Week 6: Linear Regression with Two Regressors

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Princeton

October 17, 19, 2016

[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn and Jens Hainmueller.

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- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression

Questions?

1. Two Examples
2. Adding a Binary Variable
3. Adding a Continuous Covariate
4. Once More With Feeling
5. OLS Mechanics and Partialing Out
6. Fun With Red and Blue
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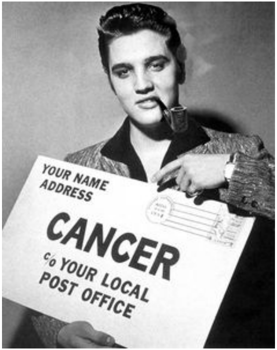

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Example 2: Berkeley Graduate Admissions

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- What about the conditional relationship within departments?

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- Marginal relationships (admissions and gender) $\neq$ conditional relationship given third variable (department)

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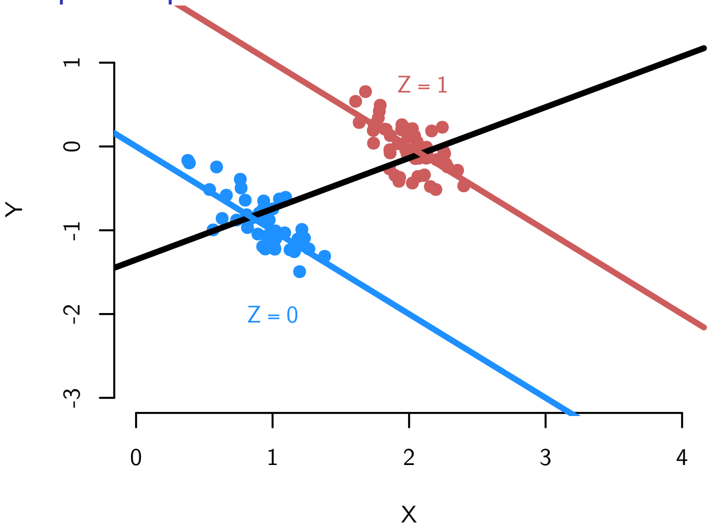

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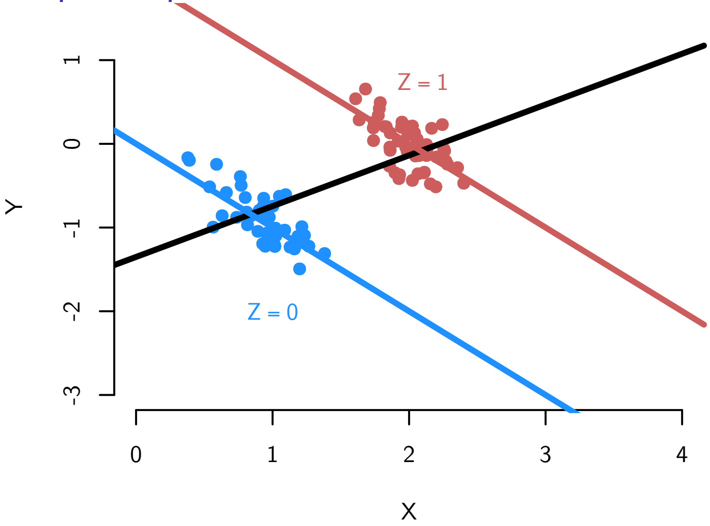

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Instance of a more general problem called the ecological inference fallacy

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- β's are the population parameters we want to estimate

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- The rest of this lecture is designed to explain what is meant by "controlling for another variable" with linear regression.

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- Let's look at the bivariate regression of Democracy on Income:

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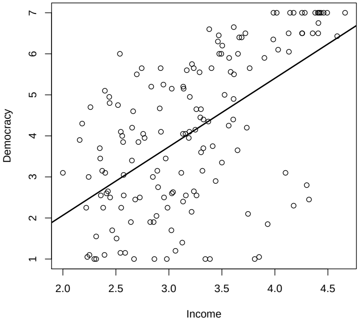

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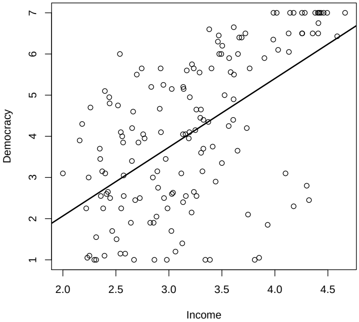

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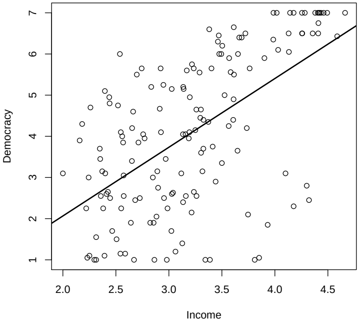

Interpretation: A one percent increase in GDP is associated with a .016 point increase in democracy.

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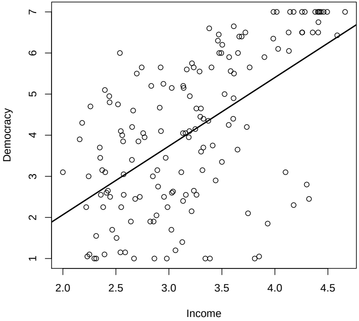
- But we can use more information in our prediction equation.



Simple Regression of Democracy on Income

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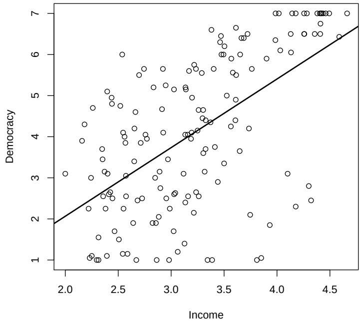

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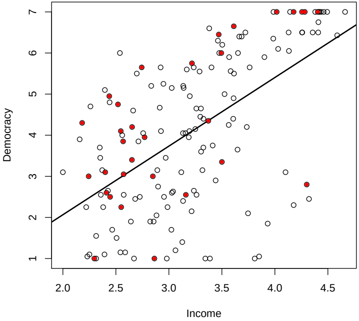

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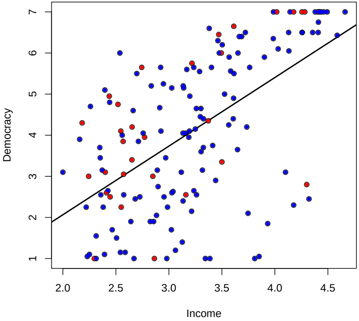

Adding a Covariate

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How do we do this? We can generalize the prediction equation:

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{1i} + \widehat{\beta}_2 x_{2i}$$

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In words:

$$\widehat{Democracy} = \widehat{\beta}_0 + \widehat{\beta}_1\, Log(GDP) + \widehat{\beta}_2\, Colony$$

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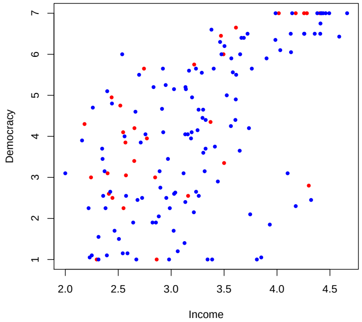
What does this mean? We are fitting two lines with the same slope but different intercepts.

Regression of Democracy on Income

From R, we obtain estimates
$\widehat{\beta}_0$, $\widehat{\beta}_1$, $\widehat{\beta}_2$:

```
Coefficients:
              Estimate
(Intercept)   -1.5060
GDP90LGN       1.7059
BRITCOL        0.5881
```



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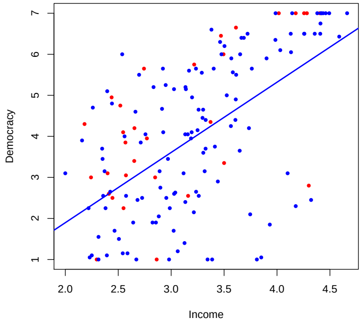

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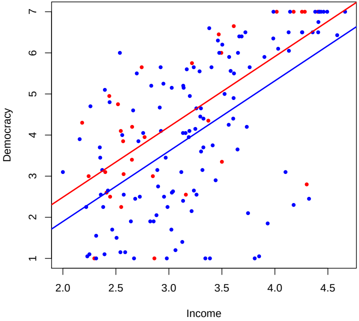


- Non-British colonies:

$$\widehat{y} = \widehat{\beta}_0 + \widehat{\beta}_1 x_1$$
$$\widehat{y} = -1.5 + 1.7\, x_1$$

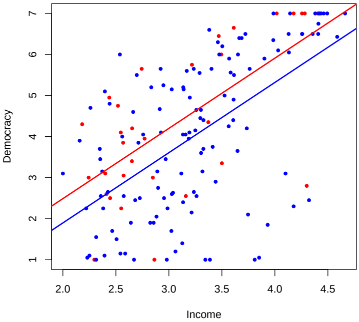
- Former British colonies:

$$\widehat{y} = (\widehat{\beta}_0 + \widehat{\beta}_2) + \widehat{\beta}_1 x_1$$
$$\widehat{y} = -.92 + 1.7\, x_1$$

Regression of Democracy on Income

Our prediction equation is:
$\widehat{y} = -1.5 + 1.7\, x_1 + .58\, x_2$

Where do these quantities appear on the graph?



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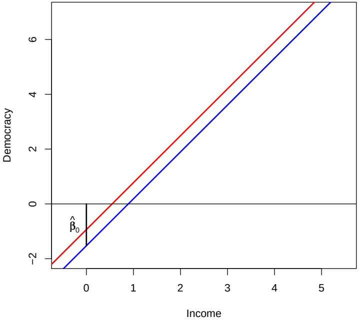

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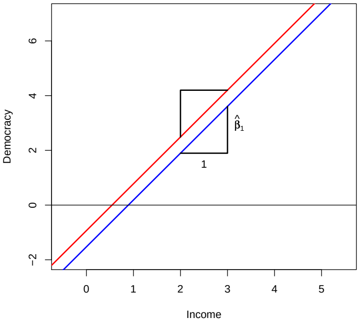

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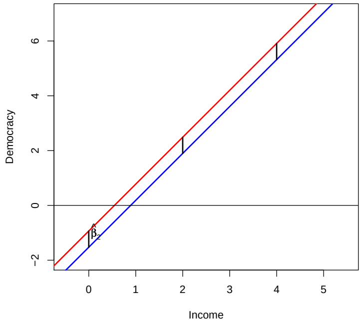
- $\widehat{\beta}_2 = .58$ is the vertical distance between the two lines for Ex-British colonies and non-colonies respectively



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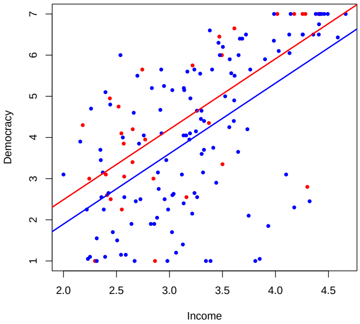
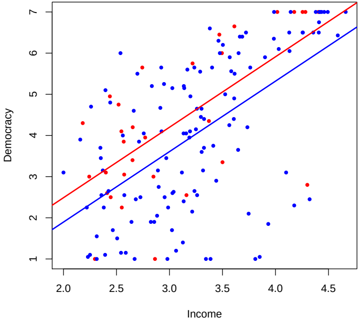
Fitting a regression plane

- We have considered an example of multiple regression with one continuous explanatory variable and one binary explanatory variable.



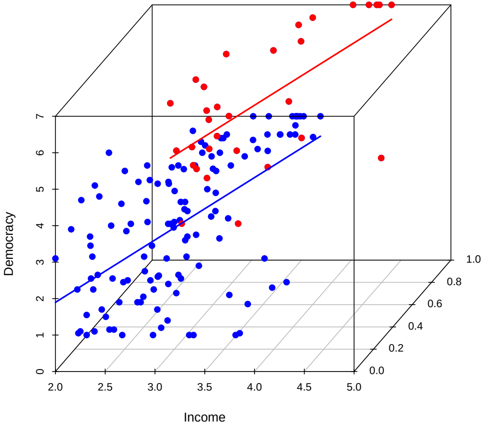
Fitting a regression plane

- We have considered an example of multiple regression with one continuous explanatory variable and one binary explanatory variable.
- This is easy to represent graphically in two dimensions because we can use colors to distinguish the two groups in the data.



Regression of Democracy on Income

- These observations are actually located in a three-dimensional space.



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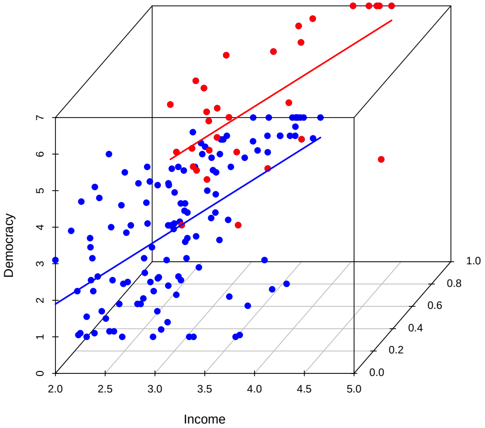

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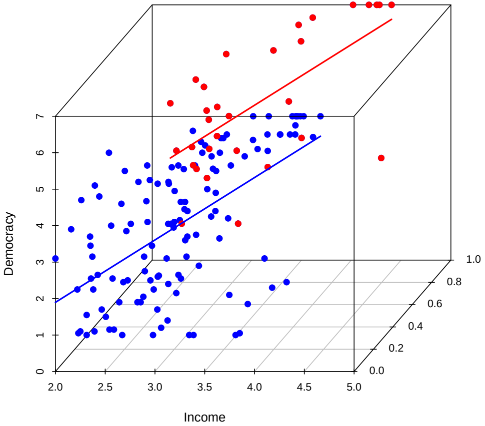
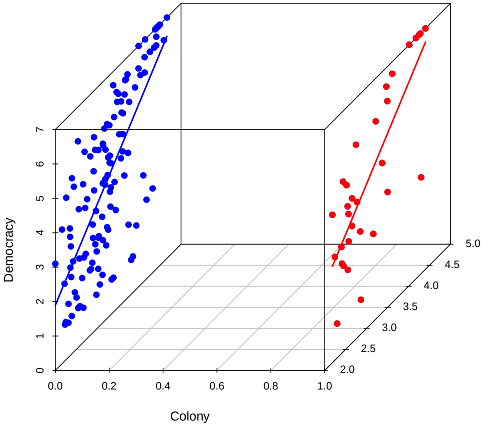
- We can try to represent this using a 3D scatterplot.
- In this view, we are looking at the data from the Income side; the two regression lines are drawn in the appropriate locations.



Regression of Democracy on Income

- We can also look at the 3D scatterplot from the British colony side.



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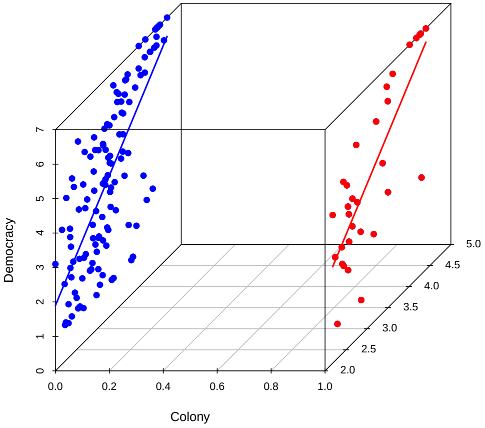

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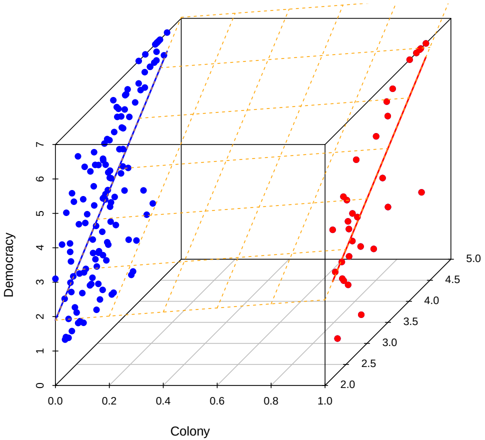
- In fact, the prediction equation defines a regression plane that connects the lines when $x_2 = 0$ and $x_2 = 1$.



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- Since we fit a regression plane to the data whenever we have two explanatory variables, it is easy to move to a case with two continuous explanatory variables.

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- For example, we might want to use:
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  - Y Democracy

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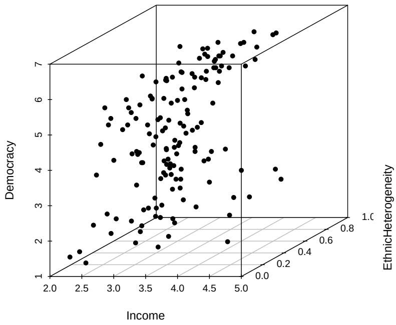

Regression of Democracy on Income

- We can plot the points in a 3D scatterplot.
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  - $\widehat{\beta}_0 = -.71$
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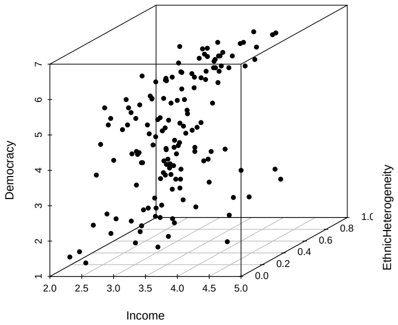
How does this look graphically?



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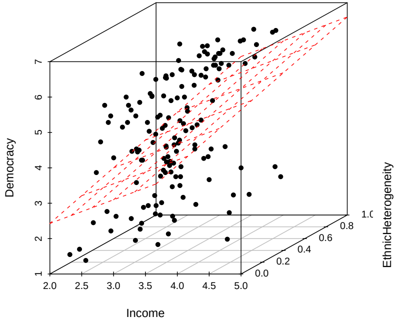

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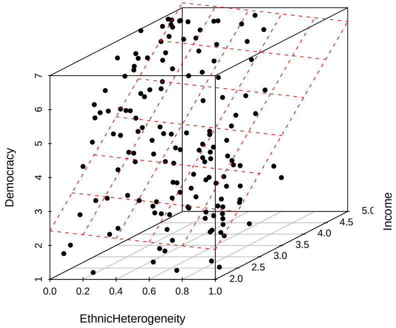

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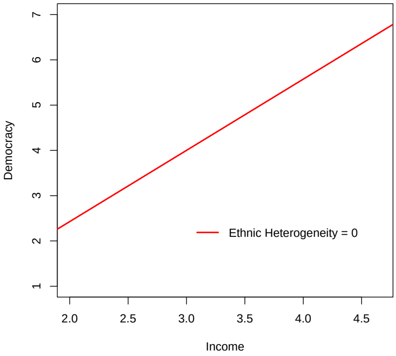

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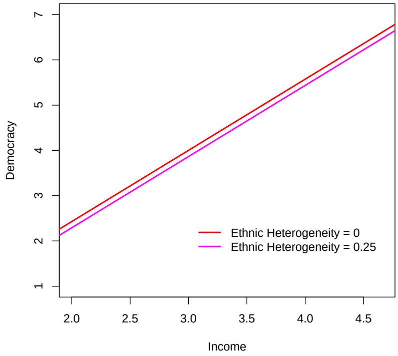

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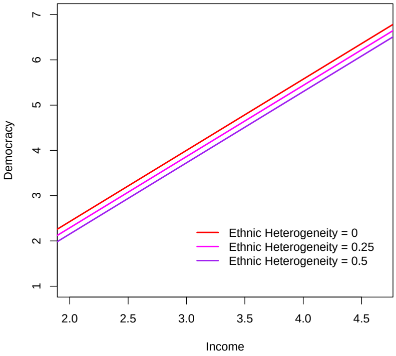

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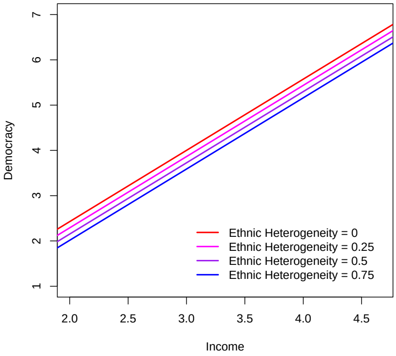

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  Predicted difference is thus: 1.8 or $(3.5 - 2.5)\widehat{\beta}_1 + (.06 - .5)\widehat{\beta}_2$

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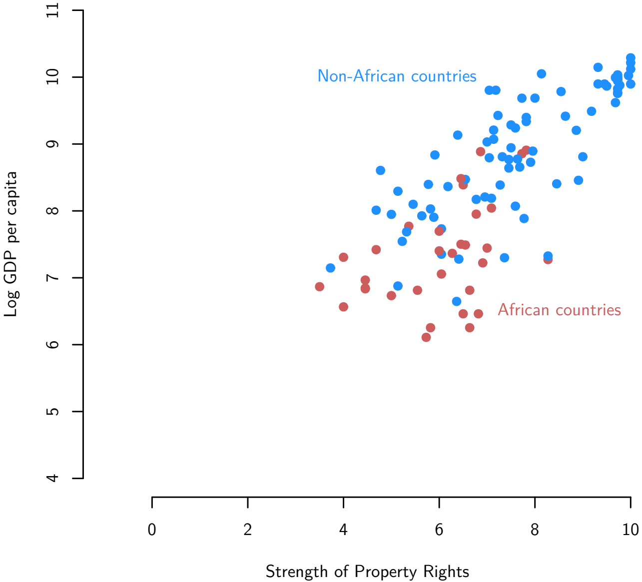
AJR Example


Non-African countries
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Log GDP per capita
Strength of Property Rights
4
5
6
7
8
9
10
11
0
2
4
6
8
10

Basics

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- New model:
$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

AJR model

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.65556    0.31344  18.043  < 2e-16 ***
## avexpr       0.42416    0.03971  10.681  < 2e-16 ***
## africa      -0.87844    0.14707  -5.973 3.03e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6253 on 108 degrees of freedom
##   (52 observations deleted due to missingness)
## Multiple R-squared:  0.7078, Adjusted R-squared:  0.7024
## F-statistic: 130.8 on 2 and 108 DF,  p-value: < 2.2e-16
```

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- Two different intercepts, same slope

Example interpretation of the coefficients

- Let's review what we've seen so far:

	Intercept for X_i	Slope for X_i
Non-African country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
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$$\widehat{Y}_i = 5.656 + 0.424 \times X_i - 0.878 \times Z_i$$

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  - $\widehat{\beta}_1$: A one-unit increase in property rights is associated with a 0.424 increase in average log incomes for two African countries (or for two non-African countries)

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  - $\widehat{\beta}_0$: average log income for non-African country ($Z_i = 0$) with property rights measured at 0 is 5.656
  - $\widehat{\beta}_1$: A one-unit increase in property rights is associated with a 0.424 increase in average log incomes for two African countries (or for two non-African countries)
  - $\widehat{\beta}_2$: there is a -0.878 average difference in log income per capita between African and non-African counties **conditional on** property rights

General interpretation of the coefficients

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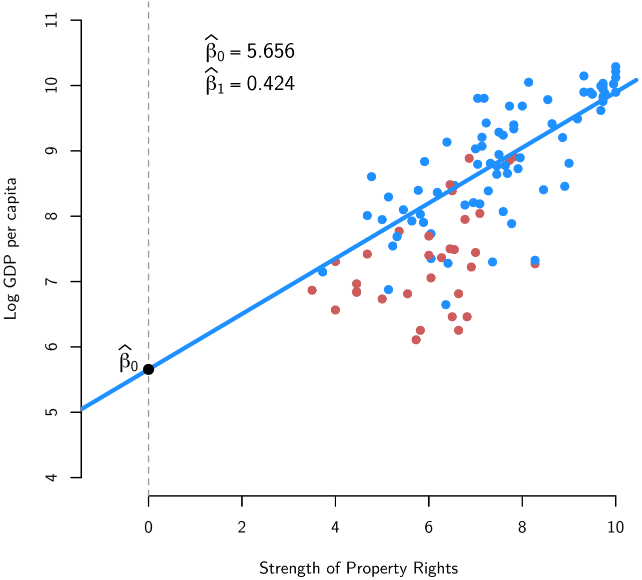
- $\widehat{\beta}_2$: average difference in Y_i between $Z_i = 1$ group and $Z_i = 0$ group conditional on X_i

Adding a binary variable, visually



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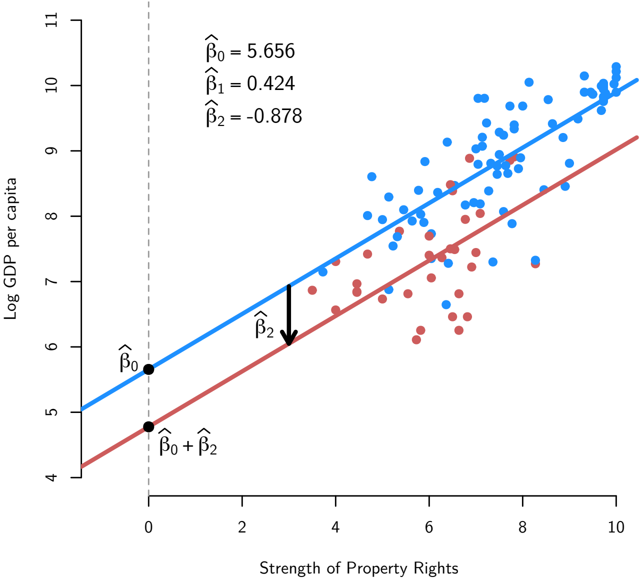

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- New model:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

AJR model, revisited

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  6.80627    0.75184   9.053 1.27e-12 ***
## avexpr       0.40568    0.06397   6.342 3.94e-08 ***
## meantemp    -0.06025    0.01940  -3.105  0.00296 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6435 on 57 degrees of freedom
##   (103 observations deleted due to missingness)
## Multiple R-squared:  0.6155, Adjusted R-squared:  0.602
## F-statistic: 45.62 on 2 and 57 DF,  p-value: 1.481e-12
```

Interpretation with a continuous Z

	Intercept for X_i	Slope for X_i
$Z_i = 0\,^\circ\mathrm{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$

Interpretation with a continuous Z

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$Z_i = 0\,^\circ\mathrm{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
$Z_i = 21\,^\circ\mathrm{C}$	$\widehat{\beta}_0 + \widehat{\beta}_2 \times 21$	$\widehat{\beta}_1$

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$Z_i = 0\,^\circ\mathrm{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
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- In this example we have:

$$\widehat{Y}_i = 6.806 + 0.406 \times X_i + -0.06 \times Z_i$$

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- $\widehat{\beta}_0$: average log income for a country with property rights measured at 0 and a mean temperature of 0 is 6.806

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- $\widehat{\beta}_1$: A one-unit change in property rights is associated with a 0.406 change in average log incomes conditional on a country's mean temperature

Interpretation with a continuous Z

	Intercept for X_i	Slope for X_i
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- $\widehat{\beta}_1$: A one-unit change in property rights is associated with a 0.406 change in average log incomes conditional on a country's mean temperature
- $\widehat{\beta}_2$: A one-degree increase in mean temperature is associated with a -0.06 change in average log incomes conditional on strength of property rights

General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

- The coefficient $\widehat{\beta}_1$ measures how the predicted outcome varies in X_i for a fixed value of Z_i.

General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

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- The coefficient $\widehat{\beta}_2$ measures how the predicted outcome varies in Z_i for a fixed value of X_i.

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3. Adding a Continuous Covariate
4. Once More With Feeling
5. OLS Mechanics and Partialing Out
6. Fun With Red and Blue
7. Omitted Variables
8. Multicollinearity
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Fitted values and residuals

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- Residuals for $i = 1, \ldots, n$:

$$\widehat{u}_i = Y_i - \widehat{Y}_i$$

Least squares is still least squares

- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?

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- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?
- Minimize the sum of the squared residuals, just like before:

$$(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2) = \underset{b_0, b_1, b_2}{\arg\min} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i - b_2 Z_i)^2$$

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- The calculus is the same as last week, with 3 partial derivatives instead of 2

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- The calculus is the same as last week, with 3 partial derivatives instead of 2
- Let's start with a simple recipe and then rigorously show that it holds

OLS estimator recipe using two steps

- "Partialling out" OLS recipe:

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  1. Run regression of X_i on Z_i:
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OLS estimator recipe using two steps

- "Partialling out" OLS recipe:
  1. Run regression of X_i on Z_i:
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  2. Calculate residuals from this regression:
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OLS estimator recipe using two steps

- "Partialling out" OLS recipe:
  1. Run regression of X_i on Z_i:
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  3. Run a simple regression of Y_i on residuals, $\widehat{r}_{xz,i}$:
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- Estimate of $\widehat{\beta}_1$ will be the same as running:
$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

Regression property rights on mean temperature

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  9.95678    0.82015  12.140  < 2e-16 ***
## meantemp    -0.14900    0.03469  -4.295 6.73e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.321 on 58 degrees of freedom
##   (103 observations deleted due to missingness)
## Multiple R-squared:  0.2413, Adjusted R-squared:  0.2282
## F-statistic: 18.45 on 1 and 58 DF,  p-value: 6.733e-05
```

Regression of log income on the residuals

```
## (Intercept)  avexpr.res
##   8.0542783   0.4056757

## (Intercept)      avexpr    meantemp
##  6.80627375  0.40567575 -0.06024937
```

Residual/partial regression plot

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Useful for plotting the conditional relationship between property rights and income given temperature:

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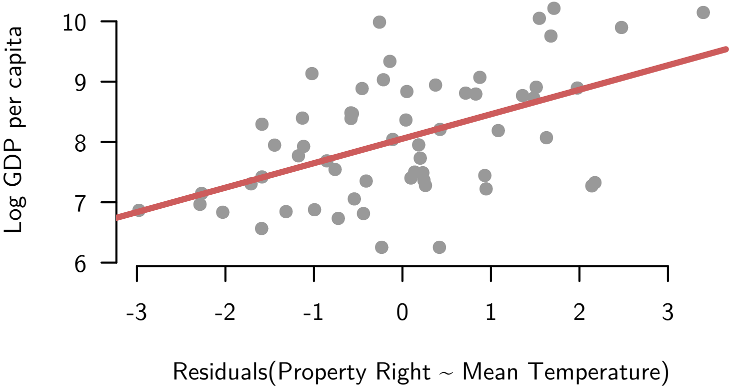

Deriving the Linear Least Squares Estimator

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- We use the same principle for picking $(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2)$ for regression with two regressors (x_i and z_i):

$$
\begin{aligned}
(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) &= \underset{\tilde{\beta}_0,\tilde{\beta}_1,\tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} \widehat{u}_i^2 = \underset{\tilde{\beta}_0,\tilde{\beta}_1,\tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \\
&= \underset{\tilde{\beta}_0,\tilde{\beta}_1,\tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \tilde{\beta}_0 - x_i\tilde{\beta}_1 - z_i\tilde{\beta}_2)^2
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- (The same works more generally for k regressors, but this is done more easily with matrices as we will see next week)

Deriving the Linear Least Squares Estimator

We want to minimize the following quantitity with respect to $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2)$:

$$S(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2) = \sum_{i=1}^{n} (y_i - \tilde{\beta}_0 - \tilde{\beta}_1 x_i - \tilde{\beta}_2 z_i)^2$$

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Plan is conceptually the same as before

1. Take the partial derivatives of S with respect to $\tilde{\beta}_0, \tilde{\beta}_1$ and $\tilde{\beta}_2$.
2. Set each of the partial derivatives to 0 to obtain the first order conditions.
3. Substitute $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ for $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2$ and solve for $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ to obtain the OLS estimator.

First Order Conditions

Setting the partial derivatives equal to zero leads to a system of 3 linear equations in 3 unknowns: $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\beta}_2$

$$\begin{aligned}
\frac{\partial S}{\partial \tilde{\beta}_0} &= \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0 \\
\frac{\partial S}{\partial \tilde{\beta}_1} &= \sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0 \\
\frac{\partial S}{\partial \tilde{\beta}_2} &= \sum_{i=1}^{n} z_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0
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When will this linear system have a unique solution?

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\end{aligned}
$$

When will this linear system have a unique solution?

- More observations than predictors (i.e. $n > 2$)
- x and z are linearly independent, i.e.,
  - neither x nor z is a constant
  - x is not a linear function of z (or vice versa)
- Wooldridge calls this assumption no perfect collinearity

The OLS Estimator

The OLS estimator for $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ can be written as

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} - \hat{\beta}_2 \bar{z} \\
\hat{\beta}_1 &= \frac{Cov(x,y)Var(z) - Cov(z,y)Cov(x,z)}{Var(x)Var(z) - Cov(x,z)^2} \\
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Condition fails if:

1. If x or z is a constant ($\Rightarrow Var(x)Var(z) = Cov(x, z) = 0$)
2. One explanatory variable is an exact linear function of another ($\Rightarrow \ Cor(x, z) = 1 \ \Rightarrow \ Var(x)Var(z) = Cov(x, z)^2$)

"Partialling Out" Interpretation of the OLS Estimator

Assume $Y = \beta_0 + \beta_1 X + \beta_2 Z + u$. Another way to write the OLS estimator is:

$$\hat{\beta}_1 = \frac{\sum_i^n \hat{r}_{xz,i}\, y_i}{\sum_i^n \hat{r}_{xz,i}^2}$$

where $\hat{r}_{xz,i}$ are the residuals from the regression of X on Z:

$$X = \lambda + \delta Z + r_{xz}$$

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In other words, both of these regressions yield identical estimates $\hat{\beta}_1$:

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- That is, same as the simple regresson of Y on X alone.

Origin of the Partial Out Recipe

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- Residuals $\hat{r}_{xz}$ are the part of X that is uncorrelated with Z. Put differently, $\hat{r}_{xz}$ is X, after the effect of Z on X has been partialled out or netted out.

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- δ measures the correlation between X and Z.
- Residuals $\hat{r}_{xz}$ are the part of X that is uncorrelated with Z. Put differently, $\hat{r}_{xz}$ is X, after the effect of Z on X has been partialled out or netted out.
- Can use same equation with k explanatory variables; $\hat{r}_{xz}$ will then come from a regression of X on all the other explanatory variables.

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2. Random/iid sample
3. No perfect collinearity
4. Zero conditional mean error
$$\mathbb{E}[u_i | X_i, Z_i] = 0$$

New assumption

Assumption 3: No perfect collinearity

(1) No explanatory variable is constant in the sample and (2) there are no exactly linear relationships among the explanatory variables.

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- What's the correlation between Z_i and X_i? 1!

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  - $X_i =$ income
  - $Z_i = X_i^2$
- Do we have to worry about collinearity here?

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- But, clearly we have the following:

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- These two variables are perfectly collinear.
- What about the following:
  - $X_i =$ income
  - $Z_i = X_i^2$
- Do we have to worry about collinearity here?
- No! Because while Z_i is a deterministic function of X_i, it is not a linear function of X_i.

R and perfect collinearity

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```
##
## Coefficients: (1 not defined because of singularities)
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  8.71638    0.08991  96.941  < 2e-16 ***
## africa      -1.36119    0.16306  -8.348 4.87e-14 ***
## nonafrica         NA         NA      NA       NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9125 on 146 degrees of freedom
##   (15 observations deleted due to missingness)
## Multiple R-squared:  0.3231, Adjusted R-squared:  0.3184
## F-statistic: 69.68 on 1 and 146 DF,  p-value: 4.87e-14
```

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```
## (Intercept)    meantemp  meantemp.f
##  10.8454999  -0.1206948          NA
```

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$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 Z_i + u_i$$

2. Random/iid sample
3. No perfect collinearity
4. Zero conditional mean error

$$\mathbb{E}[u_i | X_i, Z_i] = 0$$

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- $\rightsquigarrow$ small adjustments to the critical values and the t-values for our hypothesis tests and confidence intervals.

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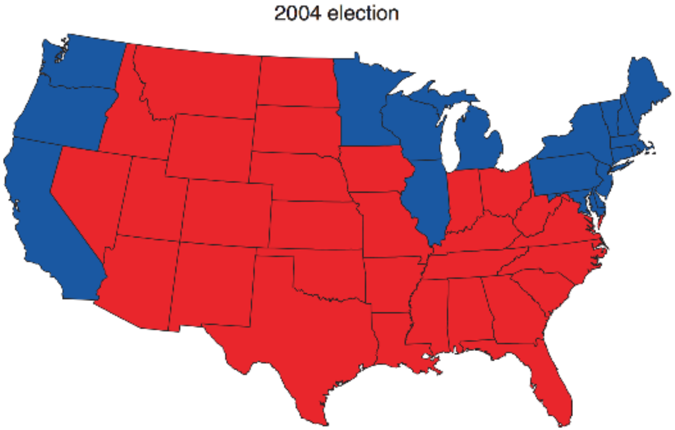
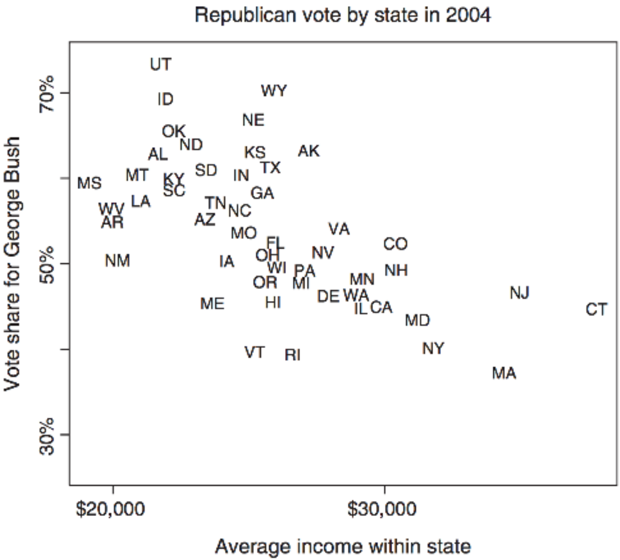
Red State Blue State

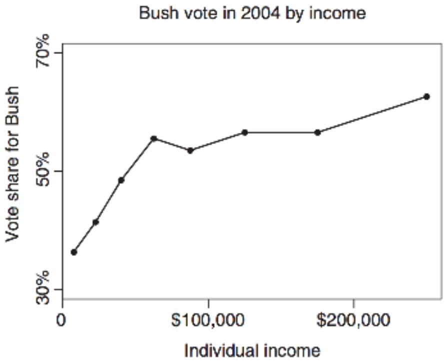
Red and Blue States

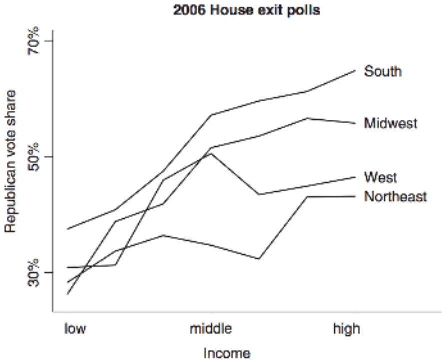


Rich States are More Democratic

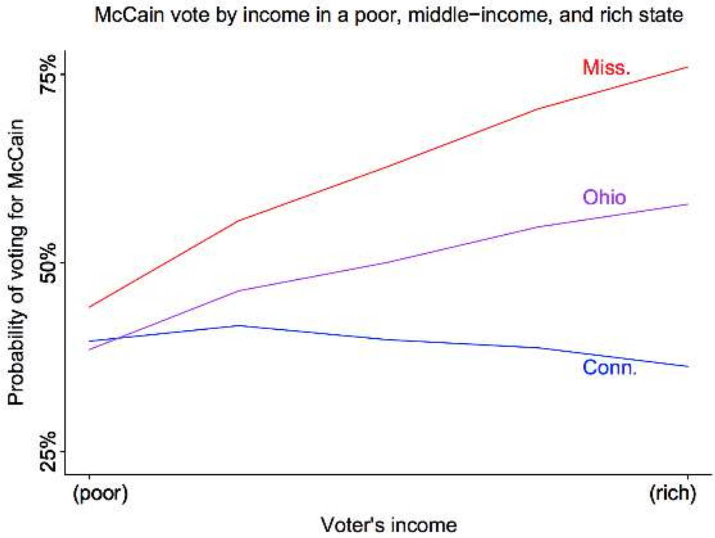


But Rich People are More Republican

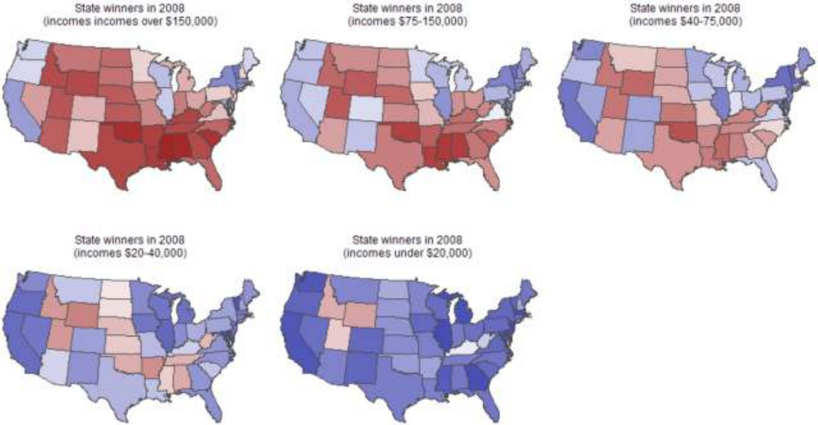




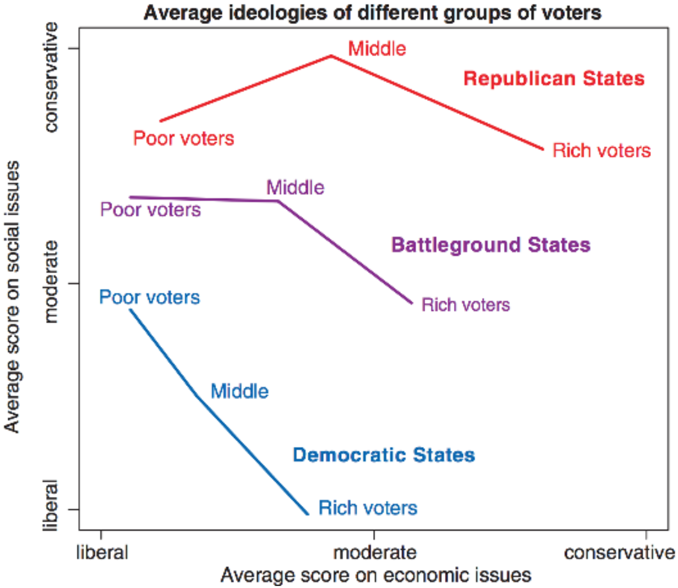
Paradox Resolved



If Only Rich People Voted, it Would Be a Landslide



A Possible Explanation



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Remember This?

- **Identification / Data Description**
  - Variation in X
- **Unbiasedness / Consistency**
  - Variation in X
  - Random Sampling
  - Linearity in Parameters
  - Zero Conditional Mean
- **Gauss-Markov (BLUE) / Asymptotic Inference (z and χ^2)**
  - Variation in X
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- **Classical LM (BUE) / Small-Sample Inference (t and F)**
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- OLS estimates from the misspecified model:

$$\widehat{Y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 X_i$$

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True Population Model:

$$\text{Voted Republican} = \beta_0 + \beta_1 \text{Watch Fox News} + \beta_2 \text{Strong Republican} + u$$

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Answer: $\tilde{\beta}_1$ is upward biased since being a strong republican is positively correlated with both watching fox news and voting republican. We have $\beta_1 < \tilde{\beta}_1$.

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Answer: The negative coefficient $\tilde{\beta}_1$ is downward biased compared to the true β_1 so $\beta_1 > \tilde{\beta}_1$. Being hospitalized is negatively correlated with health, and health is positively correlated with survival.

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- $\hat{\beta}_2$ is from the true regression and measures the relationship between x_2 and y, conditional on x_1.

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- $\tilde{\delta}$ is the slope of a regression of x_2 on x_1. If $\tilde{\delta} > 0$ then $\text{cor}(x_1, x_2) > 0$ and if $\tilde{\delta} < 0$ then $\text{cor}(x_1, x_2) < 0$.
- $\hat{\beta}_2$ is from the true regression and measures the relationship between x_2 and y, conditional on x_1.

Q. When will $\tilde{\beta}_1 = \hat{\beta}_1$?

Omitted Variable Bias: Simple Case

True Population Model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + u$$

Underspecified Model that we use:

$$\tilde{y} = \tilde{\beta}_0 + \tilde{\beta}_1 x_1$$

We can show that the relationship between $\tilde{\beta}_1$ and $\hat{\beta}_1$ is:

$$\tilde{\beta}_1 = \hat{\beta}_1 + \hat{\beta}_2 \cdot \tilde{\delta}$$

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Q. When will $\tilde{\beta}_1 = \hat{\beta}_1$?
A. If $\tilde{\delta} = 0$ or $\hat{\beta}_2 = 0$.

Omitted Variable Bias: Simple Case

We take expectations to see what the bias will be:

$$\begin{aligned} \tilde{\beta}_1 &= \hat{\beta}_1 + \hat{\beta}_2 \cdot \tilde{\delta} \\ E[\tilde{\beta}_1 \mid X] &= \end{aligned}$$

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Any variable that is correlated with an included X and the outcome Y is called a confounder.

Omitted Variable Bias: Simple Case

Direction of the bias of $\tilde{\beta}_1$ compared to β_1 is given by:

	$\text{cov}(X_1, X_2) > 0$	$\text{cov}(X_1, X_2) < 0$	$\text{cov}(X_1, X_2) = 0$
$\beta_2 > 0$	Positive bias	Negative Bias	No bias
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- Magnitude of the bias matters too

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Further points:

- Magnitude of the bias matters too
- If you miss an important confounder, your estimates are biased and inconsistent.
- In the more general case with more than two covariates the bias is more difficult to discern. It depends on all the pairwise correlations.

Including an Irrelevant Variable: Simple Case

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True Population Model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u \quad \text{where} \quad \beta_2 = 0$$

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Q: Which statement is correct?

1. $\beta_1 > \tilde{\beta}_1$
2. $\beta_1 < \tilde{\beta}_1$
3. $\beta_1 = \tilde{\beta}_1$
4. Can't tell

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and thus including the irrelevant variable does not generally affect the unbiasedness. The sampling distribution of $\hat{\beta}_2$ will be centered about zero.

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Sampling variance for simple linear regression

- Under simple linear regression, we found that the distribution of the slope was the following:

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- Regression with an additional independent variable:

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- What happens with perfect collinearity? $R_1^2 = 1$ and the variances are infinite.

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- Given the symmetry, it will also increase $\text{var}(\widehat{\beta}_2)$ as well.

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- Low variation in an independent variable (here, $\widehat{r}_{xz,i}$) $\rightsquigarrow$ high SEs
- Basically, there is less residual variation left in X_i after "partialling out" the effect of Z_i

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In R, `vif()` in the `car` package.

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So How Should I Think about Multicollinearity?

- Multicollinearity does NOT lead to bias; estimates will be unbiased and consistent.
- Multicollinearity should in fact be seen as a problem of micronumerosity, or "too little data." You can't ask the OLS estimator to distinguish the partial effects of X_1 and X_2 if they are essentially the same.
- If X_1 and X_2 are almost the same, why would you want a unique β_1 and a unique β_2? Think about how you would interpret that?

So How Should I Think about Multicollinearity?

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- If X_1 and X_2 are almost the same, why would you want a unique β_1 and a unique β_2? Think about how you would interpret that?
- Relax, you got way more important things to worry about!
- If possible, get more data
- Drop one of the variables, or combine them
- Or maybe linear regression is not the right tool

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Why Dummy Variables?

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    - E.g. does the effect of education differ by gender?

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- Let's regress GDP on this dummy variable and a constant:
  $Y = \beta_0 + \beta_1 D + u$

Example: GDP per capita on Electoral System

R Code

```
> summary(lm(REALGDPCAP ~ MAJORITARIAN, data = D))

Call:
lm(formula = REALGDPCAP ~ MAJORITARIAN, data = D)

Residuals:
   Min     1Q Median     3Q    Max
 -5982  -4592  -2112   4293  13685

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    7097.7      763.2    9.30 1.64e-14 ***
MAJORITARIAN  -1053.8     1224.9   -0.86    0.392
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 5504 on 83 degrees of freedom
Multiple R-squared: 0.008838,       Adjusted R-squared: -0.003104
F-statistic: 0.7401 on 1 and 83 DF,  p-value: 0.3921
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Dummy Variables for Multiple Categories

- More generally, let's say X measures which of m categories each unit i belongs to. E.g. the type of electoral system or region of country i is given by:
  - $X_i \in \{Proportional, Majoritarian\}$ so $m = 2$
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- The omitted category is our baseline case (also called a reference category) against which we compare the conditional means of Y for the other $m - 1$ categories.

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Category	D_1	D_2	D_3	D_4
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Our regression equation is:

$$Y = \beta_0 + \beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3 + \beta_4 D_4 + u$$

1. Two Examples
2. Adding a Binary Variable
3. Adding a Continuous Covariate
4. Once More With Feeling
5. OLS Mechanics and Partialing Out
6. Fun With Red and Blue
7. Omitted Variables
8. Multicollinearity
9. Dummy Variables
10. Interaction Terms
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    - two or more continuous variables
- Interactions often confuses researchers and mistakes in use and interpretation occur frequently (even in top journals)

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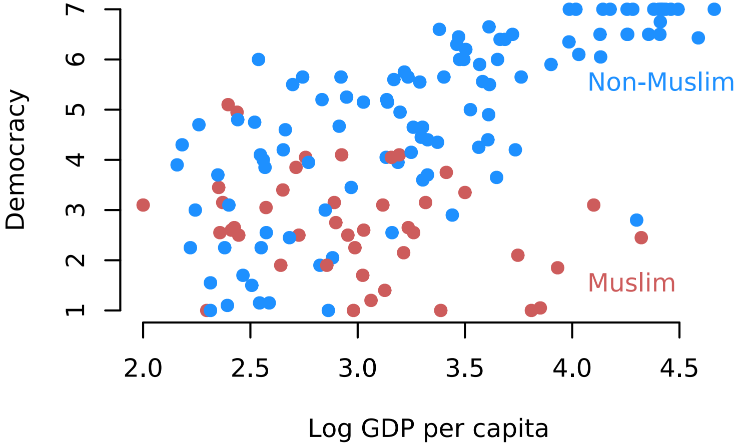
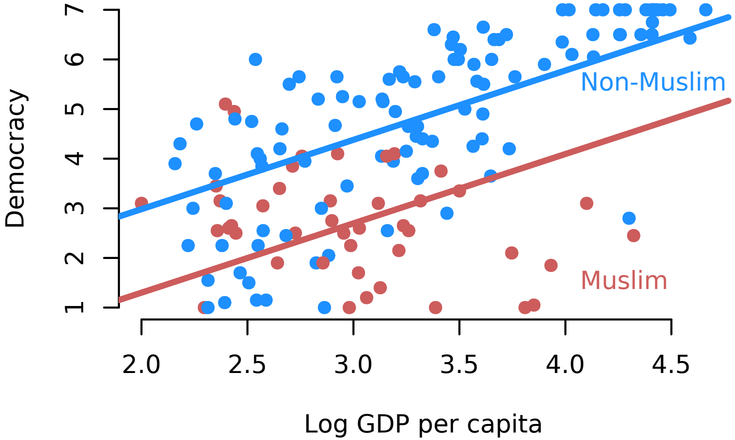
- Basic relationship: does more economic development lead to more democracy?
- We measure economic development with log GDP per capita
- We measure democracy with a Freedom House score, 1 (less free) to 7 (more free)

Let's see the data



Fish argues that Muslim countries are less likely to be democratic no matter their economic development

Controlling for Religion Additively



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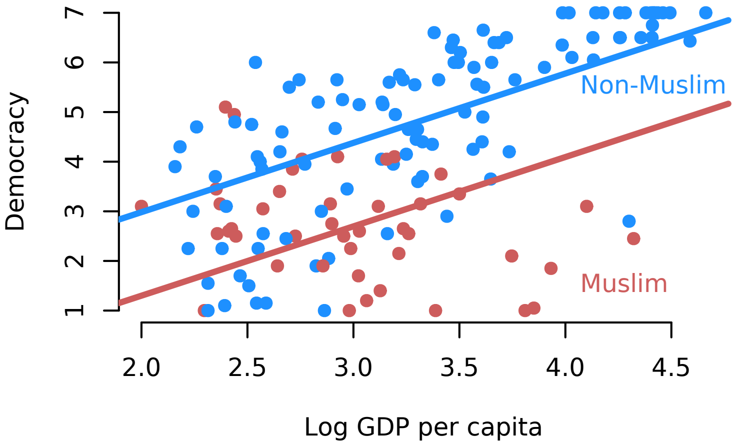

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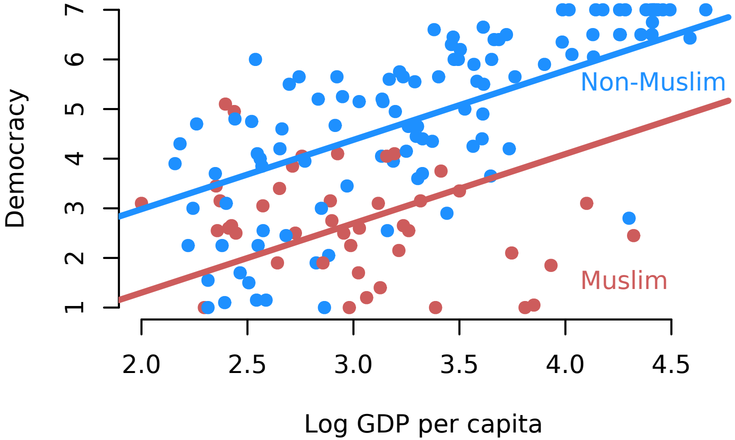

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Can we allow for different slopes for each group?

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- This covariate is called an interaction term and it is the product of the two marginal variables of interest: $income_i \times muslim_i$
- Here is the model with the interaction term:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

Two lines in one regression

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Example interpretation of the coefficients

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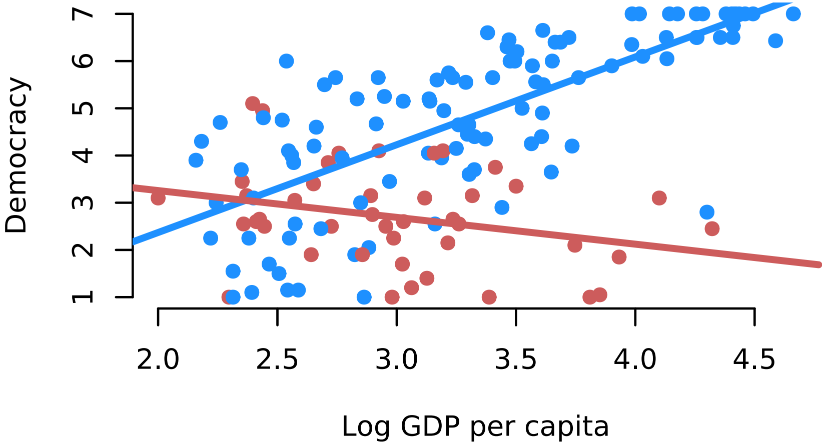
	Intercept for X_i	Slope for X_i
Non-Muslim country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
Muslim country ($Z_i = 1$)	$\widehat{\beta}_0 + \widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_3$



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- $\widehat{\beta}_2$: average difference in Y_i between $Z_i = 1$ group and $Z_i = 0$ group when $X_i = 0$
- $\widehat{\beta}_3$: change in the effect of X_i on Y_i between $Z_i = 1$ group and $Z_i = 0$

Lower order terms

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- Imagine we omitted the lower order term for `muslim`:

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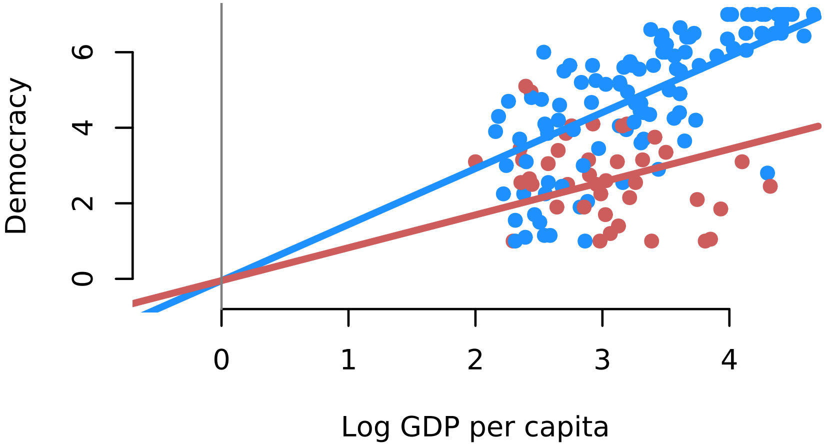

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- Distorts slope estimates.
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- With a continuous Z_i, we can have more than two values that it can take on:

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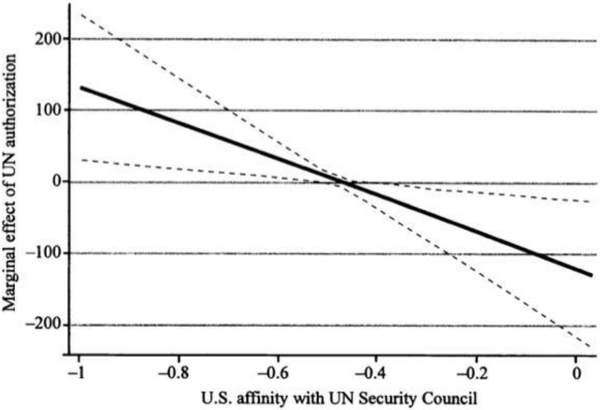
We will talk about checking these assumptions in a few weeks.

Example: Common Support

Chapman 2009 analysis

example and reanalysis from Hainmueller, Mummolo, Xu 2016



Note: Dashed lines give 95 percent confidence interval.

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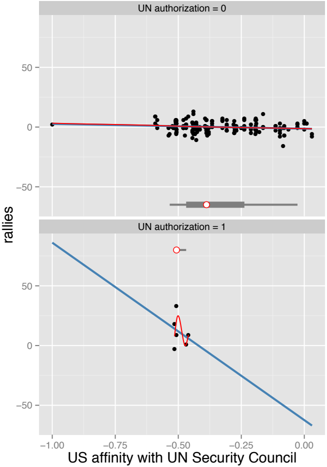

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Further Reading: Brambor, Clark, and Golder. 2006. Understanding Interaction Models: Improving Empirical Analyses. *Political Analysis* 14 (1): 63-82.

Hainmueller, Mummolo, Xu. 2016. How Much Should We Trust Estimates from Multiplicative Interaction Models? Simple Tools to Improve Empirical Practice. *Working Paper*

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Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + u \\
Y &= \beta_0 + (\beta_1 + \beta_2) X_1 + \beta_3 X_1 X_1 + u \\
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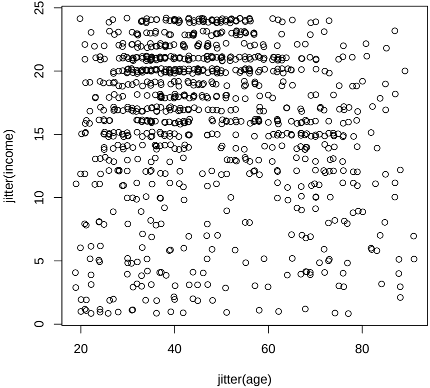
- This is called a second order polynomial in X_1
- A third order polynomial is given by:
  $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + u$

Polynomial Example: Income and Age

- Let's look at data from the U.S. and examine the relationship between Y: income and X: age



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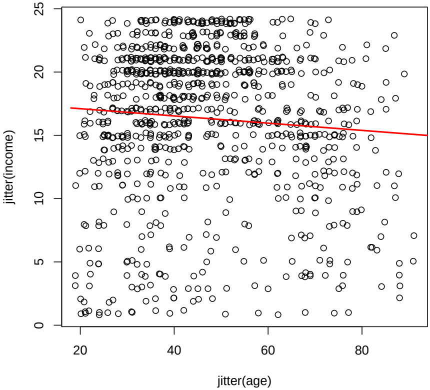

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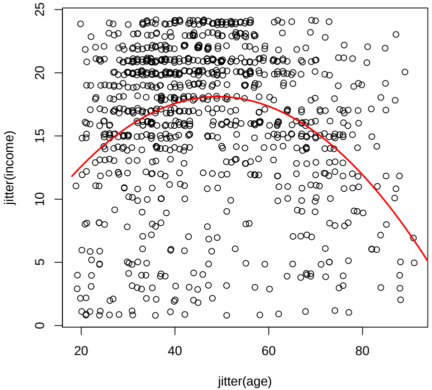

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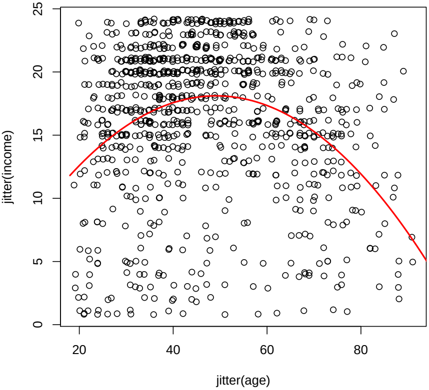

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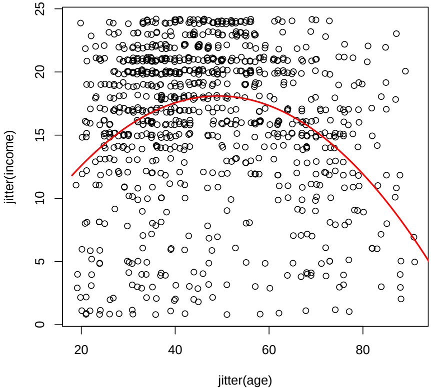

Polynomial Example: Income and Age

- $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$
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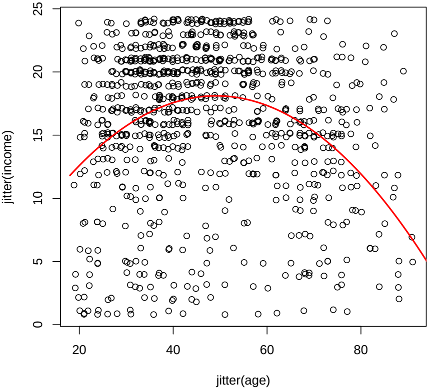
- No! The marginal effect of age depends on the level of age: $\frac{\partial Y}{\partial X_1} = \widehat{\beta}_1 + 2\,\widehat{\beta}_2\,X_1$
  Here the effect of age changes monotonically from positive to negative with income.



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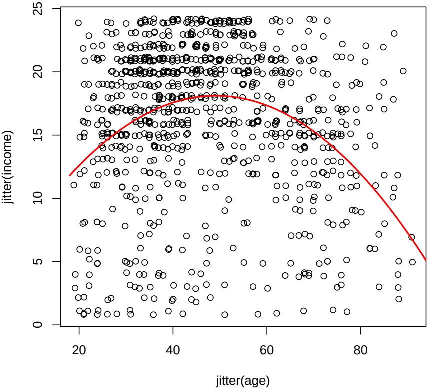

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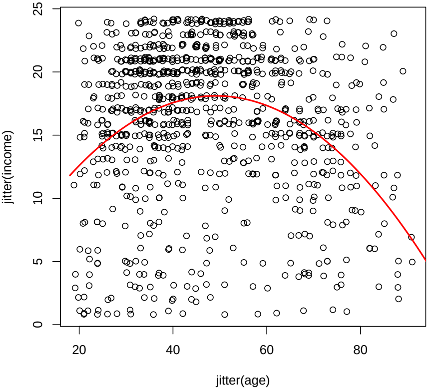
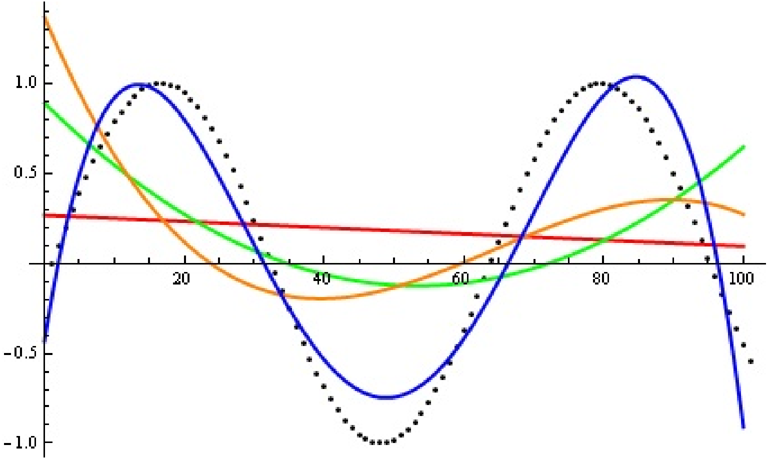
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.
- Maximum/Minimum occurs at $|\frac{\beta_1}{2\beta_2}|$. Here turning point is at $X_1 = 50$.



Higher Order Polynomials



Approximating data generated with a sine function. Red line is a first degree polynomial, green line is second degree, orange line is third degree and blue is fourth degree

Conclusion

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In this brave new world with 2 independent variables:

1. β's have slightly different interpretations
2. OLS still minimizing the sum of the squared residuals
3. Small adjustments to OLS assumptions and inference
4. Adding or omitting variables in a regression can affect the bias and the variance of OLS
5. We can optionally consider interactions, but must take care to interpret them correctly

Next Week

Next Week

- OLS in its full glory

Next Week

- OLS in its full glory
- Reading:
  - Practice up on matrices
  - Fox Chapter 9.1-9.4 (skip 9.1.1-9.1.2) Linear Models in Matrix Form
  - Aronow and Miller 4.1.2-4.1.4 Regression with Matrix Algebra
  - Optional: Fox Chapter 10 Geometry of Regression
  - Optional: Imai Chapter 4.3-4.3.3
  - Optional: Angrist and Pischke Chapter 3.1 Regression Fundamentals

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Breznau (2015) "The Missing Main Effect of Welfare State Regimes: A Replication of 'Social Policy Responsiveness in Developed Democracies."' *Sociological Science*.

Original Argument

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- but. . . they leave out a main effect.

Omitted Term

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- They mean center so the 0 represents the average over the entire sample

What Happens?

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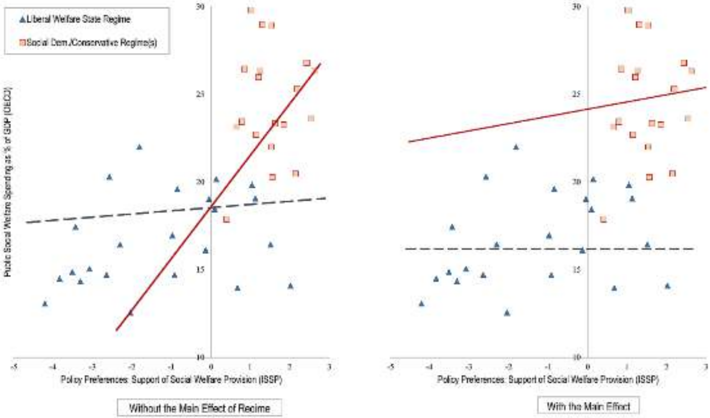

Figure 1: Predicted Regression Lines for the Effect of Policy Preferences on Social Welfare Spending, without and with the Main Effect of Regime

Moral of the Story

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Seriously

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Moral of the Story

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<drops mic>

References

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