Week 5: Simple Linear Regression

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Princeton

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn and Jens Hainmueller. Illustrations by Shay O'Brien.

Where We've Been and Where We're Going...

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  - what is regression

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Questions?

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A brief comment on exams, midterm week etc.

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- β_0, β_1 = population intercept and population slope (what we want to estimate)

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- You can think of the residuals as the prediction errors of our estimates.

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- Interpretation: how do we interpret our estimates?

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- In words, the OLS estimates are the intercept and slope that minimize the sum of the squared residuals.

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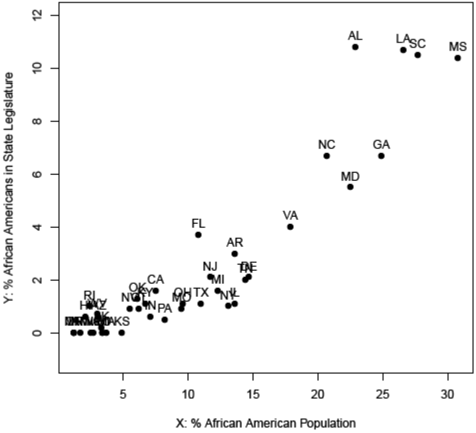

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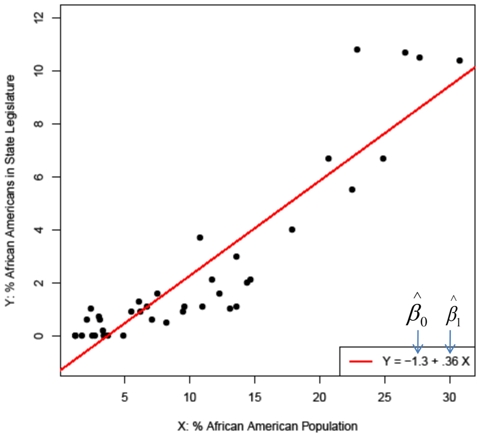

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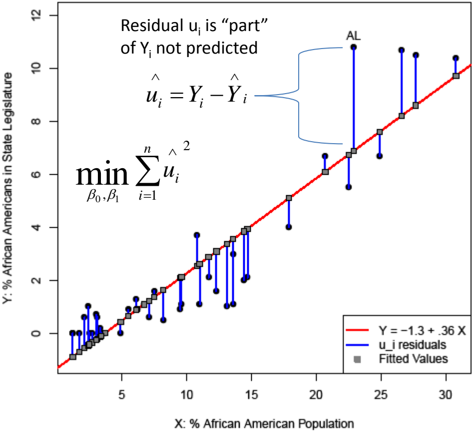
Answer: We will minimize the squared sum of residuals



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- To the board we go!

The OLS estimator

- Now we're done! Here are the **OLS estimators**:

$$\widehat{\beta}_0 = \overline{Y} - \widehat{\beta}_1 \overline{X}$$

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^n (X_i - \overline{X})^2}$$

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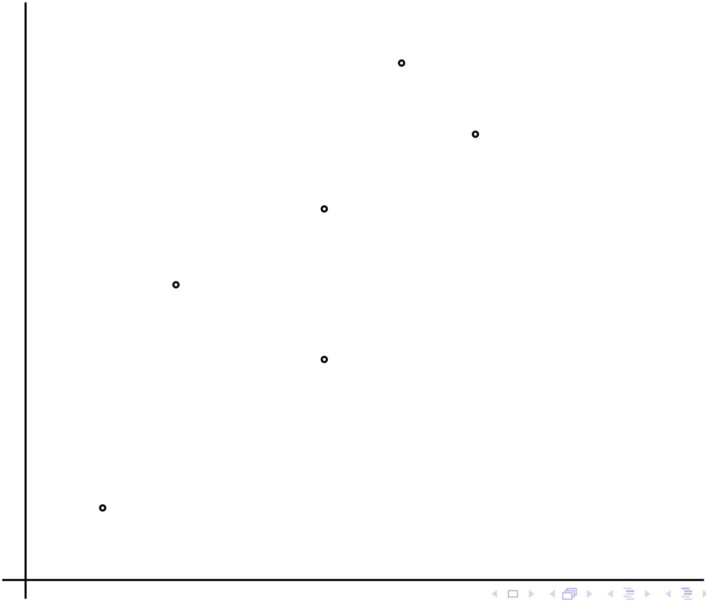

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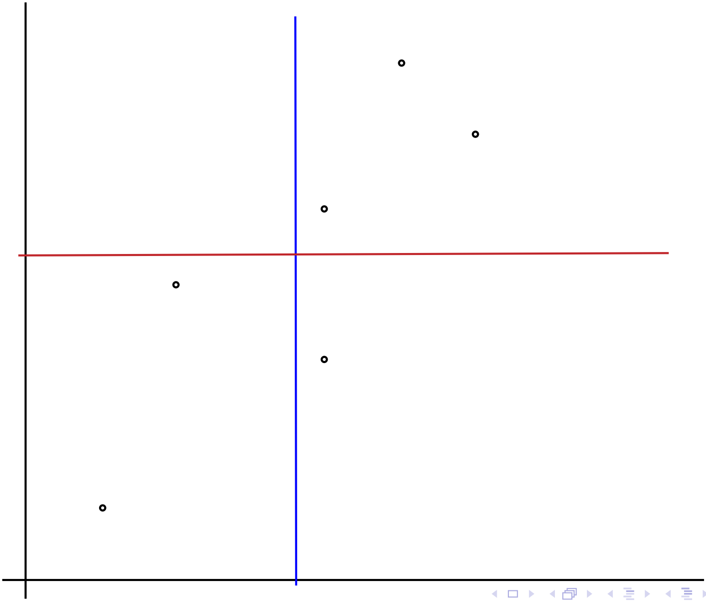

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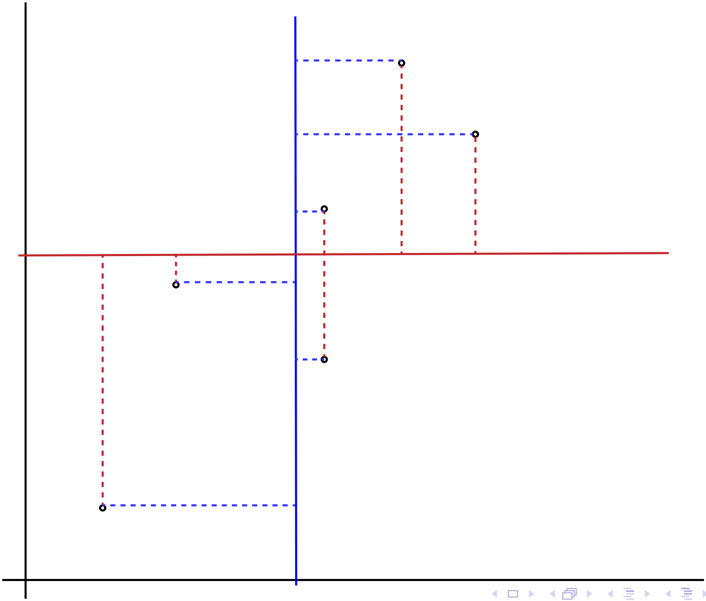

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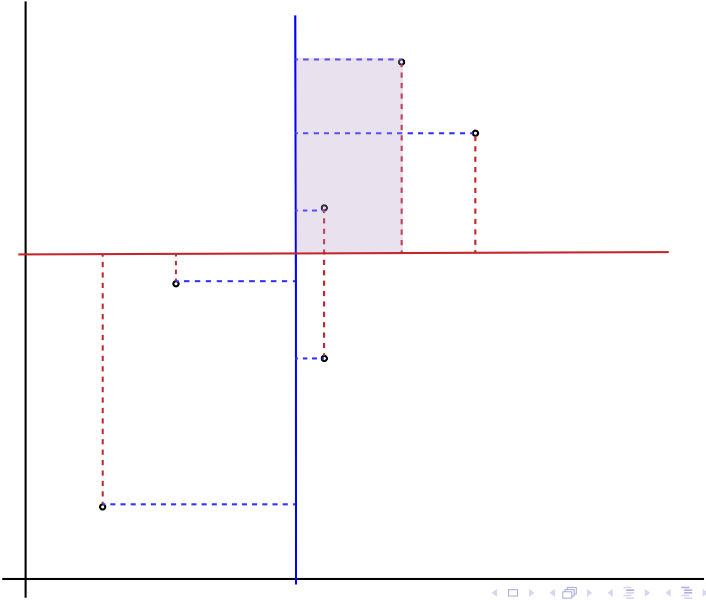

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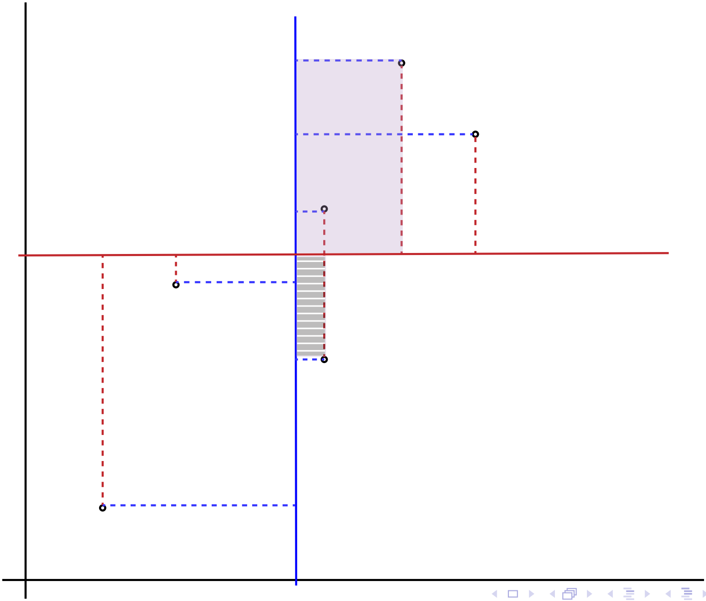

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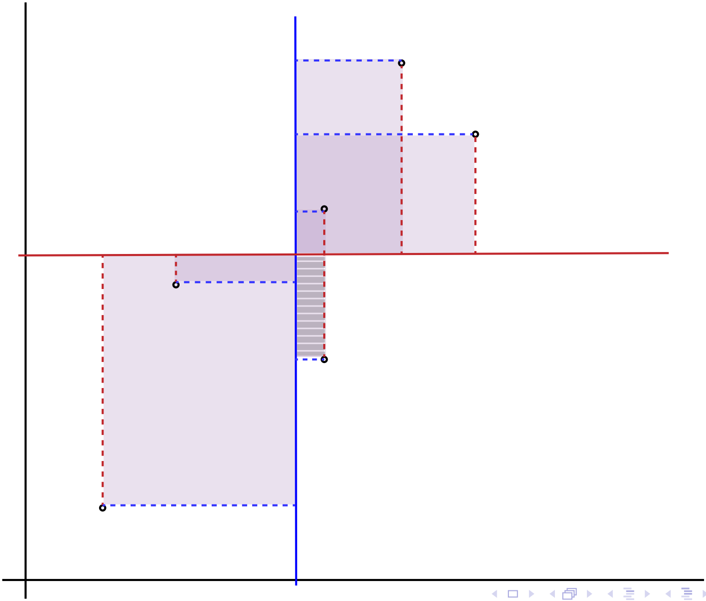

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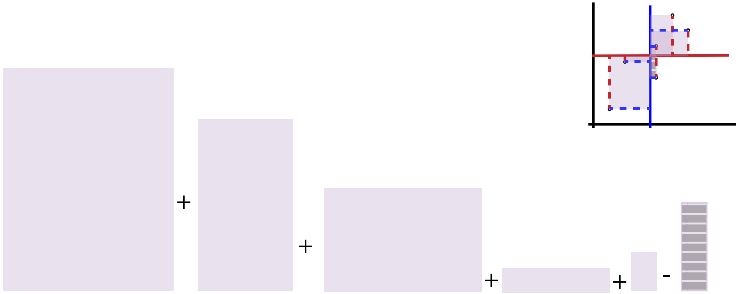

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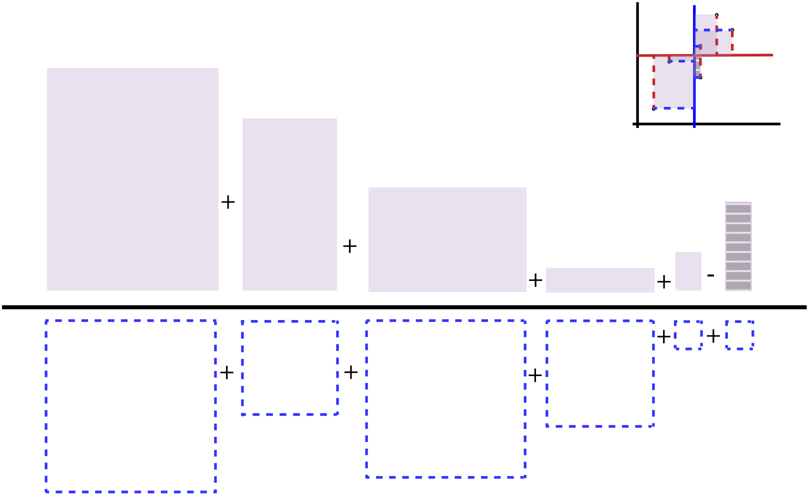

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Sample 1: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\} \rightarrow \boxed{\text{OLS}} \rightarrow (\widehat{\beta}_0, \widehat{\beta}_1)_1$

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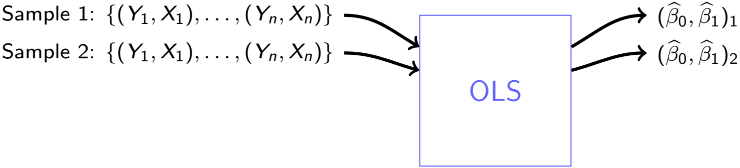

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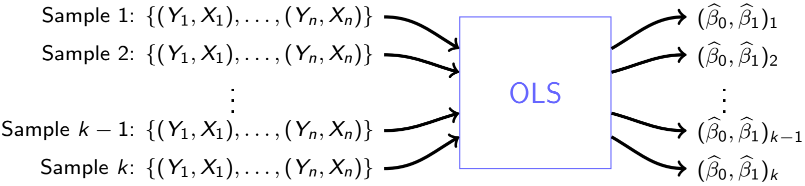

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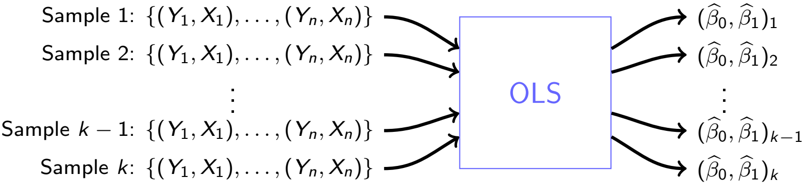

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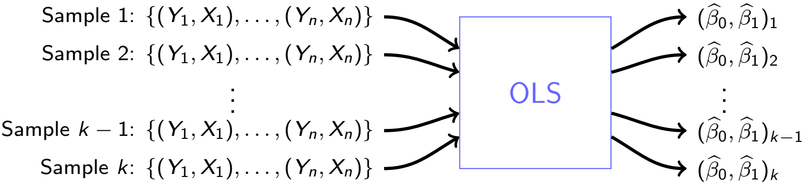

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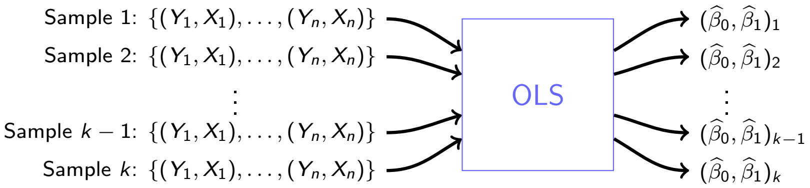

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Sample 1: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$ → OLS → $(\widehat{\beta}_0, \widehat{\beta}_1)_1$

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$\vdots$

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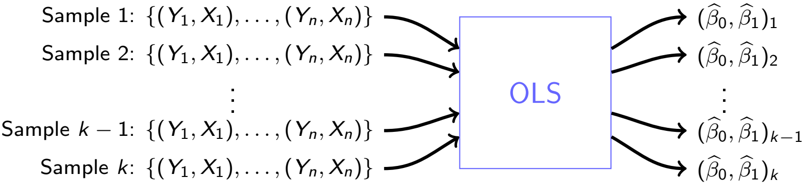

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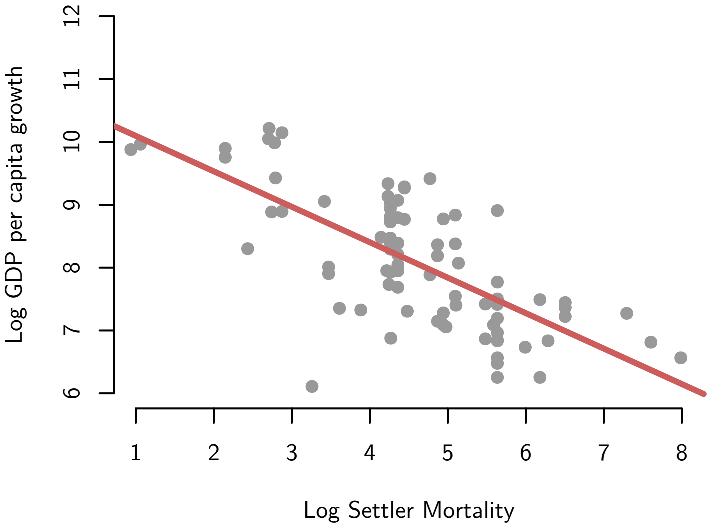
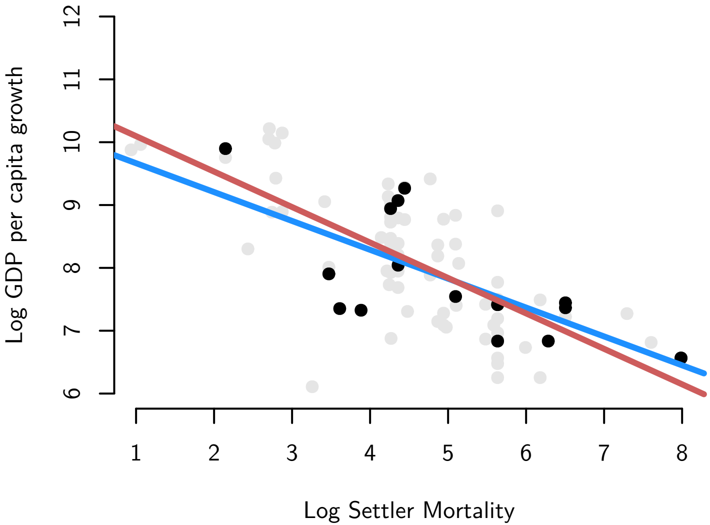


Randomly sample from AJR



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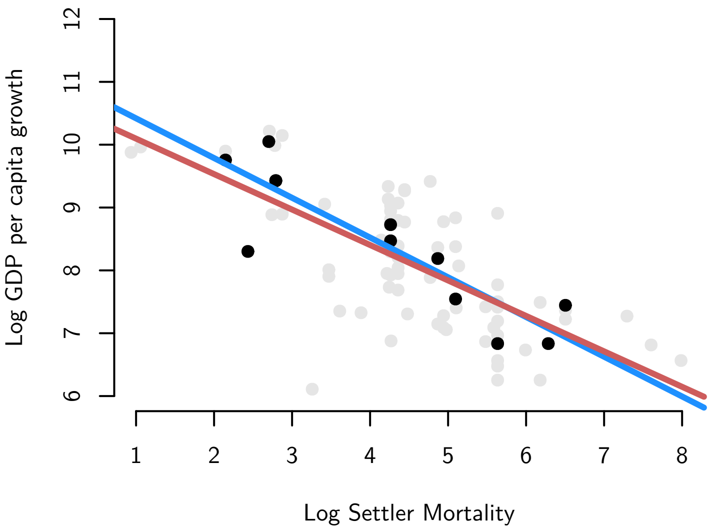

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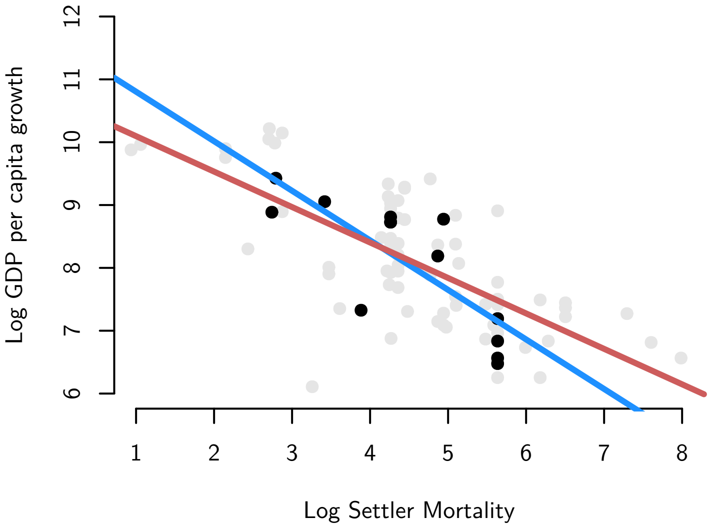

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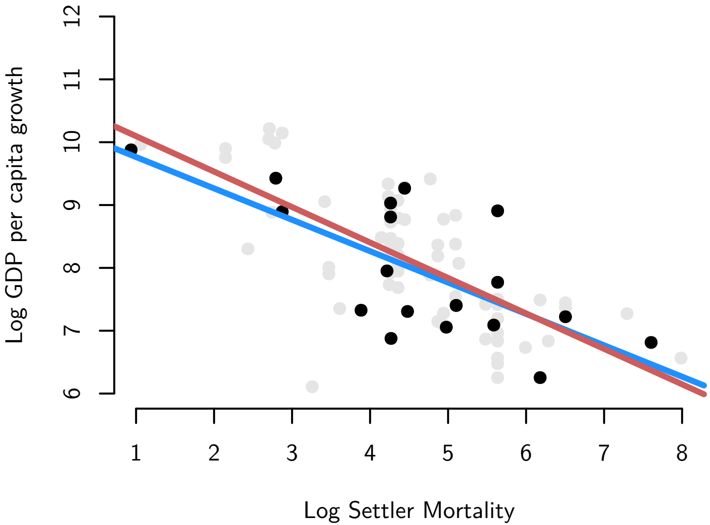

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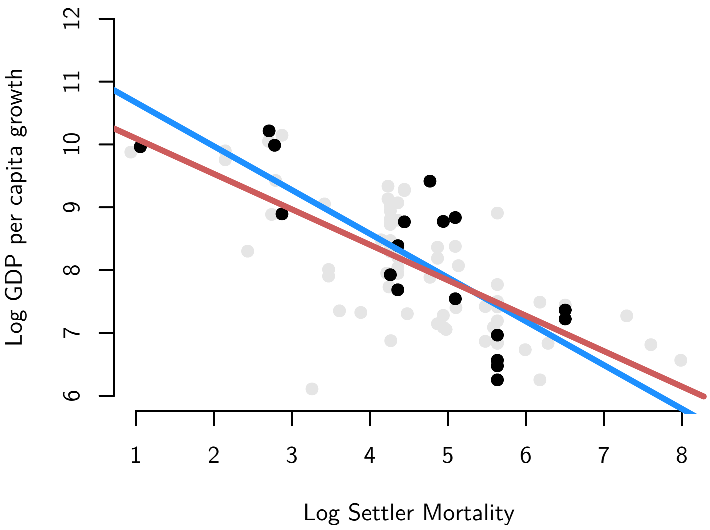

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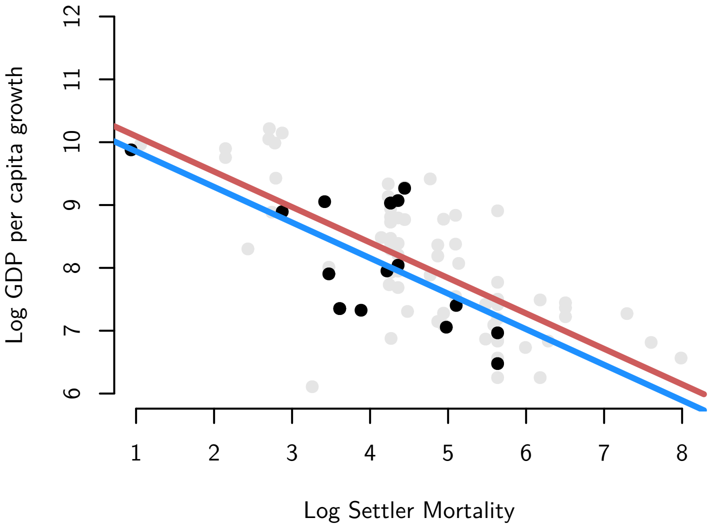

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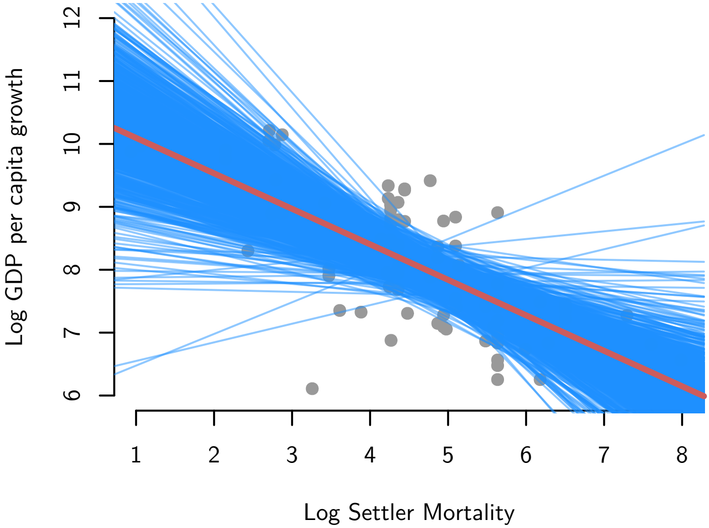

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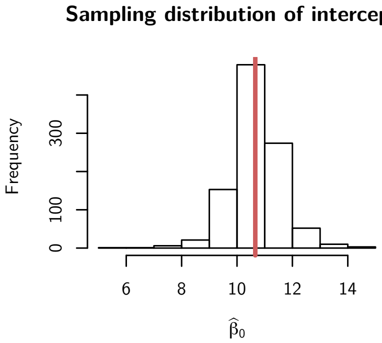

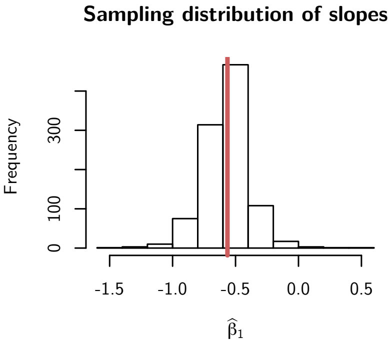

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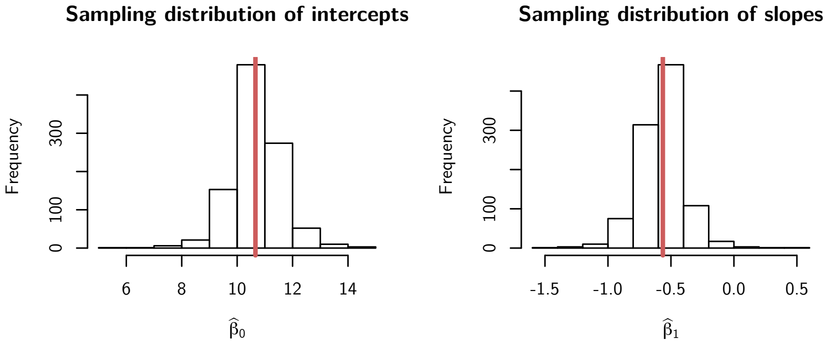

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Assumptions for unbiasedness of the sample mean

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Hierarchy of OLS Assumptions

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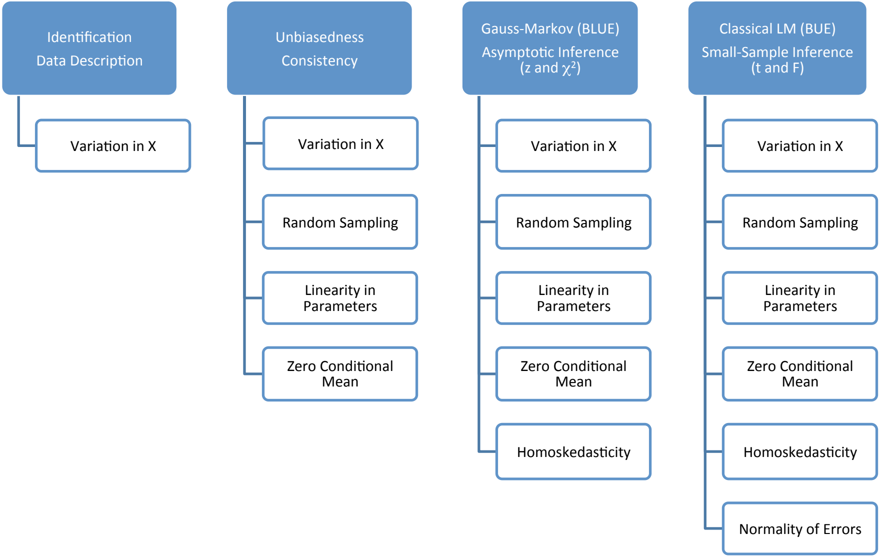

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- We assume this to be the structural model, i.e., the model describing the true process generating Y

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In fact, this is the only assumption needed for using OLS as a pure data summary.

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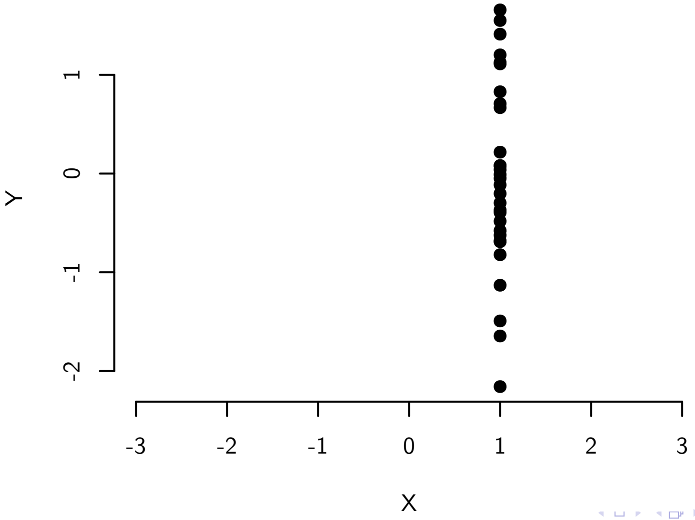
- Why does this matter? How would you draw the line of best fit through this scatterplot, which is a violation of this assumption?



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  $\longrightarrow$ It must be assumed $E[ability|educ = low] = E[ability|educ = high]$

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2. No relationship between them (satisfies the assumption)

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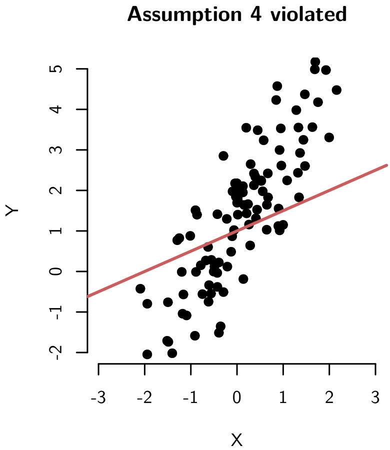

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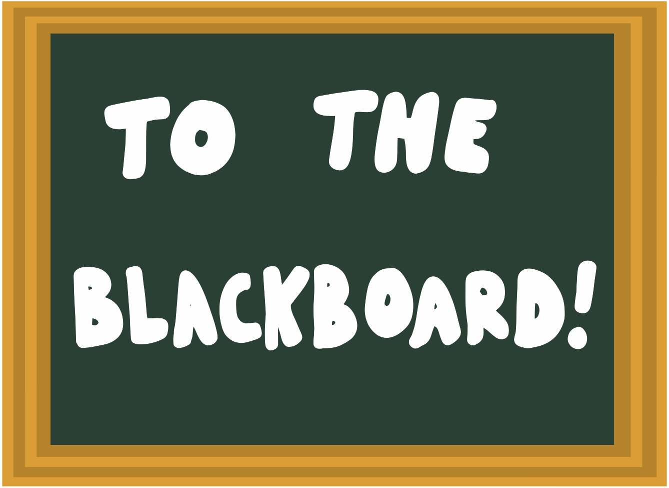

Unbiasedness of OLS

Theorem (Unbiasedness of OLS)

Given OLS Assumptions I–IV:

$$E[\hat{\beta}_0] = \beta_0 \quad \textit{and} \quad E[\hat{\beta}_1] = \beta_1$$

The sampling distributions of the estimators $\hat{\beta}_1$ and $\hat{\beta}_0$ are centered about the true population parameter values β_1 and β_0.

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- That is we know that the sampling distribution is centered on the true population slope, but we don't know the population variance.

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- Assumptions I–V are collectively known as the Gauss-Markov assumptions

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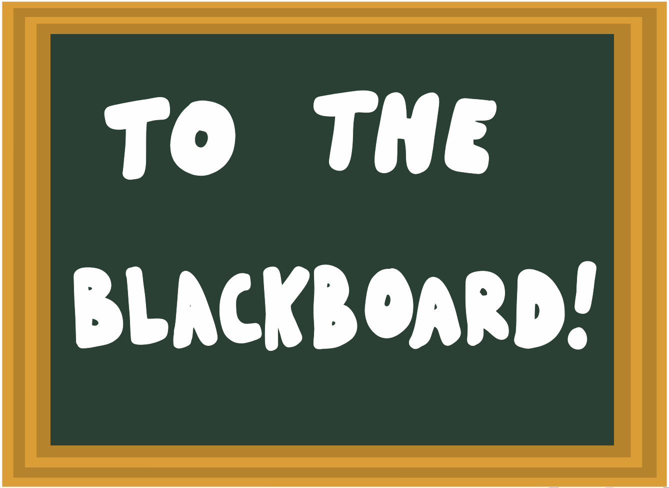

Variance of OLS Estimators

Theorem (Variance of OLS Estimators)

Given OLS Assumptions I–V (Gauss-Markov Assumptions):

$$Var[\hat{\beta}_1 \mid X] = \frac{\sigma_u^2}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\sigma_u^2}{SST_x}$$

$$Var[\hat{\beta}_0 \mid X] = \sigma_u^2 \left\{ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right\}$$

where $Var[u \mid X] = \sigma_u^2$ *(the error variance).*

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  - As we increase n, the denominator gets large, while the numerator is fixed and so the sampling variance shrinks to 0.

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Intuitively, which line is likely to be closer to the observed sample values on X and Y, the true line $y_i = \beta_0 + \beta_1 x_i$ or the fitted regression line $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$?

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- Thus, an unbiased estimator for the error variance is:

$$\hat{\sigma}_u^2 = \frac{n}{n-2} MSD(\hat{u}) = \frac{n}{n-2}\frac{1}{n}\sum_{i=1}^{n} \hat{u}_i = \frac{1}{n-2}\sum_{i=1}^{n} \hat{u}_i^2$$

We plug this estimate into the variance estimators for $\hat{\beta}_0$ and $\hat{\beta}_1$.

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Questions?

Example: Epstein and Mershon SCOTUS data

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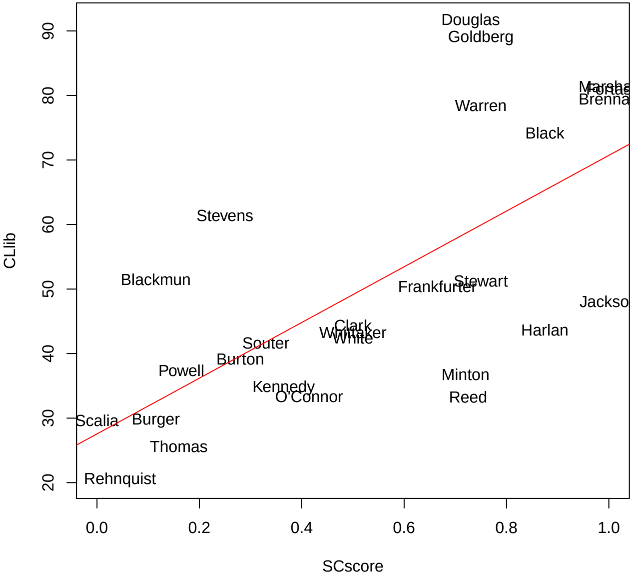

Douglas
Goldberg
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SCscore
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1.0
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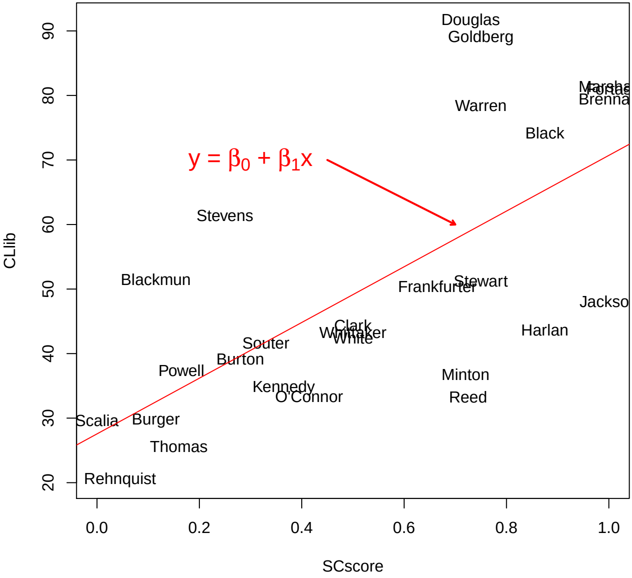

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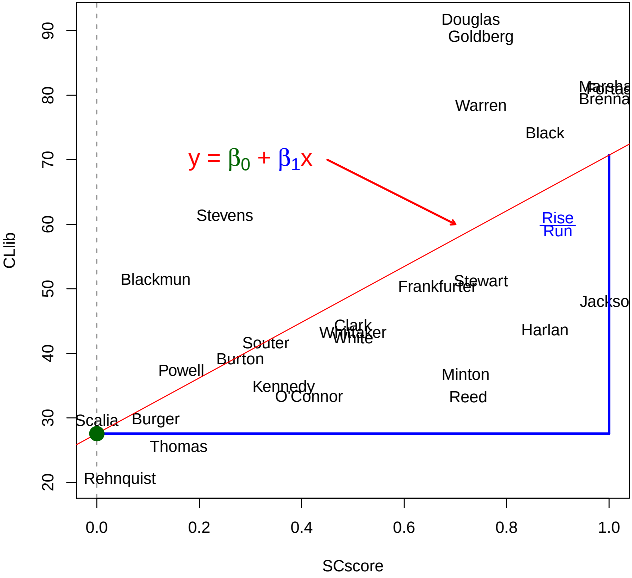

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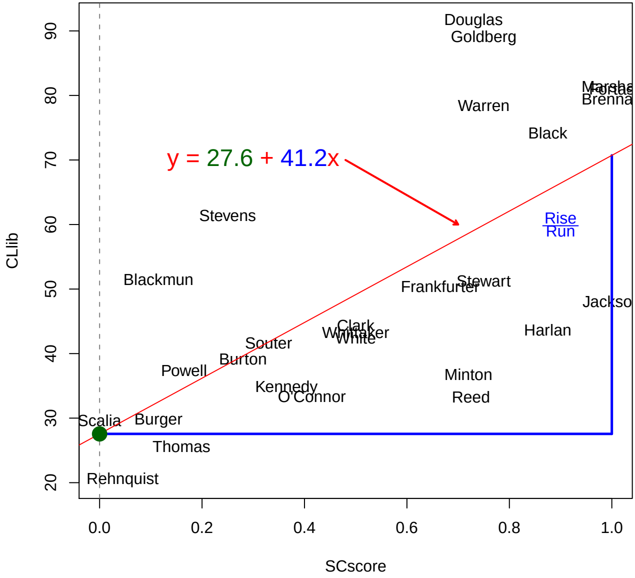

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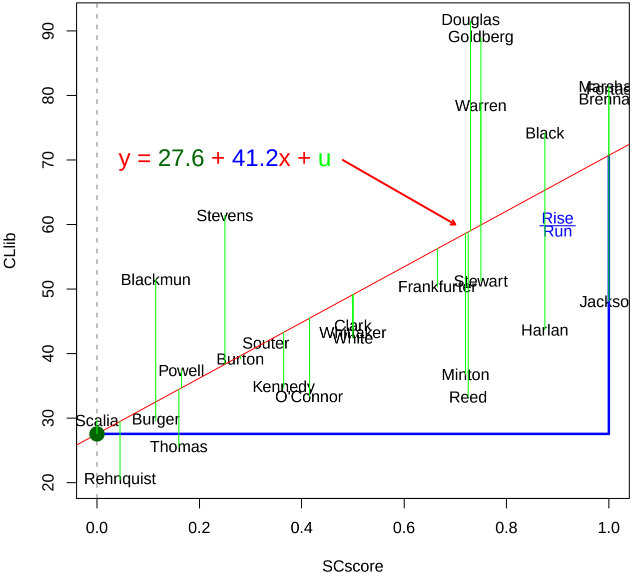
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How to get β_0 and β_1

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

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1. Mechanics of OLS
2. Properties of the OLS estimator
3. Example and Review
4. Properties Continued
5. Hypothesis tests for regression
6. Confidence intervals for regression
7. Goodness of fit
8. Wrap Up of Univariate Regression
9. Fun with Non-Linearities

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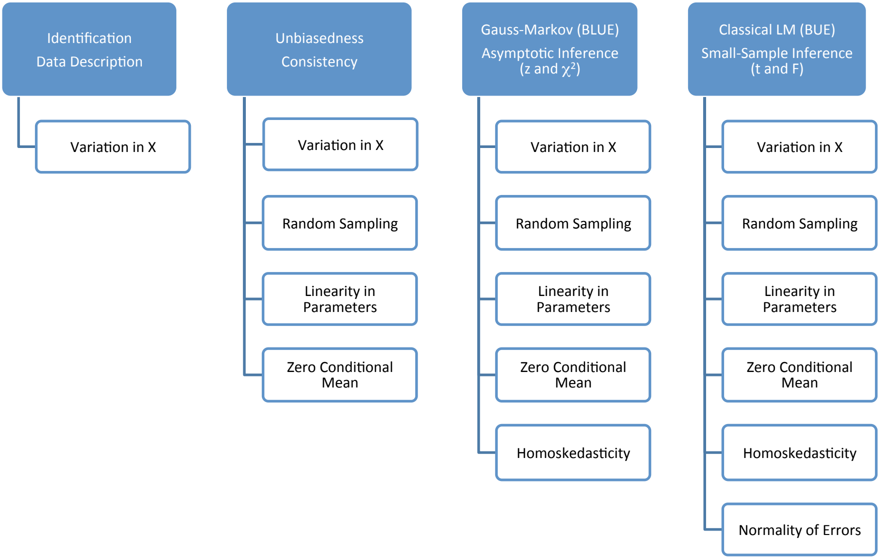

Identification
Data Description
Variation in X
Unbiasedness
Consistency
Variation in X
Random Sampling
Linearity in Parameters
Zero Conditional Mean
Gauss-Markov (BLUE)
Asymptotic Inference
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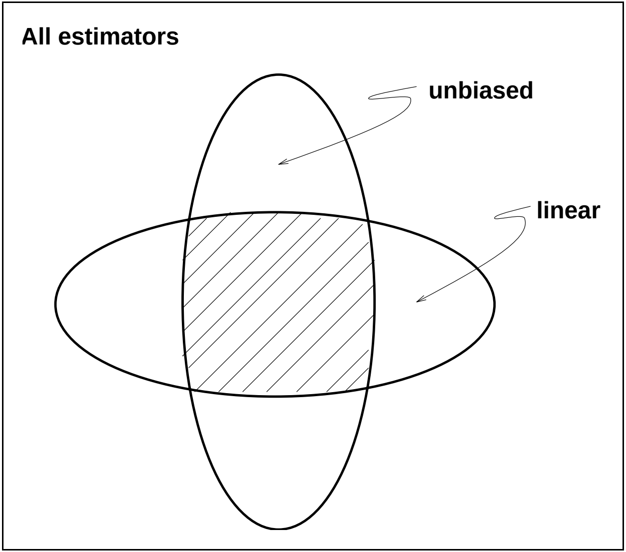
- Fails to hold when the assumptions are violated!

Gauss-Markov Theorem



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Assumption (VI. Normality)

The population error term is independent of the explanatory variable, $u \perp\!\!\!\perp X$, and is normally distributed with mean zero and variance σ_u^2:

$$u \sim N(0, \sigma_u^2), \quad \textit{which implies} \quad Y|X \sim N(\beta_0 + \beta_1 X, \sigma_u^2)$$

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Any linear combination of independent normals is normal, and we can transform/standarize any normal random variable into a standard normal by subtracting off its mean and dividing by its standard deviation. □

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- All of this depends on Normal errors! We can check to see if the error do look Normal.

The t-Test for Single Population Parameters

- $SE[\hat{\beta}_1] = \frac{\sigma_u}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$ involves the unknown population error variance σ_u^2
- Replace σ_u^2 with its unbiased estimator $\hat{\sigma}_u^2 = \frac{\sum_{i=1}^{n} \hat{u}_i^2}{n-2}$, and we obtain:

Theorem (Sampling Distribution of t-value)

Under Assumptions I–VI, the t-value for β_1 has a t-distribution with $n-2$ degrees of freedom:

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Proof.

The logic is perfectly analogous to the t-value for the population mean — because we are estimating the denominator, we need a distribution that has fatter tails than $N(0,1)$ to take into account the additional uncertainty.

This time, $\hat{\sigma}_u^2$ contains two estimated parameters ($\hat{\beta}_0$ and $\hat{\beta}_1$) instead of one, hence the degrees of freedom $= n-2$. □

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Now let's briefly return to some of the large sample properties.

Large Sample Properties: Consistency

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- Now let's take a more rigorous look at the large sample properties, i.e., how $(\hat{\beta}_0, \hat{\beta}_1)$ behaves *when* $n \to \infty$.

Theorem (Consistency of OLS Estimator)

Given Assumptions I–IV, the OLS estimator $\widehat{\beta}_1$ *is consistent for* β_1 *as* $n \to \infty$*:*

$$\operatorname*{plim}_{n\to\infty} \widehat{\beta}_1 = \beta_1$$

- Technical note: We can slightly relax Assumption IV:

$$E[u|X] = 0 \quad \text{(any function of } X \text{ is uncorrelated with } u\text{)}$$

to its implication:

$$\text{Cov}[u, X] = 0 \quad (X \text{ is uncorrelated with } u)$$

for consistency to hold (but not unbiasedness).

Large Sample Properties: Consistency

Proof.

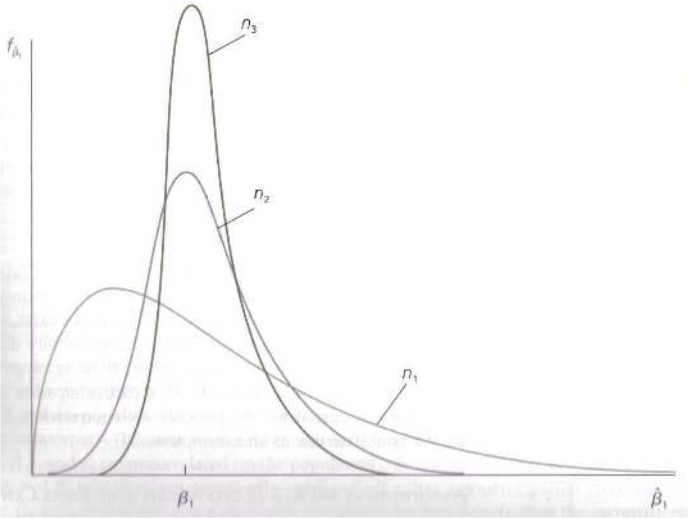
Similar to the unbiasedness proof:

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} = \beta_1 + \frac{\sum_i^n (x_i - \bar{x}) u_i}{\sum_i^n (x_i - \bar{x})^2} \\
\text{plim } \widehat{\beta}_1 &= \text{plim } \beta_1 + \text{plim } \frac{\sum_i^n (x_i - \bar{x}) u_i}{\sum_i^n (x_i - \bar{x})^2} \quad \text{(Wooldridge C.3 Property i)} \\
&= \beta_1 + \frac{\text{plim } \frac{1}{n} \sum_i^n (x_i - \bar{x}) u_i}{\text{plim } \frac{1}{n} \sum_i^n (x_i - \bar{x})^2} \quad \text{(Wooldridge C.3 Property iii)} \\
&= \beta_1 + \frac{\text{Cov}[X, u]}{\text{Var}[X]} \quad \text{(by the law of large numbers)} \\
&= \beta_1 \quad (\text{Cov}[X, u] = 0 \text{ and } \text{Var}[X] > 0)
\end{aligned}
$$

□

- OLS is inconsistent (and biased) unless $\text{Cov}[X, u] = 0$
- If $\text{Cov}[u, X] > 0$ then asymptotic bias is upward; if $\text{Cov}[u, X] < 0$ asymptotic bias is downwards

Large Sample Properties: Consistency



Sampling distributions of $\hat{\beta}_1$, for sample sizes $n_1 < n_2 < n_3$

Large Sample Properties: Asymptotic Normality

- For statistical inference, we need to know the sampling distribution of $\hat{\beta}$ when $n \to \infty$.

Theorem (Asymptotic Normality of OLS Estimator)

Given Assumptions I–V*, the OLS estimator $\widehat{\beta}_1$ is asymptotically normally distributed:*

$$\frac{\hat{\beta}_1 - \beta_1}{\widehat{SE}[\hat{\beta}_1]} \overset{approx.}{\sim} N(0,1)$$

where

$$\widehat{SE}[\hat{\beta}_1] = \frac{\hat{\sigma}_u}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

with the consistent estimator for the error variance:

$$\hat{\sigma}_u^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 \xrightarrow{p} \sigma_u^2$$

Large Sample Inference

Proof.

Proof is similar to the small-sample normality proof:

$$\hat{\beta}_1 = \beta_1 + \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{SST_x} u_i$$

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = \frac{\sqrt{n} \cdot \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}) u_i}{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

where the numerator converges in distribution to a normal random variable by CLT. Then, rearranging the terms, etc. gives you the right formula given in the theorem.

For a more formal and detailed proof, see Wooldridge Appendix 5A. □

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For a more formal and detailed proof, see Wooldridge Appendix 5A. □

- We need homoskedasticity (Assumption V) for this result, but we do not need normality (Assumption VI).
- Result implies that asymptotically our usual standard errors, t-values, p-values, and CIs remain valid even without the normality assumption! We just proceed as in the small sample case where we assume normality.
- It turns out that, given Assumptions I–V, the OLS asymptotic variance is also the lowest in class (asymptotic Gauss-Markov).

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Three ways of making statistical inference out of regression:

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For 2 and 3, we need to know more than just the mean and the variance of the sampling distribution of $\hat{\beta}_1$. We need to know the full shape of the sampling distribution of our estimators $\hat{\beta}_0$ and $\hat{\beta}_1$.

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  - The null is the straw man we want to knock down.
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- Notice these are statements about the population parameters, not the OLS estimates.

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- Thus, under the null, we know the distribution of T and can use that to formulate a rejection region and calculate p-values.

Rejection region

- Choose a level of the test, α, and find rejection regions that correspond to that value under the null distribution:

$$\mathbb{P}(-t_{\alpha/2,n-2} < T < t_{\alpha/2,n-2}) = 1 - \alpha$$

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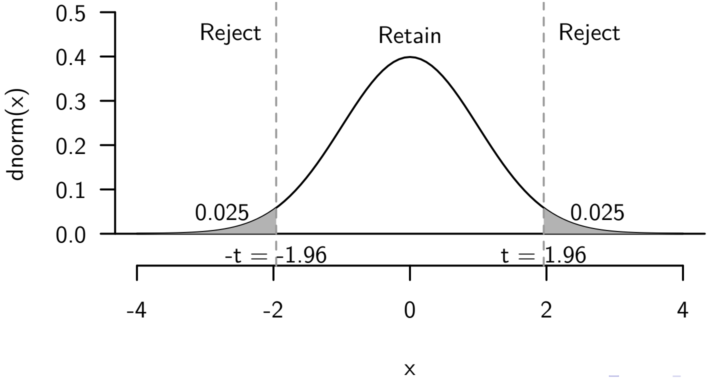
- This is exactly the same as with sample means and sample differences in means, except that the degrees of freedom on the t distribution have changed.



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- If the p-value is less than α we would reject the null at the α level.

Confidence intervals

- Very similar to the approach with sample means. By the sampling distribution of the OLS estimator, we know that we can find t-values such that:

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- We can derive these for the intercept as well:

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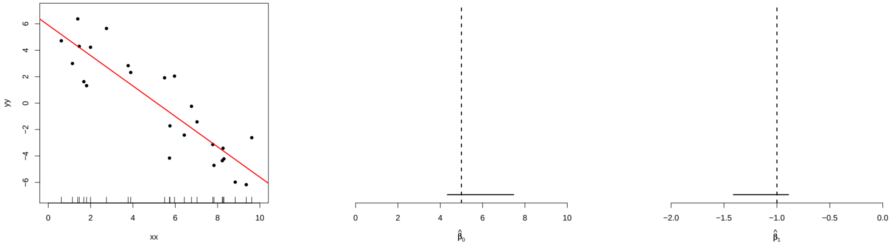
CIs Simulation Example

Returning to our simulation example we can simulate the sampling distributions of the 95 % confidence interval estimates for $\widehat{\beta}_1$ and $\widehat{\beta}_0$



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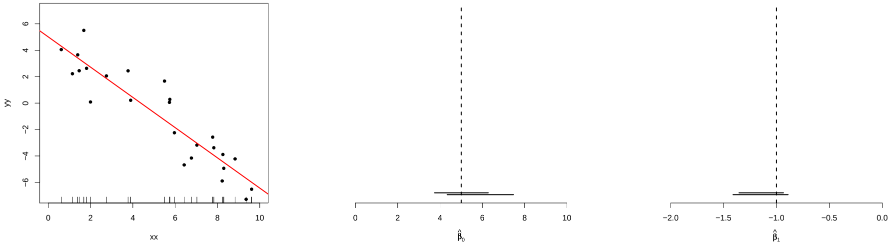

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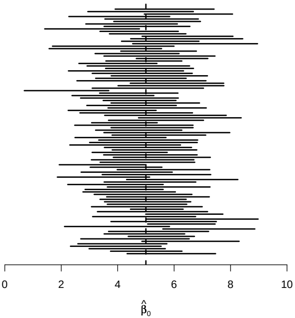

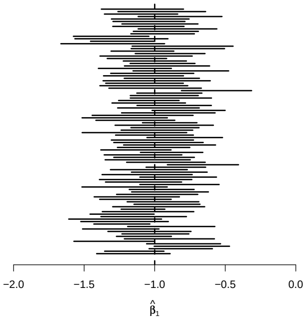

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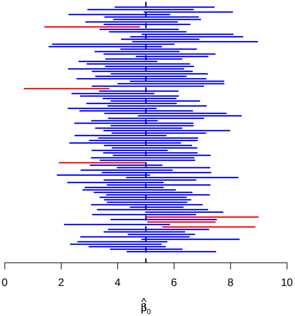

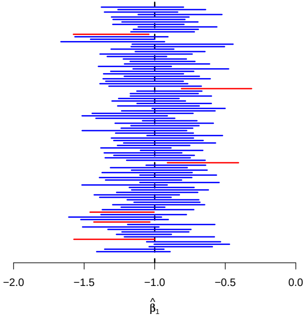

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- Once we have estimated our model, we have new prediction errors, which are just the sum of the squared residuals or SS_{res}:

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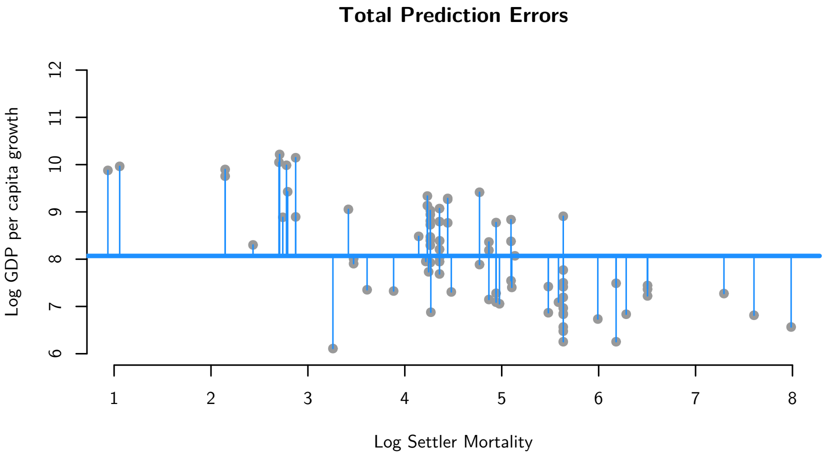
Sum of Squares



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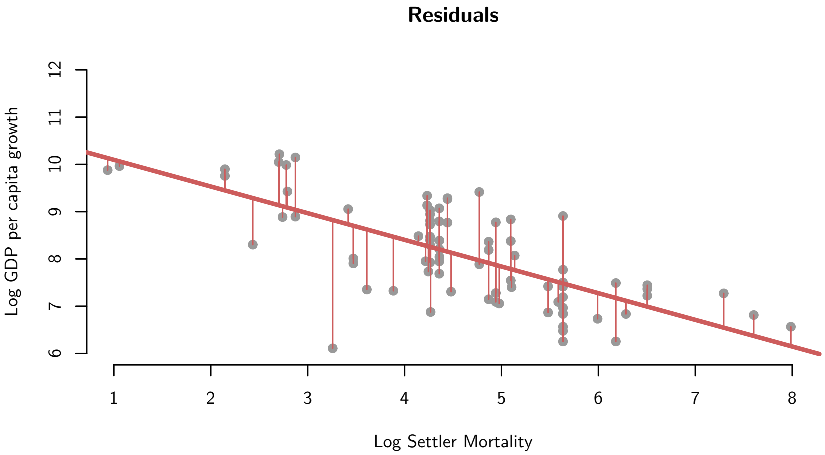

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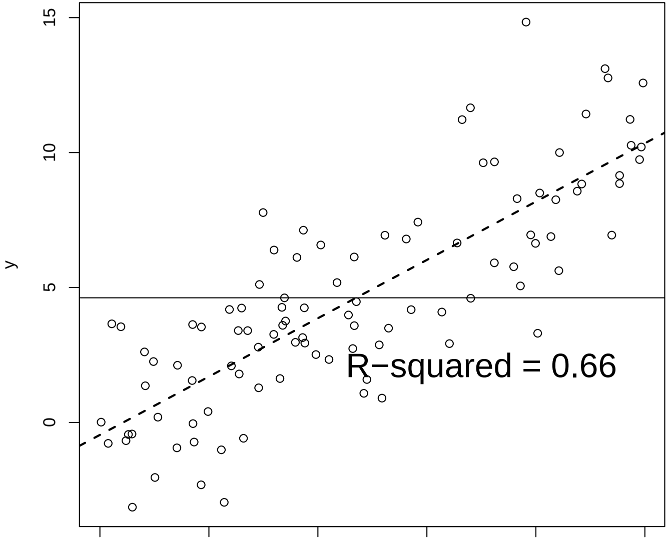
- $R^2 = 1$ implies perfect linear fit

Is R-squared useful?



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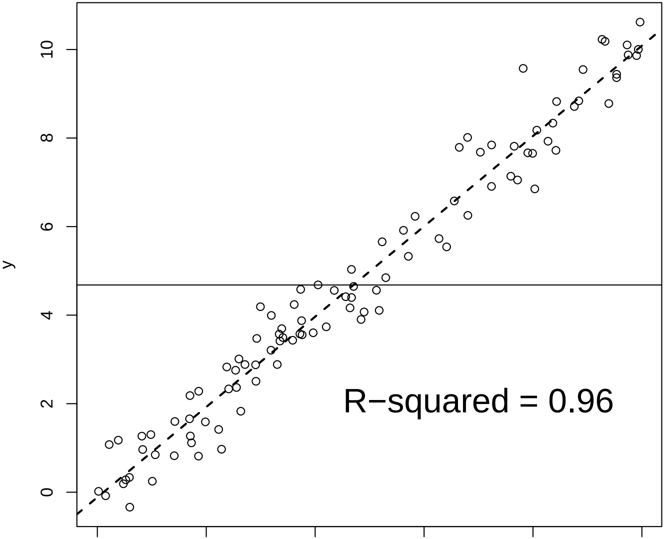

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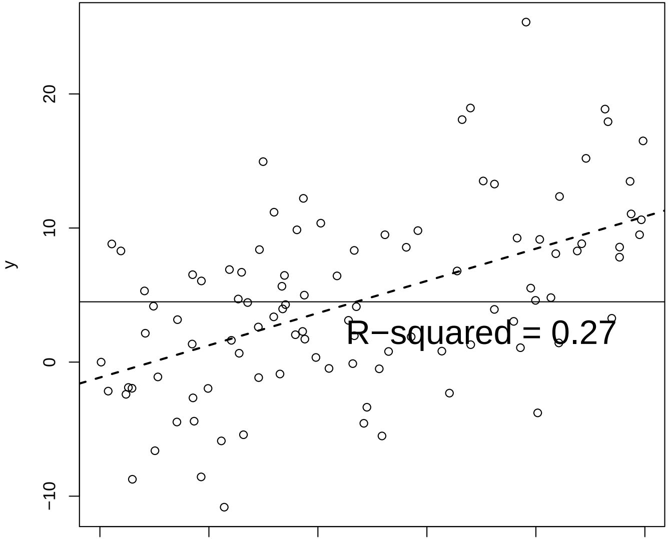

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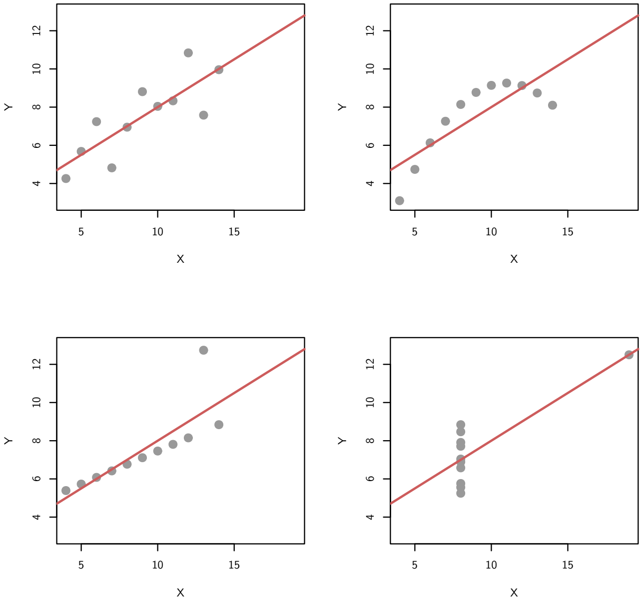


Why r^2?

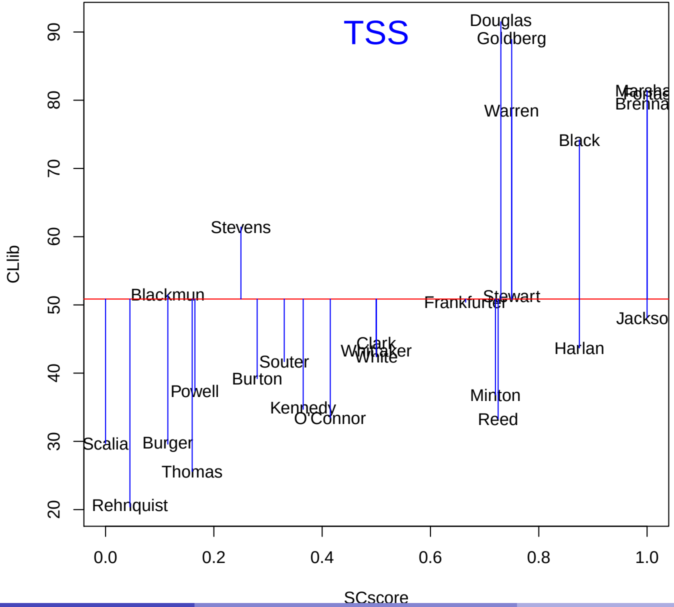
To calculate r^2, we need to think about the following two quantities:

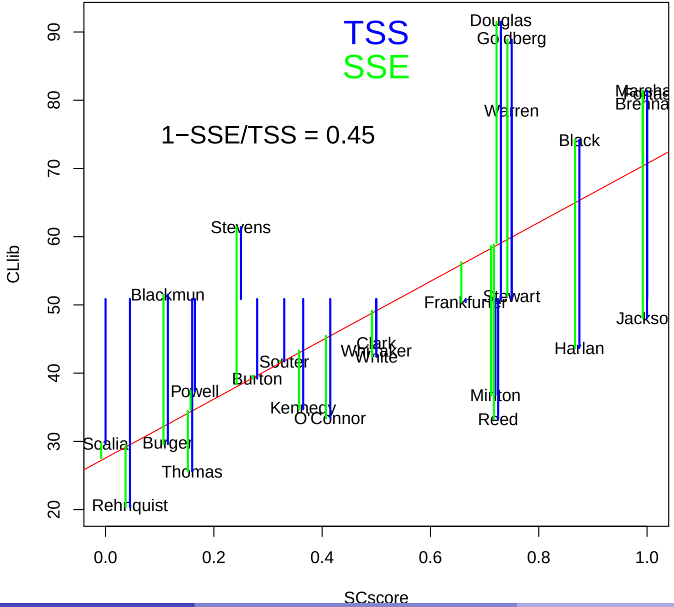
1. TSS: Total sum of squares
2. SSE: Sum of squared errors

$$TSS = \sum_{i=1}^{n}(y_i - \bar{y})^2.$$

$$SSE = \sum_{i=1}^{n} u_i^2.$$

$$r^2 = 1 - \frac{SSE}{TSS}.$$


TSS
Douglas
Goldberg
Marshall
Fortas
Brennan
Warren
Black
Stevens
Blackmun
Frankfurter
Stewart
Jackson
Clark
Whittaker
White
Harlan
Souter
Burton
Powell
Kennedy
O'Connor
Minton
Reed
Scalia
Burger
Thomas
Rehnquist
CLlib
20
30
40
50
60
70
80
90
0.0
0.2
0.4
0.6
0.8
1.0
SCscore


TSS
SSE
1−SSE/TSS = 0.45
CLlib
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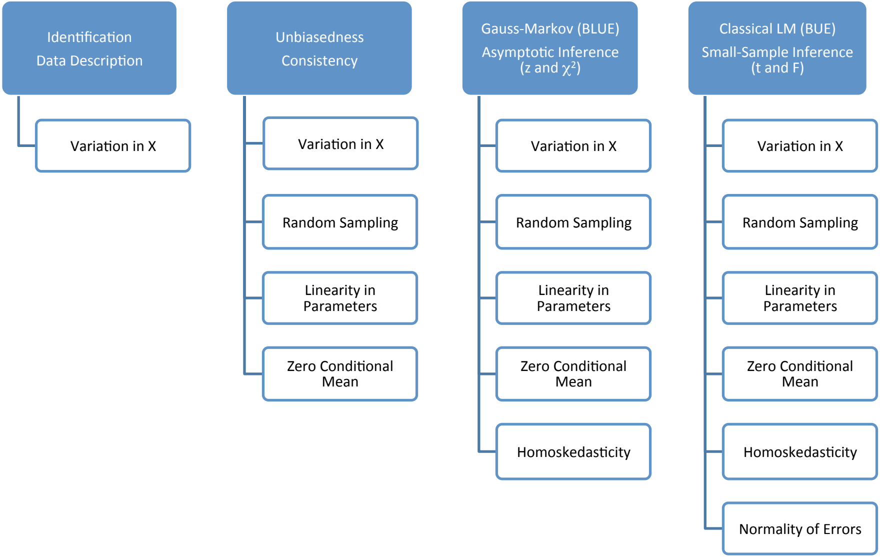
r^2 is a measure of how much of the variation in Y is accounted for by X.

OLS Assumptions Summary



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- Implications:
   - When Assumptions I–IV are all satisfied, we can estimate the structural parameters β without bias and thus make causal inference.
   - However, we can make predictive inference even if some assumptions are violated.

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- Note that Assumption I would make OLS the best, not just best linear, predictor, so it is certainly desired

State Legislators and African American Population

Interpretations of increasing quality:

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> summary(lm(beo ~ bpop, data = D))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.31489    0.32775  -4.012 0.000264 ***
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Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.317 on 39 degrees of freedom
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4. Give a short, but precise interpretation of practical significance. You want to discuss the magnitude of the slope in your particular application.

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  - The slope coefficient is fairly precisely estimated, the 95 % confidence interval ranging from 8 to 10

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- Examples:

  Turnout on education: The slope estimate suggests that a high school dropout (10th percentile of the education distribution) on average has a .3 lower probability of voting compared to a college graduate (75th percentile of schooling).

  The average probability of voting among HSDs is .2, so this corresponds to a 150 % increase for an average HSD.

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- Examples:

  Democracy on GDP: Going from the 25th to the 75th percentile of the GDP distribution (e.g. comparing Ghana and Spain) is associated with a 10 point increase in the average polity index, which corresponds to an increase from the 25th to the 52nd percentile of the democracy distribution

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  Earnings on Schooling: The standard deviation is 2.5 years for schooling and \$50,000 for annual earnings. Thus, the slope estimates suggest that a one standard deviation increase in schooling is associated with a .8 standard deviation increase in earnings.

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  - Fox Chapter 7.1-7.3 (Dummy-Variable Regression, Interactions)

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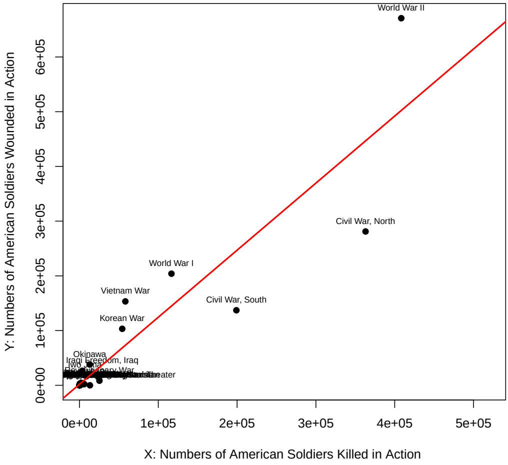
  - Note that these approximations work only for small increments
  - In particular, they do not work when X is a discrete random variable

Example from the American War Library



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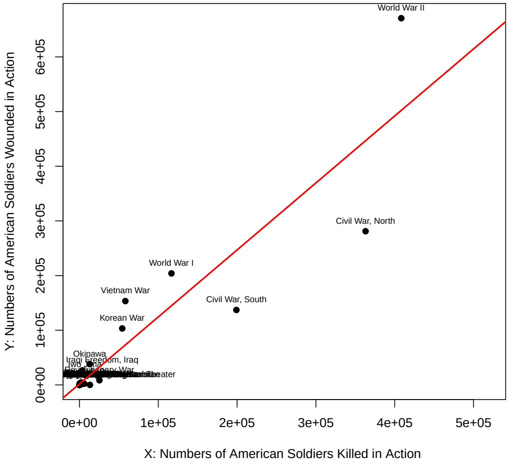
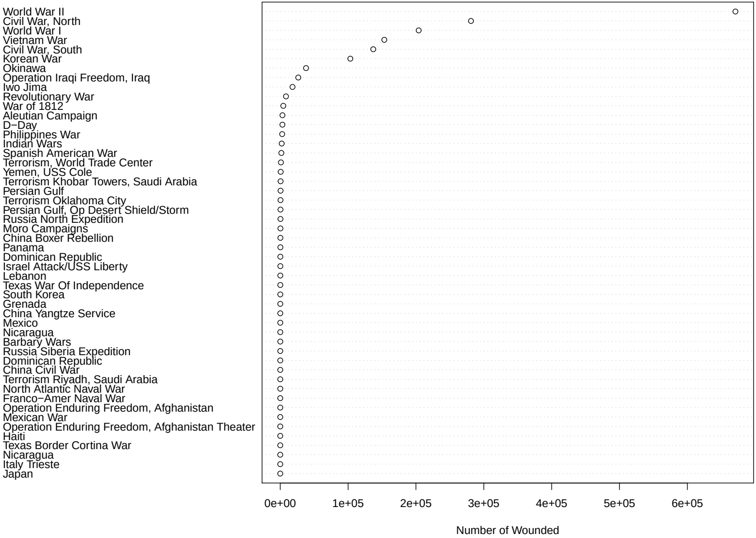
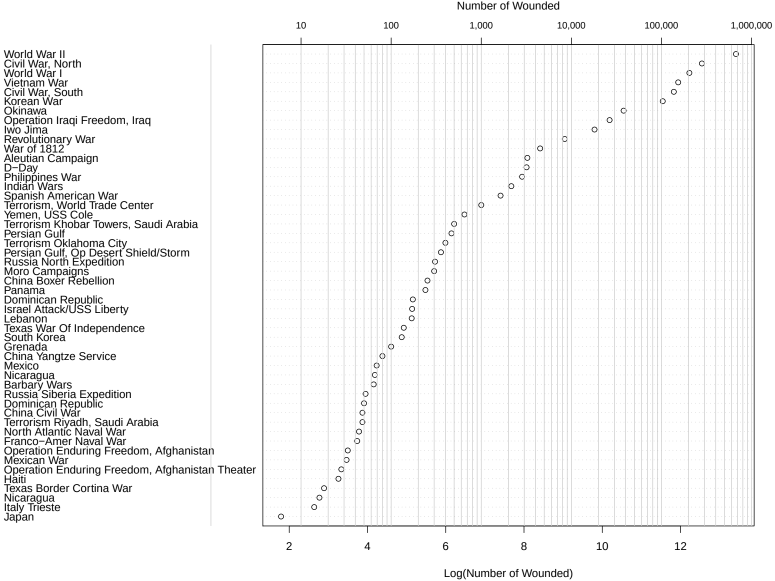


$\hat{\beta}_1 = 1.23 \longrightarrow$ One additional soldier killed predicts 1.23 additional soldiers wounded on average

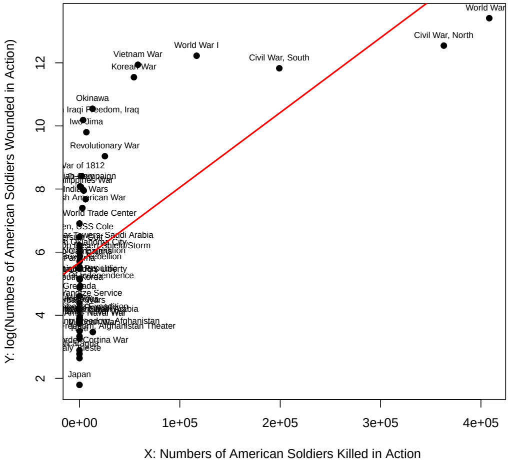
Wounded (Scale in Levels)



Wounded (Logarithmic Scale)



Regression: Log-Level



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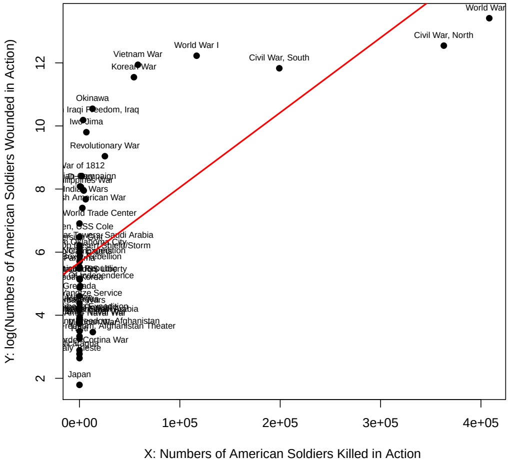
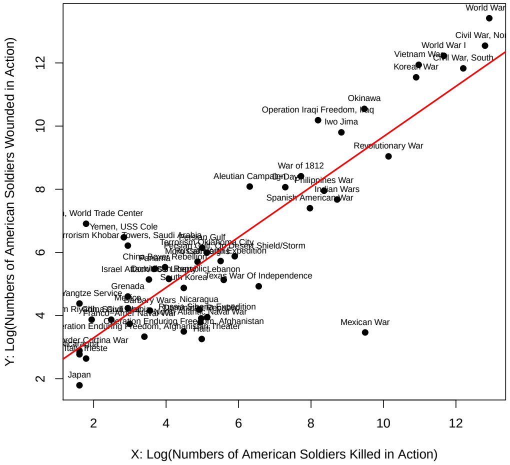


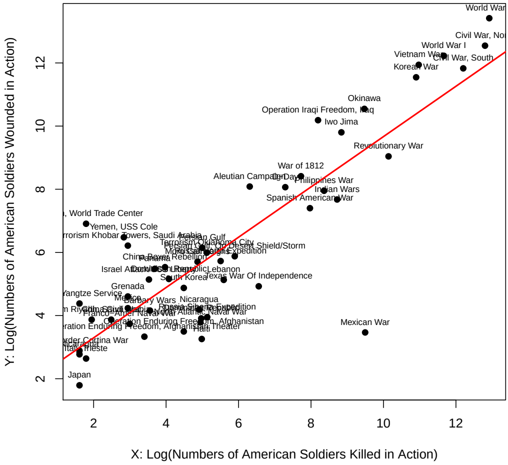
$\hat{\beta}_1 = 0.0000237 \longrightarrow$ One additional soldier killed predicts 0.0023 percent increase in the number of soldiers wounded on average

Regression: Log-Log



$\hat{\beta}_1 = 0.797 \longrightarrow$

Regression: Log-Log



$\hat{\beta}_1 = 0.797 \longrightarrow$ A percent increase in deaths predicts 0.797 percent increase in the wounded on average

References

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Wooldridge, Jeffrey. 2000. *Introductory Econometrics.* New York: South-Western.