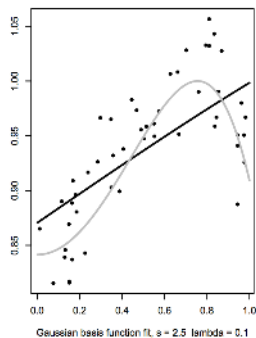
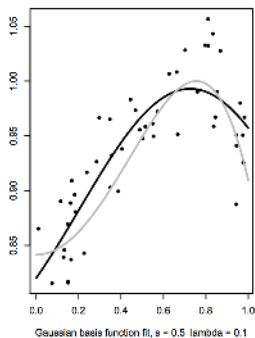
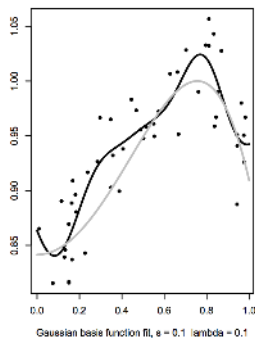
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- The trick in general is how to set λ

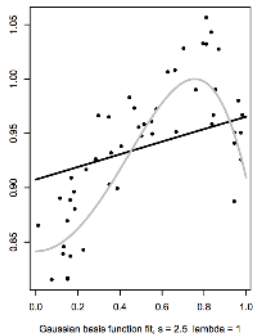
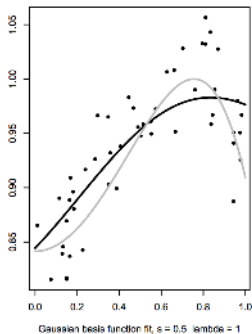
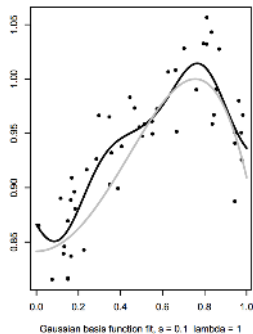
Results

Here are the results with $\lambda = 0.1$:



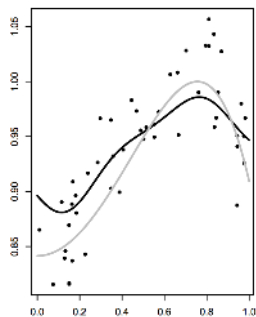
Results

Here are the results with $\lambda = 1$:

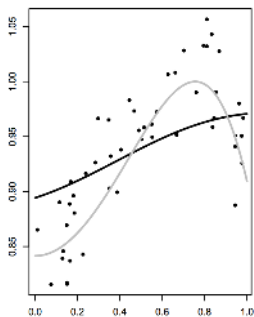


Results

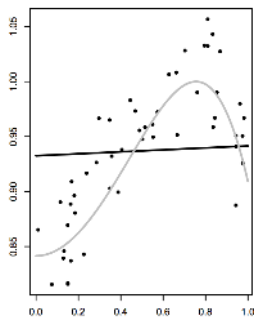
Here are the results with $\lambda = 10$:



Gaussian basis function fit, $\epsilon = 0.1$ $\lambda = 10$



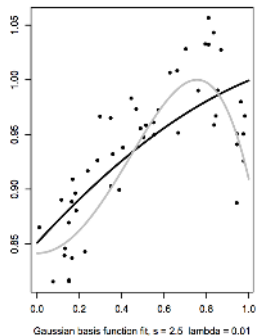
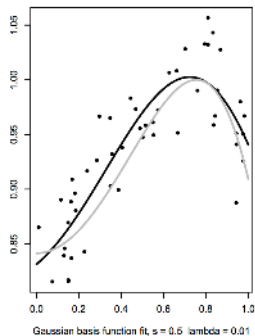
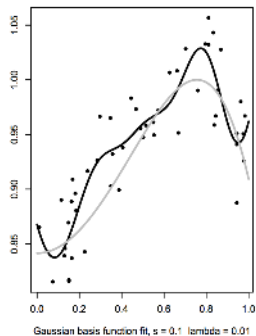
Gaussian basis function fit, $\epsilon = 0.5$ $\lambda = 10$



Gaussian basis function fit, $\epsilon = 2.5$ $\lambda = 10$

Results

Here are the results with $\lambda = 0.01$:



Conclusions from This Example

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- next up, **Generalized Additive Models**

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- Theory and estimation are somewhat involved, but they are easy to use:
 - ▶ `gam.out <- gam(y~s(x1)+s(x2)+x3)`
`plot(gam.out)`
 - ▶ Multiple functions but I recommend `mgcv` package

Generalized Additive Models (GAM)

The GAM approach can be extended to allow **interactions** ($s_{12}(\cdot)$) between explanatory variables, but this eats up degrees of freedom so you need a lot of data.

$$y_i = \beta_0 + s_{12}(x_{1i}, x_{2i}) + s_3(x_{3i}) + u_i$$

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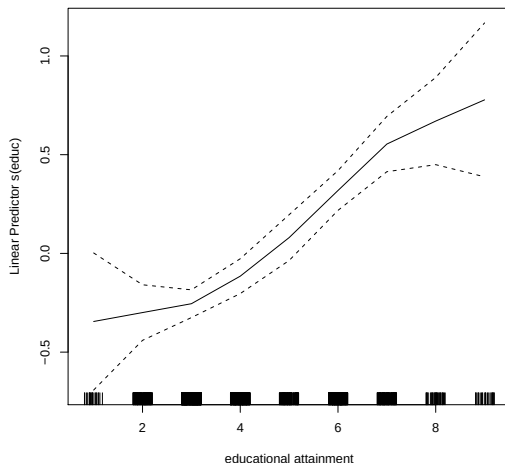
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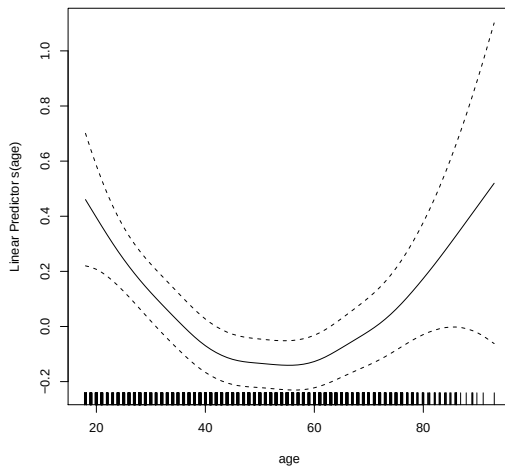
It can also be used for **hybrid models** where we model some variables as parametrically and other with a flexible function:

$$y_i = \beta_0 + \beta_1 x_{1i} + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$

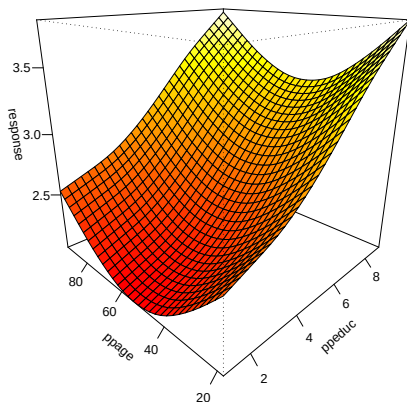
GAM Fit to Attitudes Toward Immigration



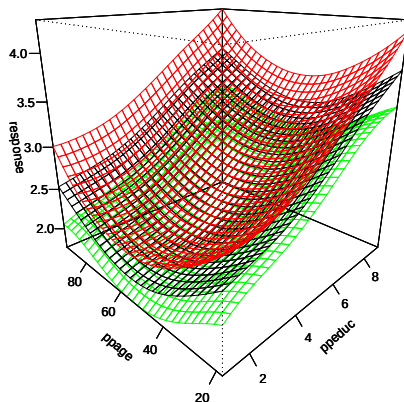
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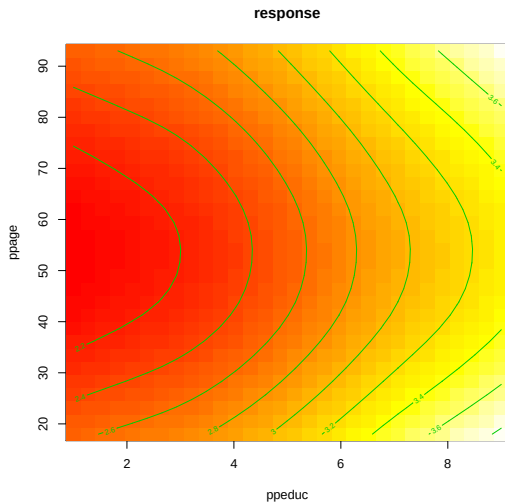


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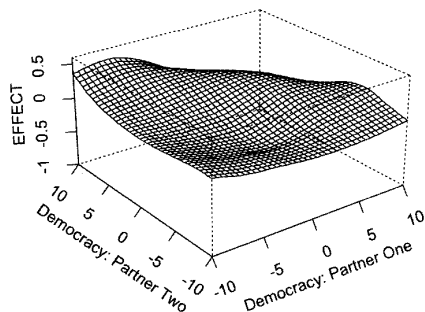
red/green are ± 2 s.e.

GAM Fit to Attitudes Toward Immigration

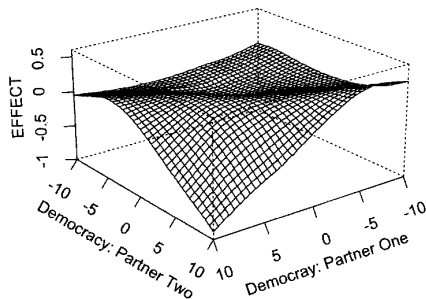


GAM Fit to Dyadic Democracy and Militarized Disputes

(a) Perspective of Non-Democracies



(b) Perspective of Democracies



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- **GAMs** are a great way to model/detect non-linearity but **transformations** are often simpler
- However, be wary of the **global** properties of transformations and polynomials
- Non-linearity concerns are most relevant for continuous covariates with a large range (age)

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Fun With Kernels

Hainmueller and Hazlett (2013). “Kernel Regularized Least Squares: Reducing Misspecification Bias with a Flexible and Interpretable Machine Learning Approach” *Political Analysis*.²

²I thank Chad Hazlett for sharing many of the slides that follow

Motivation: Misspecification Bias

Consider a data generating process such as:

```
> # Predictors
> GDP = runif(500)
> Polity = .5*GDP^2 + .2*runif(200)
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Regressing Stability on polity and GDP:

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> # OLS
> lm(Stability ~ Polity + GDP)
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	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-2.3000	0.1039	-22.145	< 2e-16 ***
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Entirely wrong conclusions!

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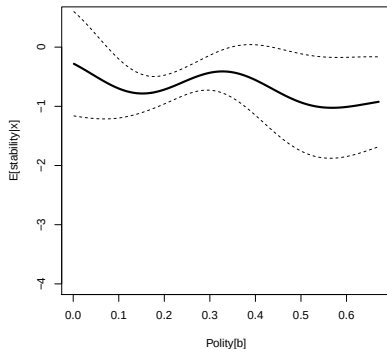
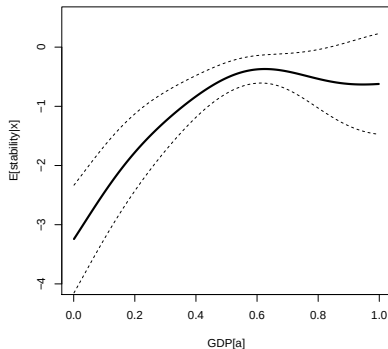
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$E[\text{stability}|\text{gdp, mean(polity)}]$ $E[\text{stability}|\text{mean(gdp), polity}]$



Kernel Basics

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For now, a kernel is a function $\mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}$

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Some kernels are naturally interpretable as a distance metric, e.g. the Gaussian:

Gaussian Kernel

$$k(\cdot, \cdot) : \mathbb{R}^D \times \mathbb{R}^P \mapsto \mathbb{R}$$

$$k(x_j, x_i) = e^{-\frac{\|x_j - x_i\|^2}{\sigma^2}}$$

where $\|x_j - x_i\|$ is the Euclidean distance between x_j and x_i

Using the Kernel Trick for Regression

- A feature map, $\phi : \mathbb{R}^P \mapsto \mathbb{R}^{P'}$, such that: $k(X_i, X_j) = \langle \phi(X_i), \phi(X_j) \rangle$

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- Solve the F.O.C.s:

$$R(\theta, \lambda) = \sum_{i=1}^N (Y_i - \phi(X_i)^T \theta)^2 + \lambda \theta^T \theta$$
$$\frac{\partial R(\theta, \lambda)}{\partial \theta} = -2 \sum_{i=1}^N \phi(X_i) (Y_i - \phi(X_i)^T \theta) + 2\lambda \theta = 0$$

How would humans learn this?



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Linear regression?

$$E[alt|lat, long] = \beta_0 + \beta_1 lat + \beta_2 long + \beta_3 lat \times long + \dots$$

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Similarity model:

$$E[alt|lat, long] = c_1(\text{similarity to obs1}) + \dots + c_5(\text{similarity to obs5})$$

Intuition: Similarity

Think of this function space as built on similarity:

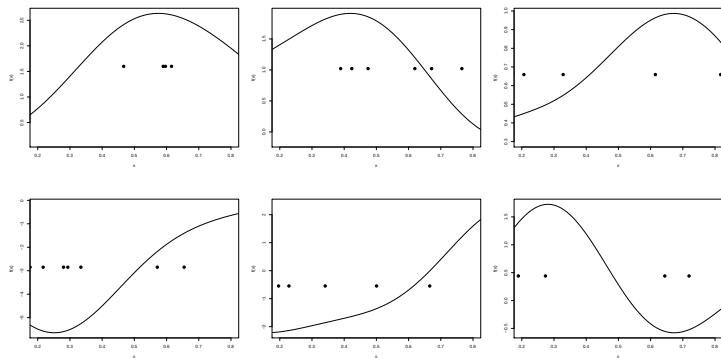
$$\begin{aligned} f(X^*) &= \sum_{i=1}^N c_i k(X^*, X_i) \\ &= c_1(\text{similarity of } X^* \text{ to } X_1) + \dots + c_N(\text{similarity of } X^* \text{ to } X_N) \end{aligned}$$

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Some random functions from this space:



A real example: Harff 2003

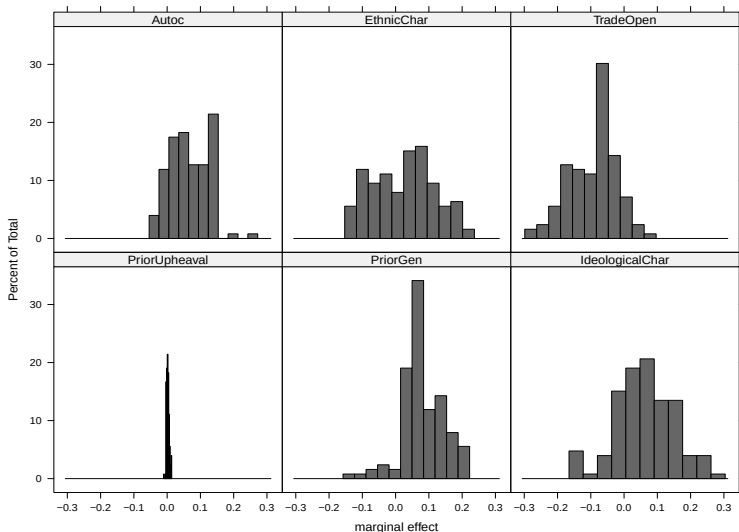
From `summary(krls(y,X))`

DV: Genocide onset		
	β_{OLS}	$E[\frac{\hat{dy}}{dx_i}]$
Prior upheaval	0.009* (0.004)	0.00 0.00
Prior genocide	0.26* (0.12)	0.19* (0.08)
Ideological char. elite	0.15* (0.084)	0.13* (0.08)
Autocracy	0.16* (0.077)	0.12* (0.07)
Ethnic char. elite	0.12 (0.084)	0.05 (0.08)
log(trade openness)	-0.17* (0.057)	-0.09* (0.03)

Behind the averages

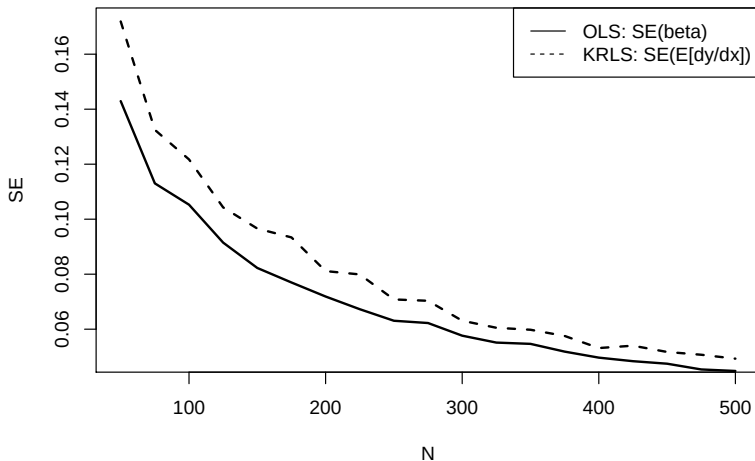
```
plot(krls(X,y))
```

Distributions of pointwise marginal effects

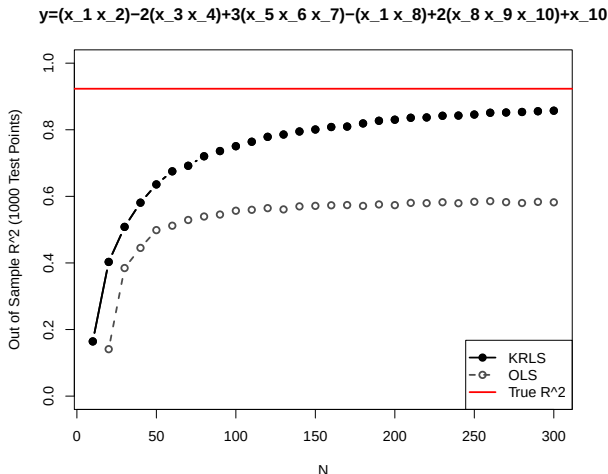


Efficiency Comparison

$$y = 2x + \epsilon, x \sim N(0, 1), \epsilon \sim N(0, 1)$$



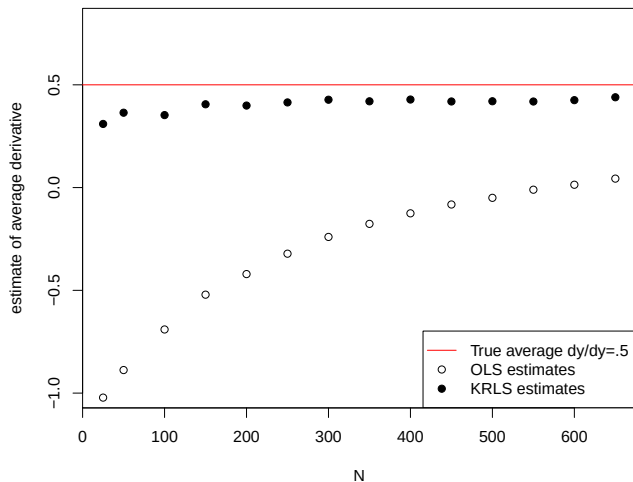
High-dimensional data with non-linearities



$y = (X_1 X_2) - 2(X_3 X_4) + 3(X_5 X_6 X_7) - (X_1 X_8) + 2(X_8 X_9 X_{10}) + X_{10} + \epsilon$ where all X are i.i.d. $Bernoulli(p)$ at varying p , $\epsilon \sim N(0, .5)$. 1,000 test points.

Linear model with bad leverage points

- $y = .5x + \varepsilon$ where $\varepsilon \sim N(0, .3)$
- One bad point, $(y_i = -5, x_i = 5)$.



Interaction or non-linearity?

Truth: $y = 5x_1^2 + \varepsilon$, $\rho(x_1, x_2) = .72$
 $\varepsilon \sim (0, .44)$. $x_1 \sim \text{Uniform}(0, 2)$

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Estimator	OLS	KRLS			
$\partial y / \partial x_{ij}$	Average	Average	1st Qu.	Median	3rd Qu.
const	-1.50 (0.34)				
x_1	7.51 (0.40)	9.22 (0.52)	5.22 (0.82)	9.38 (0.85)	14.03 (0.79)
x_2	-1.28 (0.21)	0.02 (0.13)	-0.08 (0.19)	0.00 (0.16)	0.10 (0.20)
$(x_1 \cdot x_2)$	1.24 (0.15)				
N	250				

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- ▶ it may model deep interactions but it is still hard to summarize deep interactions

References

- Hainmueller and Hazlett (2013). “Kernel Regularized Least Squares: Reducing Misspecification Bias with a Flexible and Interpretable Machine Learning Approach” *Political Analysis*.
- Beck, N. and Jackman, S. 1998. Beyond Linearity by Default: Generalized Additive Models. *American Journal of Political Science*.
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Where We've Been and Where We're Going...

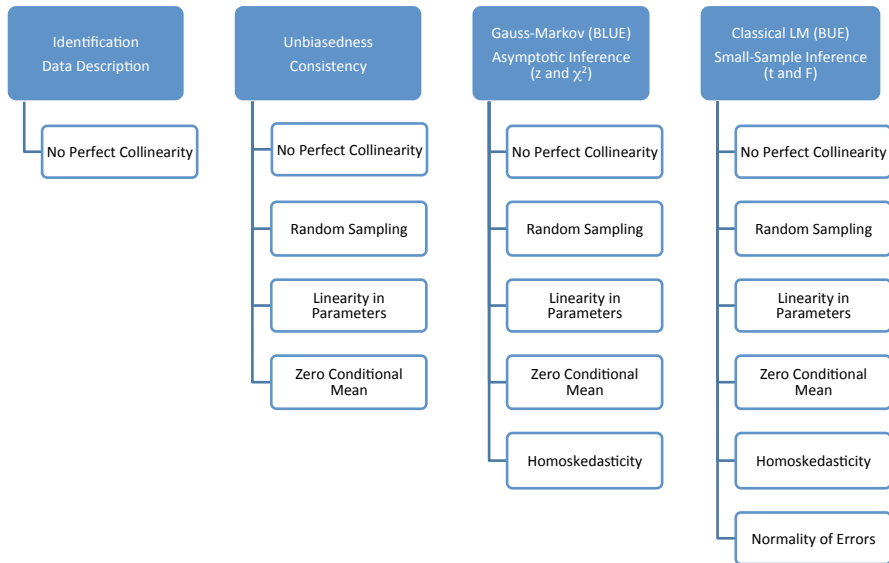
- Last Week
 - ▶ multiple regression
- This “Week”
 - ▶ Monday (5):
 - ★ unusual and influential data → robust estimation
 - ▶ Wednesday (7):
 - ★ non-linearity → generalized additive models
 - ▶ Monday (12):
 - ★ unusual errors → sandwich SEs
- Next Week
 - ▶ regression in social science
- Long Run
 - ▶ probability → inference → regression → causal inference

Questions?

- 1 Assumptions and Violations
- 2 Non-normality
- 3 Outliers
- 4 Robust Regression Methods
- 5 Fun with Outliers
- 6 Appendix: Robustness
- 7 Detecting Nonlinearity
- 8 Linear Basis Function Models
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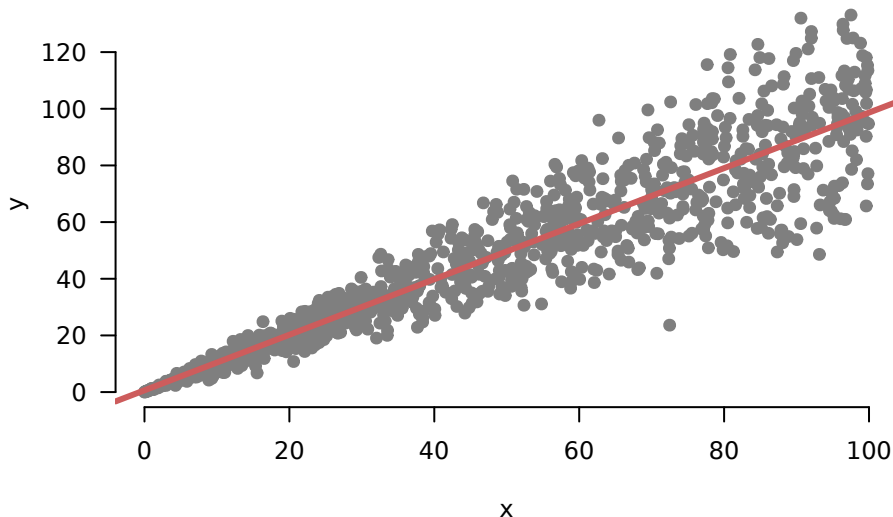
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- ① Linearity: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$
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- 1-4 give us unbiasedness/consistency
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 - 1-6 allow for small-sample inference

Today: How Do We Deal With This?



Plan for Today

Talk about different forms of error variance problems

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Then we will discuss a **contrarian** view

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- Replace σ^2 with estimate $\hat{\sigma}^2$ will give us our estimate of the covariance matrix

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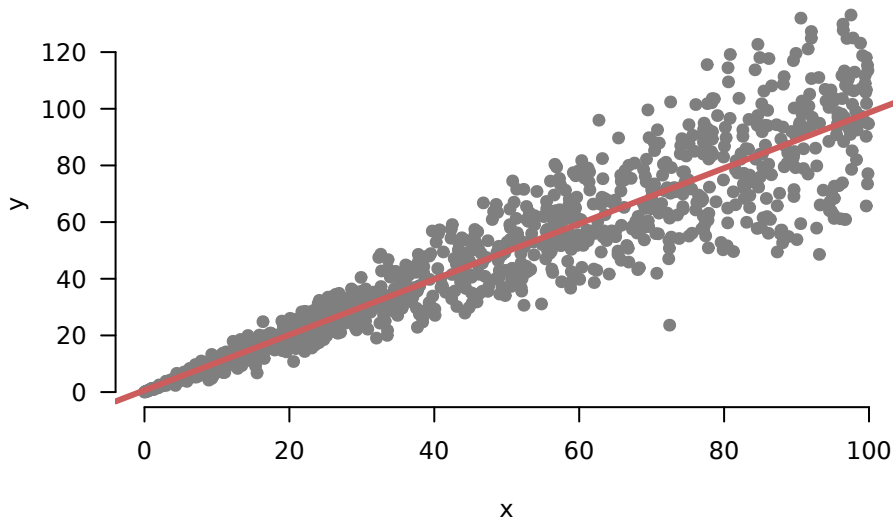
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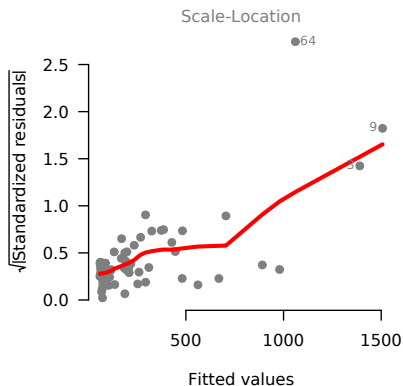
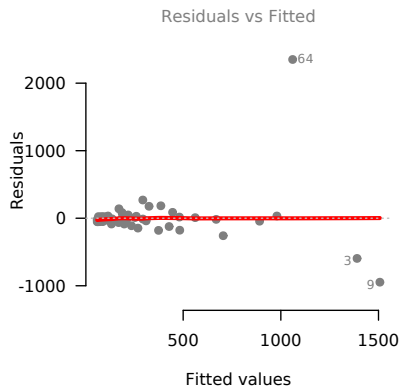
Example: Buchanan votes

```
flvote <- foreign::read.dta("flbuchan.dta")
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)

##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 5.423e+01  4.914e+01   1.104   0.274
## edaytotal    2.323e-03  3.104e-04   7.483 2.42e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 332.7 on 65 degrees of freedom
## Multiple R-squared:  0.4628, Adjusted R-squared:  0.4545
## F-statistic:      56 on 1 and 65 DF,  p-value: 2.417e-10
```


Diagnostics

```
par(mfrow = c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")  
plot(mod, which = 1, lwd = 3)  
plot(mod, which = 3, lwd = 3)
```



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 - ▶ This implies that, under H_0 , the **squared residuals** should also be unrelated to the explanatory variables

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 - ▶ In R, `bptest` in the `lmtest` package

Breusch-Pagan Example

```
library(lmtest)
bptest(mod)

##
##  studentized Breusch-Pagan test
##
## data:  mod
## BP = 12.59, df = 1, p-value = 0.0003878
```

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- 4 Admit we have the **wrong model** and use a different approach

Appendix: Variance Stabilizing Transformations

If the variance for each error (σ_i^2) is proportional to some function of the mean ($\mathbf{x}_i\boldsymbol{\beta}$), then a variance stabilizing transformation may be appropriate.

Note: Transformations will affect the other regression assumptions, as well as interpretation of the regression coefficients.

Examples:

Transformation	Mean/Variance Relationship
$\sqrt{Y}$	$\sigma_i^2 \propto \mathbf{x}_i\boldsymbol{\beta}$
$\log Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^2$
$1/Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^4$

Heteroskedasticity Consistent Estimator

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- Under non-constant error variance:

$$\text{Var}[\mathbf{u}] = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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- When $\Sigma \neq \sigma^2 \mathbf{I}$, we are stuck with this expression:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\Sigma\mathbf{X} (\mathbf{X}'\mathbf{X})^{-1}$$

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- Idea: If we can consistently estimate the components of Σ , we could directly use this expression by replacing Σ with its estimate, $\hat{\Sigma}$.

White's Heteroskedasticity Consistent Estimator

Suppose we have **heteroskedasticity of unknown form** (but zero covariance):

$$V[\mathbf{u}] = \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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$$\widehat{V[\hat{\beta}|\mathbf{X}]} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{u}_2^2 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \hat{u}_n^2 \end{bmatrix} \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

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The estimate based on the above is called the **heteroskedasticity consistent (HC)** or **robust standard errors**.

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- 3 Plug $\hat{\Sigma}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix

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- There are various **small sample corrections** to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|\mathbf{X}] = \frac{n}{n-k-1} \cdot (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\hat{\Sigma}\mathbf{X} (\mathbf{X}'\mathbf{X})^{-1}$$

Regular & Robust Standard Errors in Florida Example

R Code

```
> library(sandwich)
> library(lmtest)
> coeftest(mod1) # homoskedasticity
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.9141e+01  1.1036   0.2738
TotalVotes00  2.3229e-03 3.1041e-04  7.4831 2.417e-10 ***

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC0")) # classic White
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.0612e+01  1.3353  0.18642
TotalVotes00  2.3229e-03 8.7047e-04  2.6685  0.00961 **

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC1")) # small sample correction
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.1232e+01  1.3153  0.19304
TotalVotes00  2.3229e-03 8.8376e-04  2.6284  0.01069 *
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- For small n , performance might be poor (correction factors exist but are often insufficient)
- We can arrive at White's heteroskedasticity consistent standard errors using the **plug-in principle** and thus in some ways, these are the natural way of getting standard errors in the agnostic regression framework.

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- 2 Non-normality
- 3 Outliers
- 4 Robust Regression Methods
- 5 Fun with Outliers
- 6 Appendix: Robustness
- 7 Detecting Nonlinearity
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- Called clustering or clustered dependence

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- Ignoring clustering is “cheating”: units not independent

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$$\begin{aligned} y_{ij} &= \beta_0 + \beta_1 x_{ij} + \varepsilon_{ij} \\ &= \beta_0 + \beta_1 x_{ij} + \textcolor{blue}{v}_j + \textcolor{red}{u}_{ij} \end{aligned}$$

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- What if we ignore this structure and just use ε_{ij} as the error?
- Variance of the composite error is σ^2 :

$$\begin{aligned}\text{Var}[\varepsilon_{ij}] &= \text{Var}[v_j + u_{ij}] \\ &= \text{Var}[v_j] + \text{Var}[u_{ij}] \\ &= \rho\sigma^2 + (1 - \rho)\sigma^2 = \sigma^2\end{aligned}$$

Lack of Independence

- Covariance between two units i and s in the same cluster is $\rho\sigma^2$:

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- Zero covariance of two units i and s in different clusters j and k :

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sk}] = 0$$

Example Covariance Matrix

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{bmatrix}'$$

$$\text{Var}[\boldsymbol{\varepsilon}] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 \end{bmatrix}$$

Appendix: Example 6 Units, 2 Clusters

$$\varepsilon = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{bmatrix}'$$

$$V[\varepsilon] = \Sigma = \begin{bmatrix} V[\varepsilon_{1,1}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{1,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{1,1}] & . & . & . \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{2,1}] & V[\varepsilon_{2,1}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{2,1}] & . & . & . \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{3,1}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{3,1}] & V[\varepsilon_{3,1}] & . & . & . \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{4,2}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{4,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{4,2}] & V[\varepsilon_{4,2}] & . & . \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{5,2}] & \text{Cov}[\varepsilon_{4,2}, \varepsilon_{5,2}] & V[\varepsilon_{5,2}] & . \\ \text{Cov}[\varepsilon_{1,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{2,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{3,1}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{4,2}, \varepsilon_{6,2}] & \text{Cov}[\varepsilon_{5,2}, \varepsilon_{6,2}] & V[\varepsilon_{6,2}] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 \end{bmatrix}$$

which can be verified as follows:

- $V[\varepsilon_{ij}] = V[v_j + u_{ij}] = V[v_j] + V[u_{ij}] = \rho\sigma^2 + (1 - \rho)\sigma^2 = \sigma^2$
- $\text{Cov}[\varepsilon_{ij}, \varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] - E[\varepsilon_{ij}]E[\varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] = E[(v_j + u_{ij})(v_j + u_{lj})]$
 $= E[v_j^2] + E[v_j u_{lj}] + E[v_j u_{ij}] + E[u_{ij} u_{lj}]$
 $= E[v_j^2] + E[v_j]E[u_{lj}] + E[v_j]E[u_{ij}] + E[u_{ij}]E[u_{lj}]$
 $= E[v_j^2] = V[v_j] + (E[v_j])^2 = V[v_j] = \rho\sigma^2$
- $\text{Cov}[\varepsilon_{ij}, \varepsilon_{lk}] = E[\varepsilon_{ij}\varepsilon_{lk}] - E[\varepsilon_{ij}]E[\varepsilon_{lk}] = E[\varepsilon_{ij}\varepsilon_{lk}] = E[(v_j + u_{ij})(v_k + u_{lk})]$
 $= E[v_j v_k] + E[v_j u_{lk}] + E[v_k u_{ij}] + E[u_{ij} u_{lk}]$
 $= E[v_j]E[v_k] + E[v_j]E[u_{lk}] + E[v_k]E[u_{ij}] + E[u_{ij}]E[u_{lk}] = 0$

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- Under this clustered dependence, we can write this as:

$$\text{Var}[\hat{\boldsymbol{\beta}}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{j=1}^m \mathbf{x}'_j \boldsymbol{\Sigma}_j \mathbf{x}_j \right) (\mathbf{X}'\mathbf{X})^{-1}$$

Estimating the Variance Components: ρ and σ^2

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- 3 The within cluster correlation is then estimated as: $\hat{\rho} = \frac{\hat{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2}$

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- There are multiple implementations in R including `multiwayvcov:cluster.vcov` and `sandwich::vcovCL`

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- There are numerous alternative clustered standard error variants out there.

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- 2 Non-normality
- 3 Outliers
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- 5 Fun with Outliers
- 6 Appendix: Robustness
- 7 Detecting Nonlinearity
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A Contrarian View of Robust Standard Errors

King, Gary and Margaret E. Roberts. “How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It” *Political Analysis* (2015) 23: 159-179.³

³I thank Gary and Molly for the slides that follow.

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- RSEs are not useful, as a “fix”

Their Alternative Procedure

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Robust standard errors:

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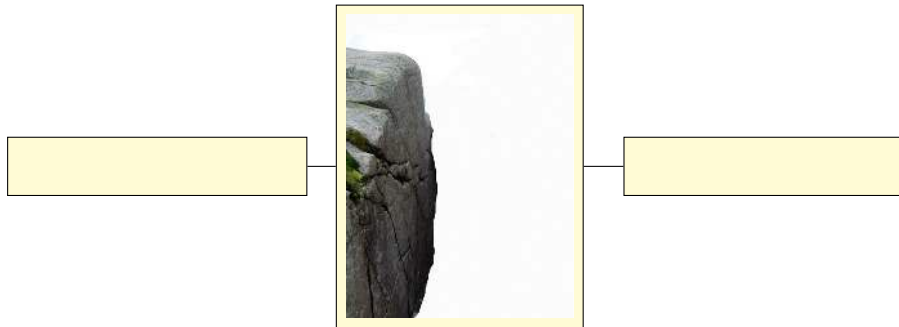
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- 3 Keeping going, until they don't differ.

For RSEs to help: Everything has to be Juuuussttt Right

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Model Correct



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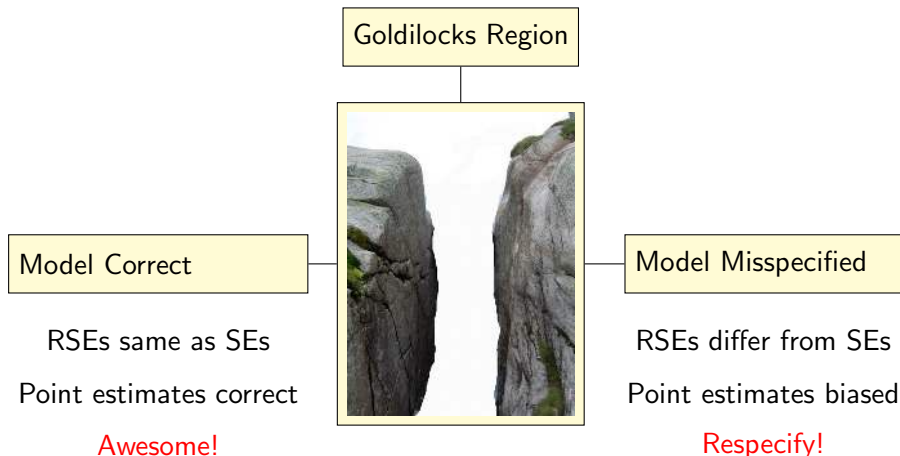
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Goldilocks Region

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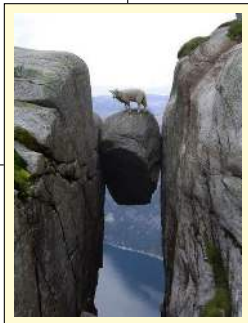
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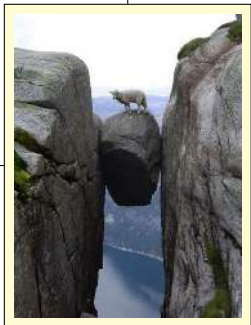
Goldilocks Region

but not so much as to
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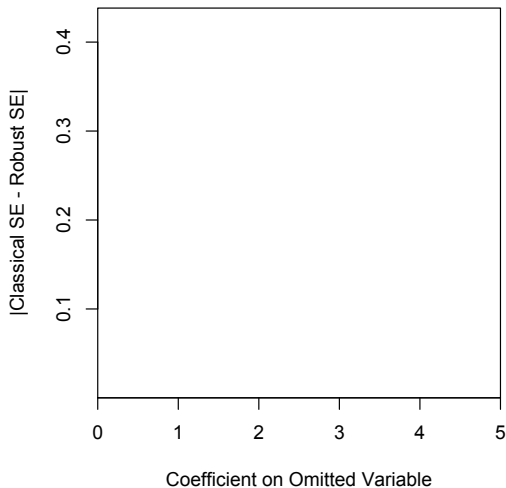
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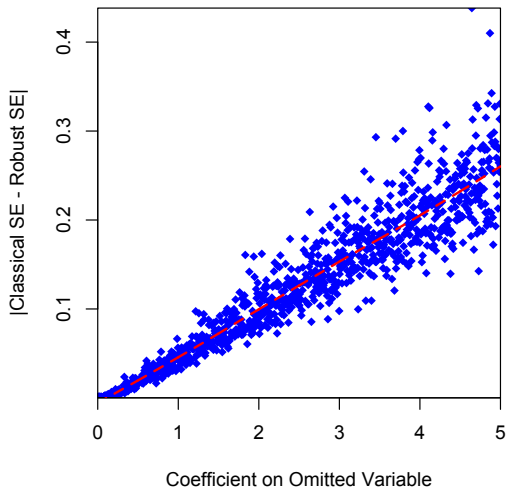


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- Parts of the model are wrong; **why do we think the rest are right?**

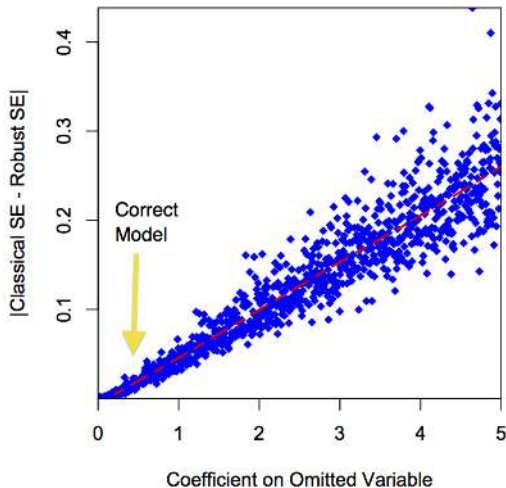
Difference Between SE and RSE Exposes Misspecification



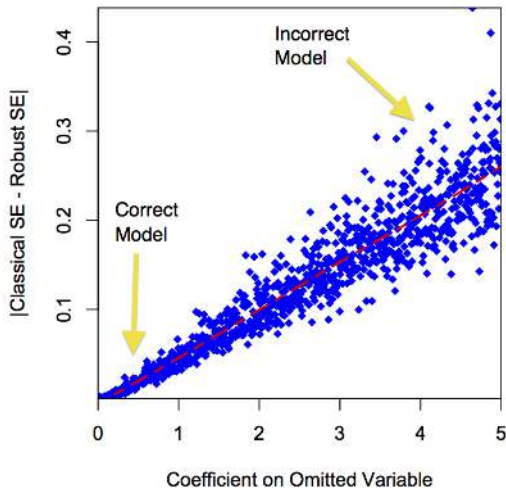
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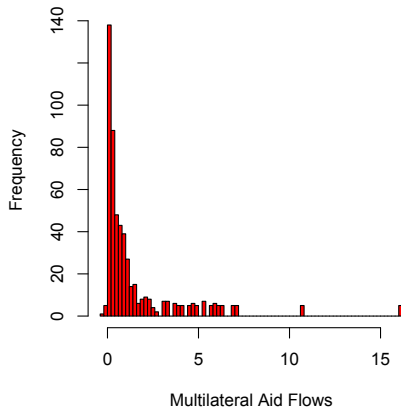
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 - ▶ $\Rightarrow$ indicates model misspecification

Problem: Highly Skewed Dependent Variable

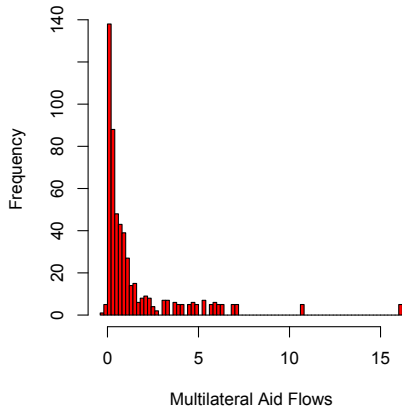
Problem: Highly Skewed Dependent Variable

Original

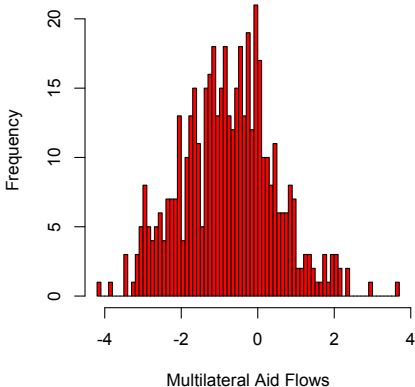


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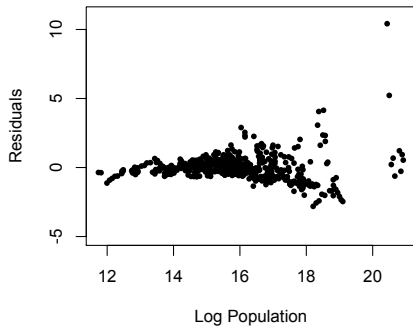
Transformed



Diagnostics: Reveal Misspecification

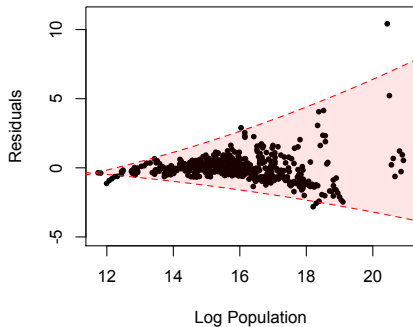
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Population vs Residuals, Author's Model



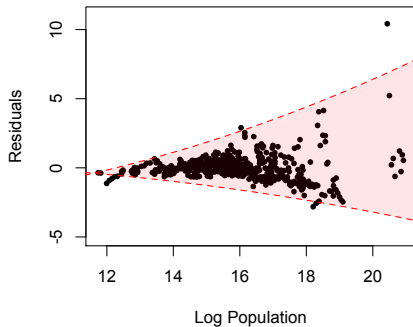
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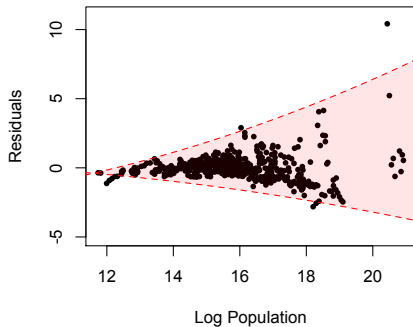
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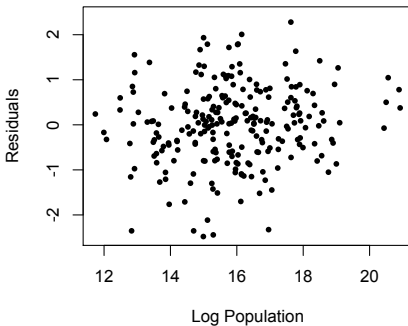
Textbook case of heteroskedasticity

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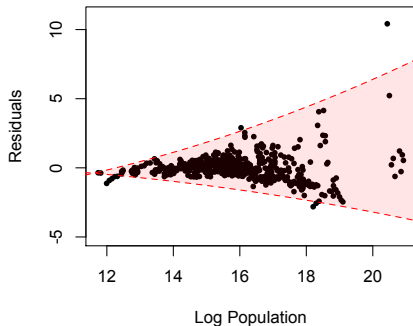
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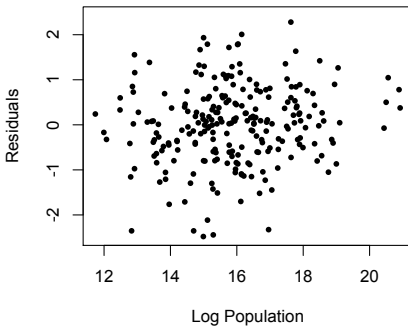
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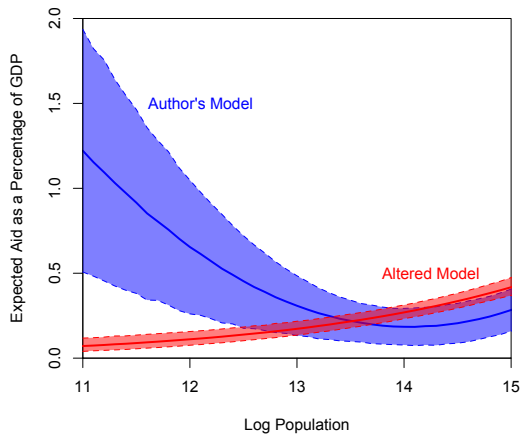


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Concluding Contrarian Thoughts

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- Fixing these problems $\Rightarrow$ **hugely** different substantive conclusions

Concluding Thoughts on Diagnostics

Residuals are **important**. Look at them.

Next 'Week'

- Regression in the Social Sciences and An Introduction to Causal Inference
- Reading:
 - ▶ Healy *Data Visualization: A practical introduction*
<http://socviz.co/> Chapter 1: Look at Data
 - ▶ Morgan and Winship Chapter 1: Causality and Empirical Research in the Social Sciences
 - ▶ Morgan and Winship Chapter 13.1: Objections to Adoption of the Counterfactual Approach
 - ▶ Angrist and Pischke Chapters 1-2
 - ▶ Hernan and Robins (2016) Chapter 1: A definition of a causal effect
<https://www.hsph.harvard.edu/miguel-hernan/causal-inference-book/>
- As a side note: if you want to read the argument against the contrarian response: Aronow (2016) "A Note on 'How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It.'" It is an interesting piece- feel free to come talk to me about this debate!

- 1 Assumptions and Violations
- 2 Non-normality
- 3 Outliers
- 4 Robust Regression Methods
- 5 Fun with Outliers
- 6 Appendix: Robustness
- 7 Detecting Nonlinearity
- 8 Linear Basis Function Models
- 9 Generalized Additive Models
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Zhukov, Yuri M. and Brandon M. Stewart. "Choosing Your Neighbors: Networks of Diffusion in International Relations" *International Studies Quarterly* 2013; 57: 271-287.

Our Main Questions

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How Do We Generally Choose Neighbors?

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- ② 75% of articles with spatial variable in the top 6 IR journals of the last 10 years

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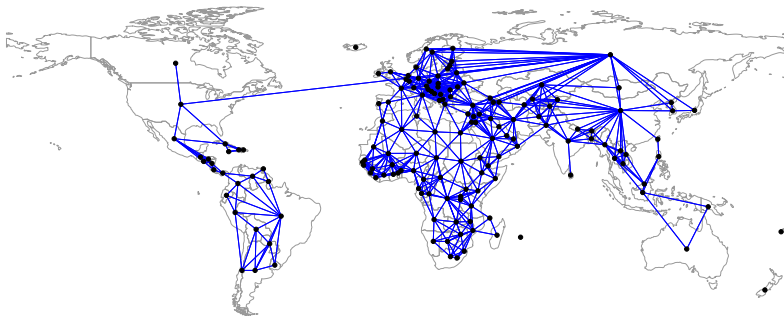
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- ④ Different types of neighbors tell different stories

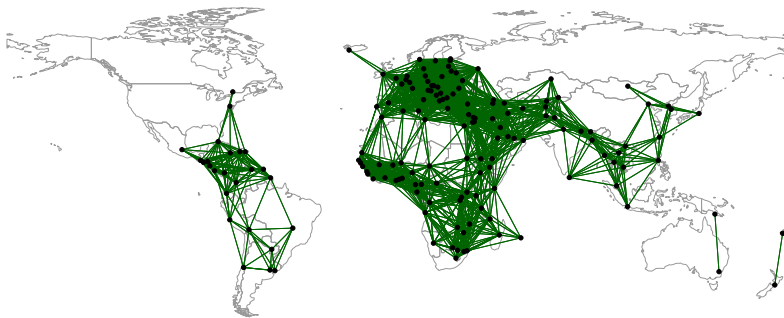
Visualization of Connections: Contiguity

Figure: Contiguity neighbors with 500 km snap distance



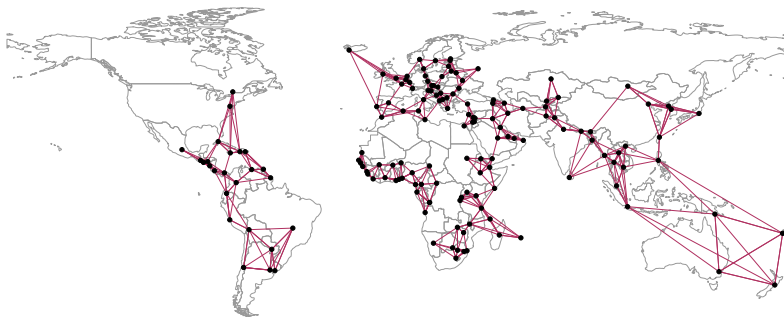
Visualization of Connections: Minimum Distance

Figure: Minimum distance neighbors (capital cities)



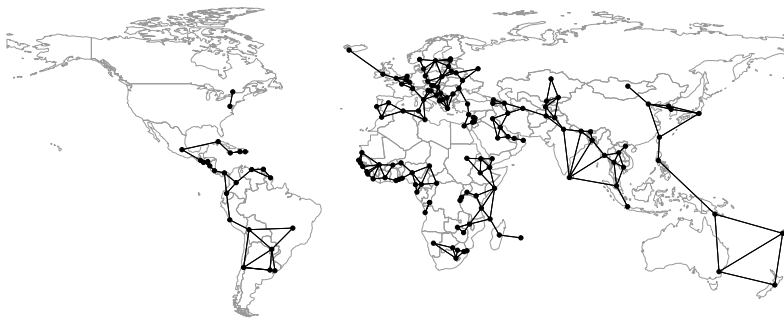
Visualization of Connections: K-Nearest Neighbors

Figure: $k = 4$ Nearest Neighbors (capital cities)



Visualization of Connections: Graph-based Neighbors

Figure: Sphere of Influence Neighbors (capital cities)



Application: Democratic Diffusion

Gleditsch and Ward (2006)

Changes of political regime modeled as a first-order Markov chain process with the transition matrix

$$\mathbf{K} = \begin{bmatrix} Pr(y_{i,t} = 0 | y_{i,t-1} = 0) & Pr(y_{i,t} = 1 | y_{i,t-1} = 0) \\ Pr(y_{i,t} = 0 | y_{i,t-1} = 1) & Pr(y_{i,t} = 1 | y_{i,t-1} = 1) \end{bmatrix}$$

where $y_{i,t} = 1$ if an (A)utocratic regime exists in country i at time t , and $y_{i,t} = 0$ if the regime is (D)emocratic.

... in other words:

$$\mathbf{K} = \begin{bmatrix} Pr(D \rightarrow D) & Pr(D \rightarrow A) \\ Pr(A \rightarrow D) & Pr(A \rightarrow A) \end{bmatrix}$$

Equilibrium Effects of Democratic Transition

If a regime transition takes place in country i , what is the change in predicted probability of a regime transition in country j (country i 's neighbor)?

$$QI = Pr(y_{j,t} | y_{i,t} = y_{i,t-1}) - Pr(y_{j,t} | y_{i,t} \neq y_{i,t-1})$$

where $y_{i,t} = 0$ if country i is a democracy at time t and $y_{i,t} = 1$ if it is an autocracy. All other covariates are held constant.

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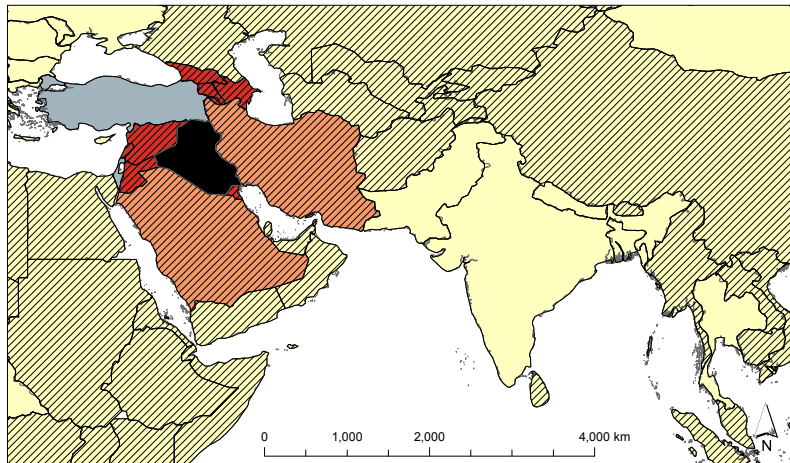
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Illustrative cases

- Iraq transitions from autocracy to democracy.
- Russia transitions from democracy to autocracy.

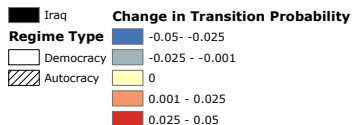
Iraq's democratization and regional regime stability



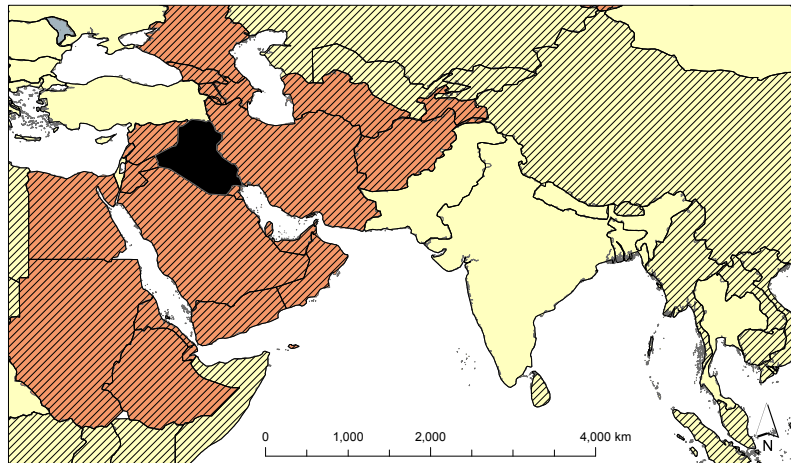
Contiguity + 500 km

Iraq transitions from autocracy to democracy
(1998 data)

Monte Carlo simulation (1,000 runs)



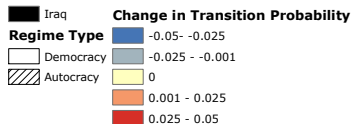
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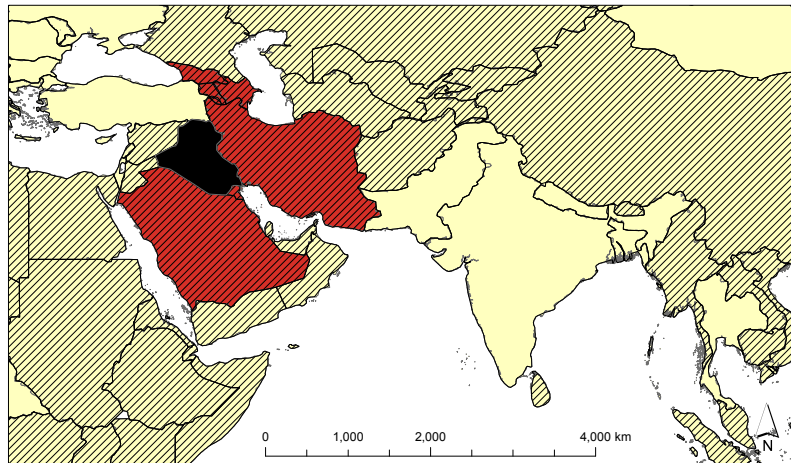
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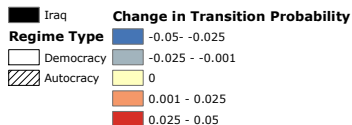
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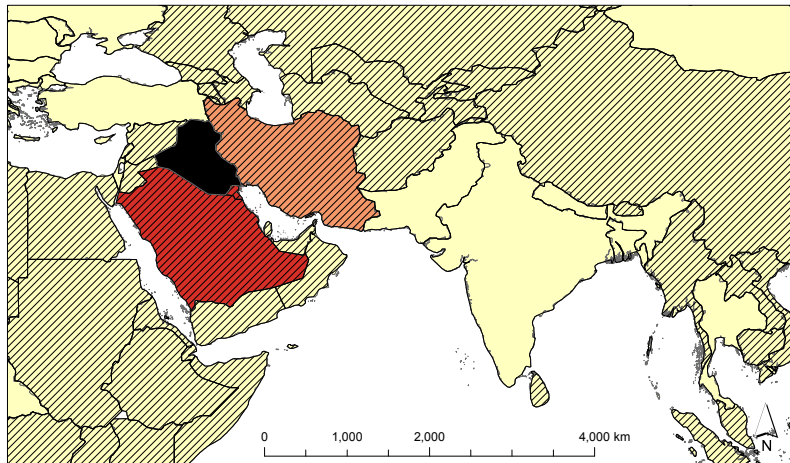
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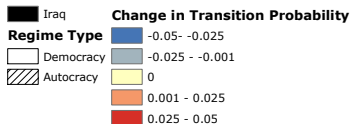
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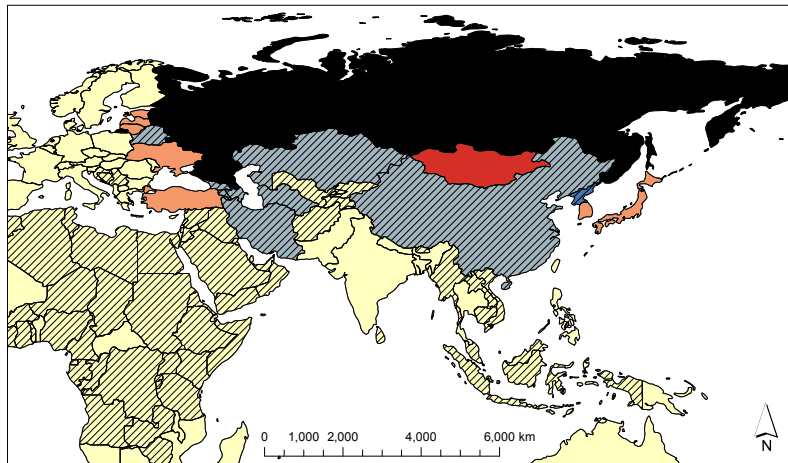
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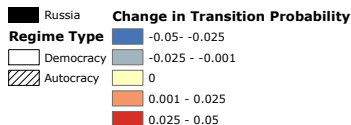
Russia's autocratization and regional regime stability



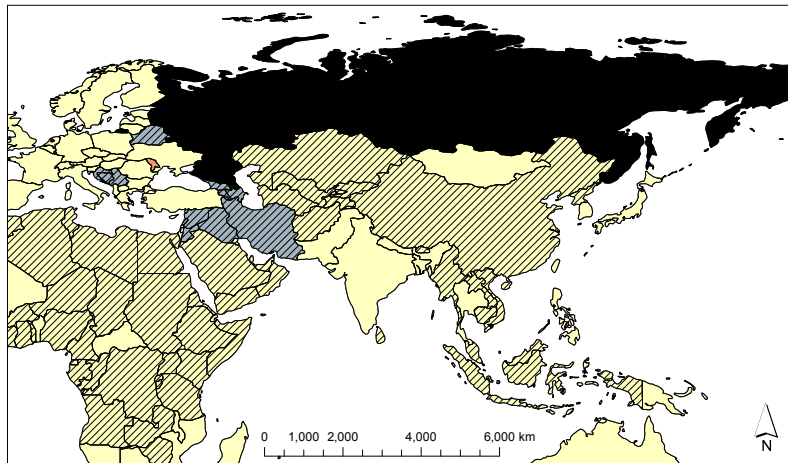
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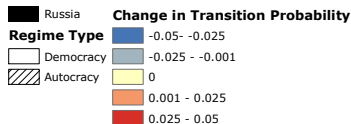
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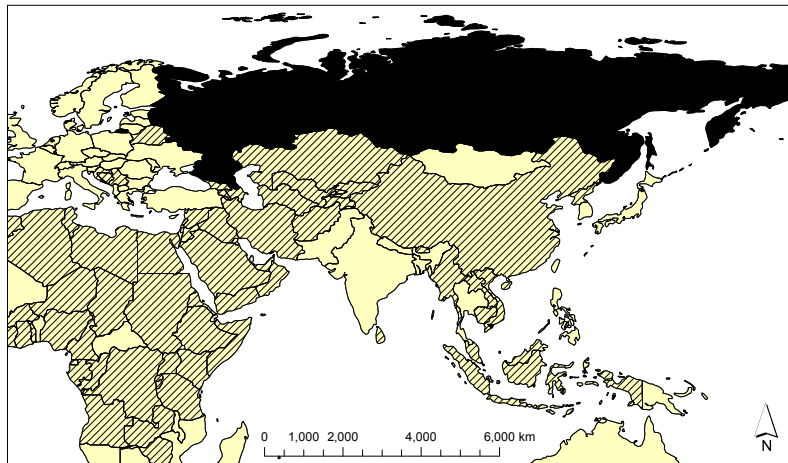
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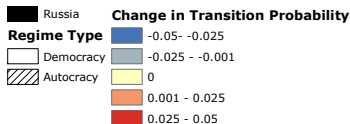
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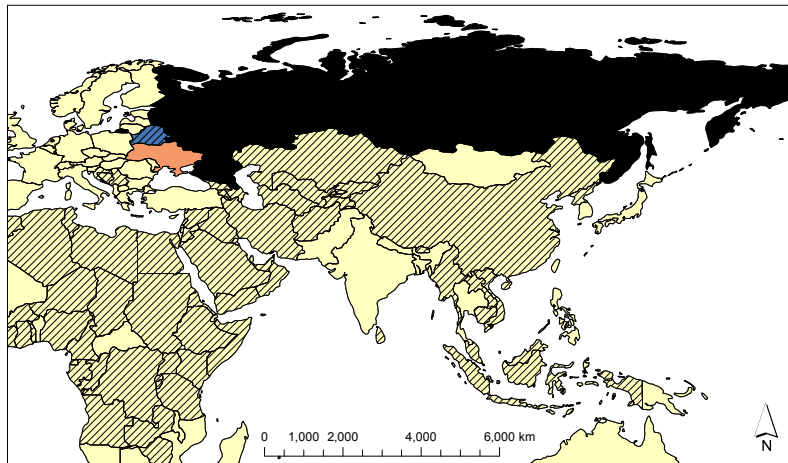
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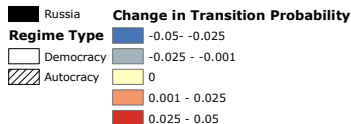
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- 2 Non-normality
- 3 Outliers
- 4 Robust Regression Methods
- 5 Fun with Outliers
- 6 Appendix: Robustness
- 7 Detecting Nonlinearity
- 8 Linear Basis Function Models
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Appendix: Weighted Least Squares

- Suppose that the heteroskedasticity is known up to a multiplicative constant:

$$\text{Var}[u_i|\mathbf{X}] = a_i\sigma^2$$

where $a_i = a_i(\mathbf{x}'_i)$ is a positive and known function of $\mathbf{x}'_i$

- WLS: multiply y_i by $1/\sqrt{a_i}$:

$$y_i/\sqrt{a_i} = \beta_0/\sqrt{a_i} + \beta_1 x_{i1}/\sqrt{a_i} + \cdots + \beta_k x_{ik}/\sqrt{a_i} + u_i/\sqrt{a_i}$$

Appendix: Weighted Least Squares Intuition

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- Rescales errors to $u_i/\sqrt{a_i}$, which maintains zero mean error
- But makes the error variance constant again:

$$\text{Var} \left[\frac{1}{\sqrt{a_i}} u_i | \mathbf{X} \right] = \frac{1}{a_i} \text{Var} [u_i | \mathbf{X}]$$

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- If you know a_i , then you can use this approach to makes the model homoskedastic and, thus, BLUE again
- When do we know a_i ?

Appendix: Weighted Least Squares procedure

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- Define the weighting matrix:

$$\mathbf{W} = \begin{bmatrix} 1/\sqrt{a_1} & 0 & 0 & 0 \\ 0 & 1/\sqrt{a_2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1/\sqrt{a_n} \end{bmatrix}$$

- Run the following regression:

$$\mathbf{W}\mathbf{y} = \mathbf{W}\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{u}$$

$$\mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \mathbf{u}^*$$

- Run regression of $\mathbf{y}^* = \mathbf{W}\mathbf{y}$ on $\mathbf{X}^* = \mathbf{W}\mathbf{X}$ and all Gauss-Markov assumptions are satisfied
- Plugging into the usual formula for $\hat{\boldsymbol{\beta}}$:

$$\hat{\boldsymbol{\beta}}_W = (\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{y}$$

Appendix: WLS Example

- In R, use `weights = argument to lm` and give the weights squared: $1/a_i$
- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

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- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

```
mod.wls <- lm(edaybuchanan ~ edaytotal, weights = 1/edaytotal,  
              data = flvot)
```

```
summary(mod.wls)
```

```
##
```

```
## Coefficients:
```

```
##           Estimate Std. Error t value Pr(>|t|)
```

```
## (Intercept) 2.707e+01  8.507e+00   3.182  0.00225 **
```

```
## edaytotal    2.628e-03  2.502e-04  10.503 1.22e-15 ***
```

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
##
```

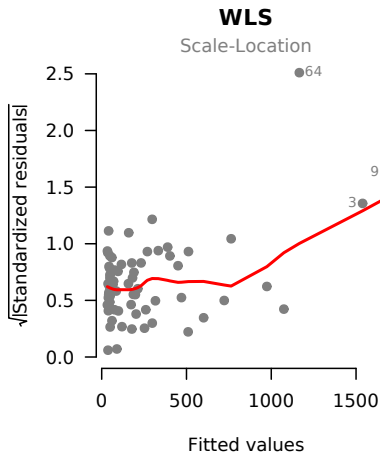
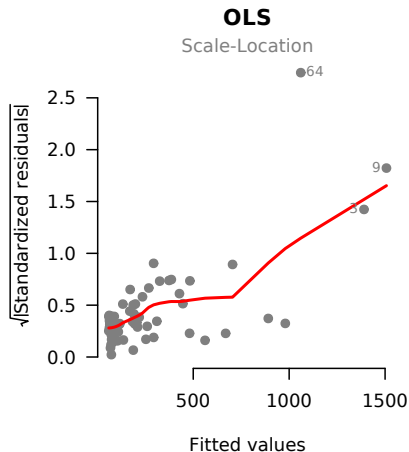
```
## Residual standard error: 0.5645 on 65 degrees of freedom
```

```
## Multiple R-squared:  0.6292, Adjusted R-squared:  0.6235
```

```
## F-statistic: 110.3 on 1 and 65 DF,  p-value: 1.22e-15
```

Appendix: Comparing WLS to OLS

```
par(mfrow=c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")  
plot(mod, which = 3, main = "OLS", lwd = 2)  
plot(mod.wls, which = 3, main = "WLS", lwd = 2)
```



Time Dependence: Serial Correlation

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 $t = 1, \dots, T$

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- Often have **serially correlated**: errors in one time period are correlated with errors in other time periods
- Many different ways for this to happen, but we often assume a very limited type of dependence called AR(1).

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$$u_t = \rho u_{t-1} + e_t \quad \text{where} \quad |\rho| < 1$$

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- Generalizes to higher order serial correlation (e.g. an AR(2) model is given by $u_t = \rho u_{t-1} + \delta u_{t-2} + e_t$).

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ρ is usually positive, which implies that we underestimate the variance if we ignore serial correlation.

How to Detect and Fix Serial Correlated Errors

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Detection:

- Plot residuals over time (or more fancy “autocorrelation” plots)

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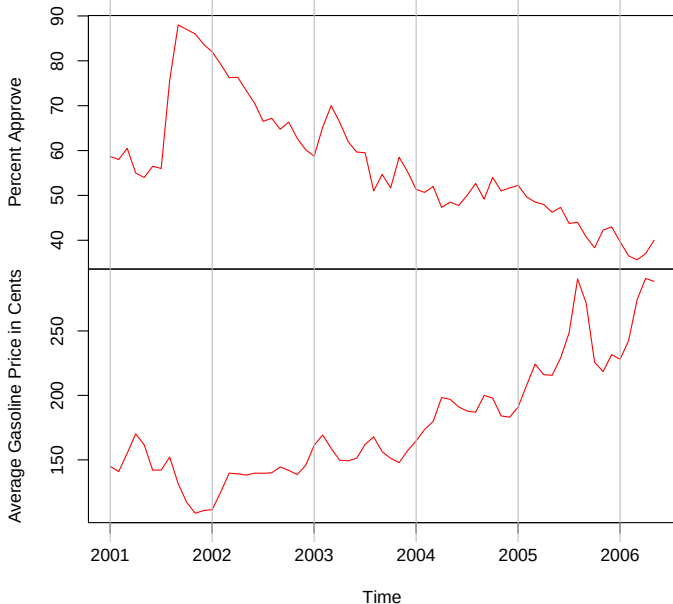
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- First-differencing the data

Monthly Presidential Approval Ratings and Gas Prices



Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(Zelig)
> data(approval)
> mod1 <- lm(approve ~ avg.price, data=approval)
> coeftest(mod1)
```

t test of coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	100.472076	3.567277	28.165	< 2.2e-16 ***
avg.price	-0.243885	0.019465	-12.529	< 2.2e-16 ***

Tests for Serial Correlation: Durbin-Watson

Recall our AR(1) model is:

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The null hypothesis (no serial correlation) is: $H_0: \rho = 0$

The alternative (positive serial correlation): $H_1: \rho > 0$

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The null hypothesis (no serial correlation) is: $H_0: \rho = 0$

The alternative (positive serial correlation): $H_1: \rho > 0$

One common test for serial correlation is the **Durbin-Watson statistic**:

$$DW = \frac{\sum_{t=2}^n \hat{u}_t - \hat{u}_{t-1}}{\sum_{t=1}^n \hat{u}_t^2} \quad \text{where} \quad DW \approx 2(1 - \hat{\rho})$$

Tests for Serial Correlation: Durbin-Watson

Recall our AR(1) model is:

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- If $DW \approx 2$ then $\hat{\rho} \approx 0$ (Note that $0 \leq DW \leq 4$)
- If $DW < 1$ we have serious positive serial correlation
- If $DW > 3$ we have serious negative serial correlation

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(lmtest)
> dwtest(approve ~ avg.price, data=approval)

Durbin-Watson test

data:  approve ~ avg.price
DW = 0.4863, p-value = 1.326e-14
alternative hypothesis: true autocorrelation is greater than 0
```

The test suggests strong positive serial correlation. Standard errors are severely downward biased.

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 - ▶ The `sandwich` package in R implements a variety of HAC estimators
 - ▶ A common option is `NeweyWest`

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> mod1 <- lm(approve~avg.price,data=approval)
> coeftest(mod1) # homoskedastic errors
t test of coefficients:

              Estimate Std. Error t value Pr(>|t|)
(Intercept) 100.472076   3.567277  28.165 < 2.2e-16 ***
avg.price    -0.243885   0.019465 -12.529 < 2.2e-16 ***

> coeftest(mod1, vcov = NeweyWest) # HAC errors
t test of coefficients:

              Estimate Std. Error t value Pr(>|t|)
(Intercept) 100.472076  14.499337   6.9294 2.652e-09 ***
avg.price    -0.243885   0.071733  -3.3999 0.001174 **
```

Once we correct for autocorrelation, standard errors increase dramatically.

Appendix: Derivation of Error Structure for the AR(1) Model

We have

$$V[u_t] = V[\rho u_{t-1} + e_t] = \rho^2 V[u_{t-1}] + \sigma^2$$

with stationarity, $V[u_t] = V[u_{t-1}]$, and so

$$V[u_t](1 - \rho^2) = \sigma^2 \Rightarrow V[u_t] = \frac{\sigma^2}{(1 - \rho^2)}$$

also

$$\text{Cov}[u_t, u_{t-1}] = E[u_t u_{t-1}] = E[(\rho u_{t-1} + e_t) e_{t-1}] = \rho V[e_{t-1}] = \rho \frac{\sigma^2}{(1 - \rho^2)}$$

or generally

$$\text{Cov}[u_t, u_{t-h}] = \rho^h \frac{\sigma^2}{(1 - \rho^2)}$$

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