Week 5: Simple Linear Regression

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Princeton

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[1]These slides are heavily influenced by Matt Blackwell, Adam Glynn, Erin Hartman and Jens Hainmueller. Illustrations by Shay O'Brien.

Where We've Been and Where We're Going...

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  - hypothesis testing
  - what is regression

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- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

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- Repeated Observations and Panel Data
- Review and Final Discussion

Narrow Goal: Understand `lm()` Output

```
Call:
lm(formula = sr ~ pop15, data = LifeCycleSavings)

Residuals:
   Min     1Q Median     3Q    Max
-8.637 -2.374  0.349  2.022 11.155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.49660    2.27972   7.675 6.85e-10 ***
pop15       -0.22302    0.06291  -3.545 0.000887 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.03 on 48 degrees of freedom
Multiple R-squared:  0.2075,Adjusted R-squared:  0.191
F-statistic: 12.57 on 1 and 48 DF,  p-value: 0.0008866
```

Reminder

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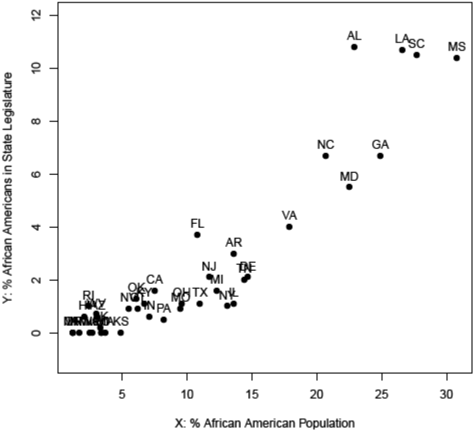
How do we fit the regression line $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$ to the data?



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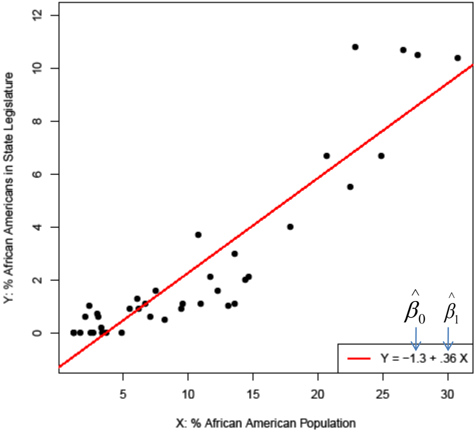

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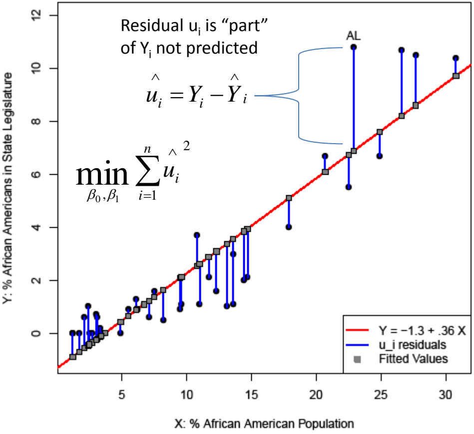
How do we fit the regression line $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$ to the data?

Answer: We will minimize the squared sum of residuals



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- The CEF has a potentially arbitrary shape but there is always a best linear predictor (BLP) or linear projection which is the line given by:

$$g(X) = \beta_0 + \beta_1 X$$

$$\beta_0 = E[Y] - \frac{\text{Cov}[X, Y]}{V[X]} E[X]$$

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- Define deviations from the BLP as

$$u = Y - g(X)$$

then, the following properties hold:

(1) $E[u] = 0$, (2) $E[Xu] = 0$, (3) $\text{Cov}[X,u] = 0$

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- There are many loss functions, but OLS uses the squared error loss which is connected to the conditional expectation function. If we chose a different loss, we would target a different feature of the conditional distribution.

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- We are going to step through this process together.

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Solving for the Intercept

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b_0 &= \overline{Y} - b_1 \overline{X}
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&= 0
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The OLS estimator

- Now we're done! Here are the **OLS estimators**:

$$\widehat{\beta}_0 = \overline{Y} - \widehat{\beta}_1 \overline{X}$$

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- The higher the covariance between X and Y, the higher the slope will be.
- Negative covariances $\rightarrow$ negative slopes;
  positive covariances $\rightarrow$ positive slopes
- If X_i doesn't vary, the denominator is undefined.
- If Y_i doesn't vary, you get a flat line.

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- This is important for two reasons. First, it'll make derivations later much easier. And second, it shows that is just the sum of a random variable. Therefore it is also a random variable.

Lemma 2: OLS as a Weighted Sum of Outcomes

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Next Time: The Classical Perspective

Where We've Been and Where We're Going...

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- Last Week
  - hypothesis testing
  - what is regression
- This Week
  - mechanics and properties of simple linear regression
  - inference and measures of model fit
  - confidence intervals for regression
  - goodness of fit
- Next Week
  - mechanics with two regressors
  - omitted variables, multicollinearity
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

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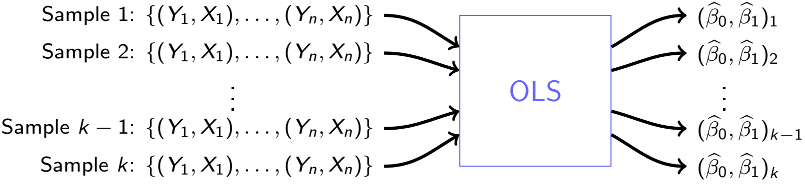

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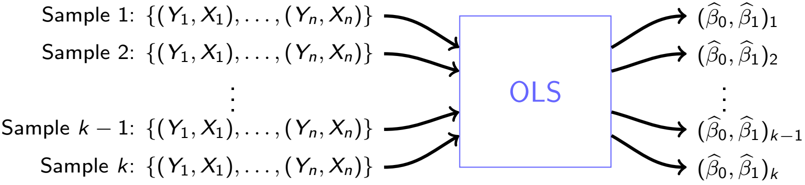

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  - See how the line varies from sample to sample

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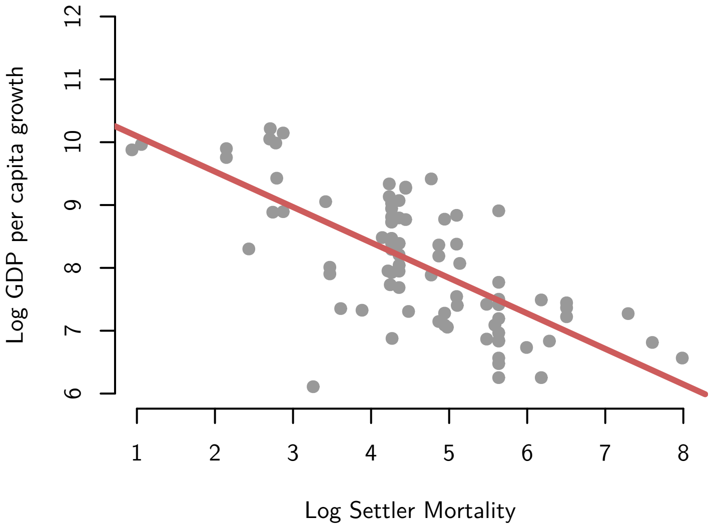
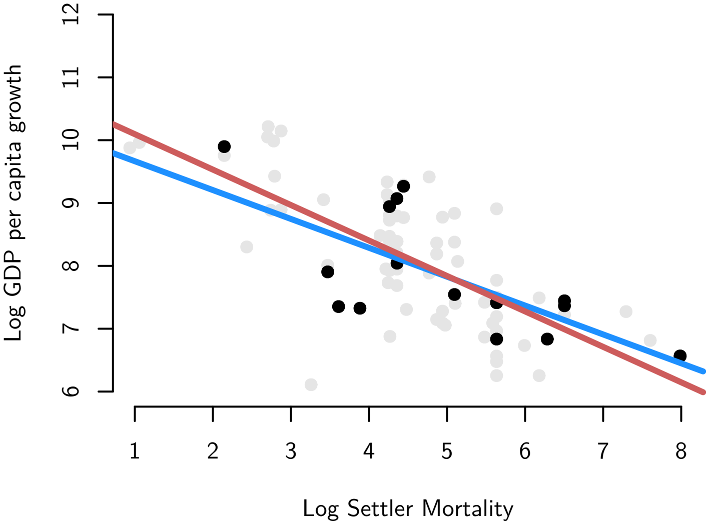
Population Regression



Randomly sample from AJR



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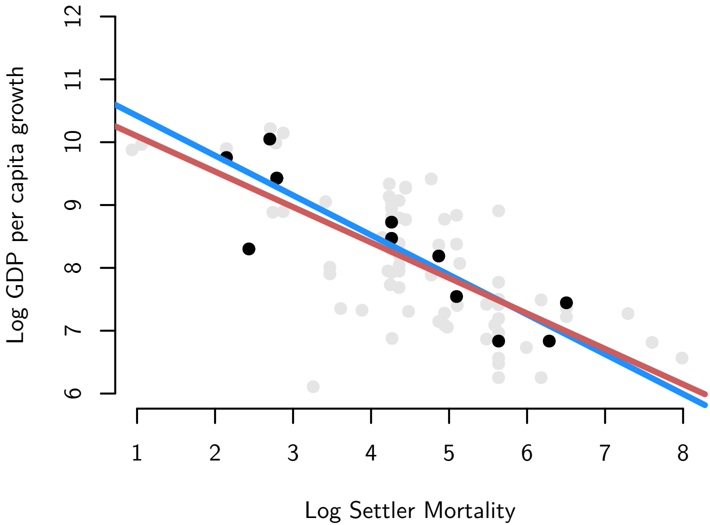

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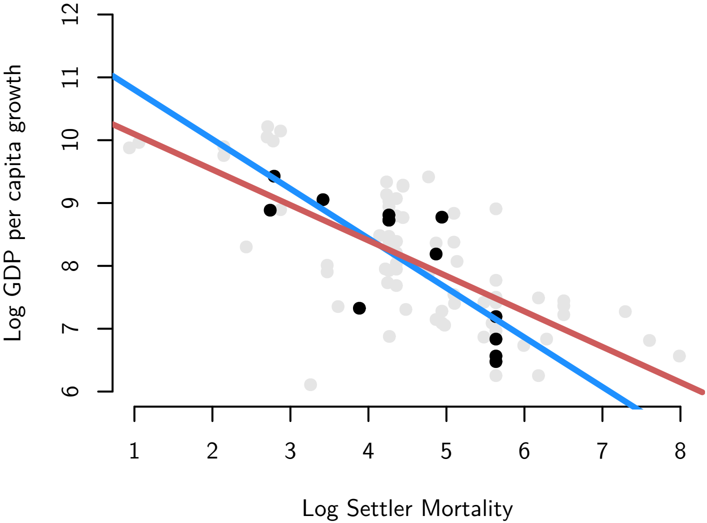

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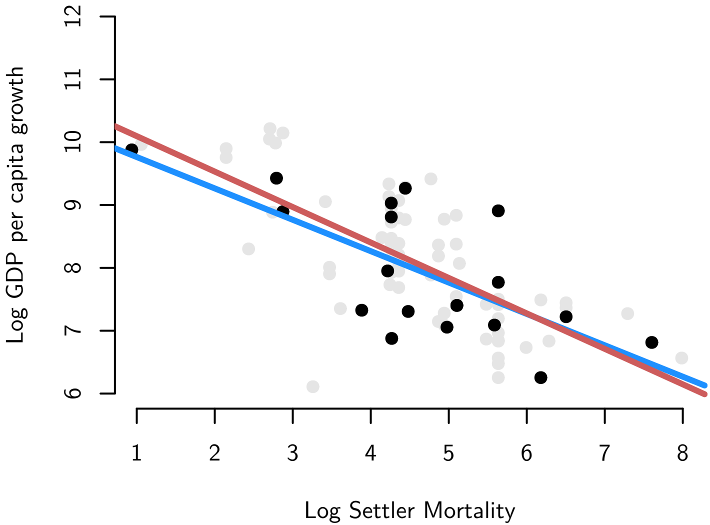

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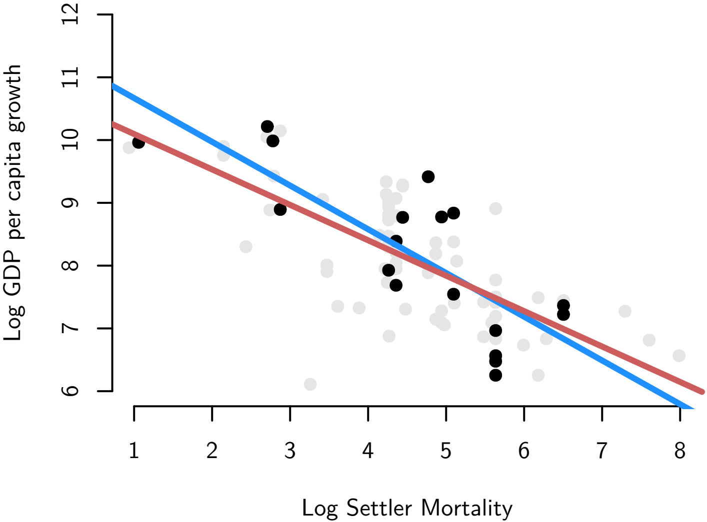

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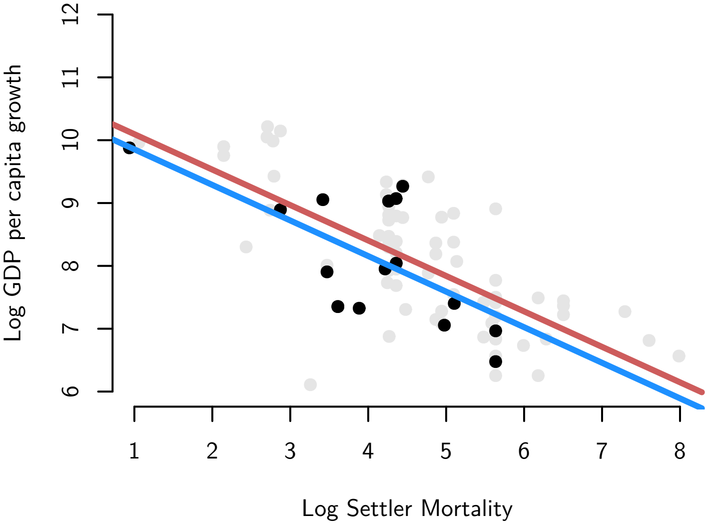

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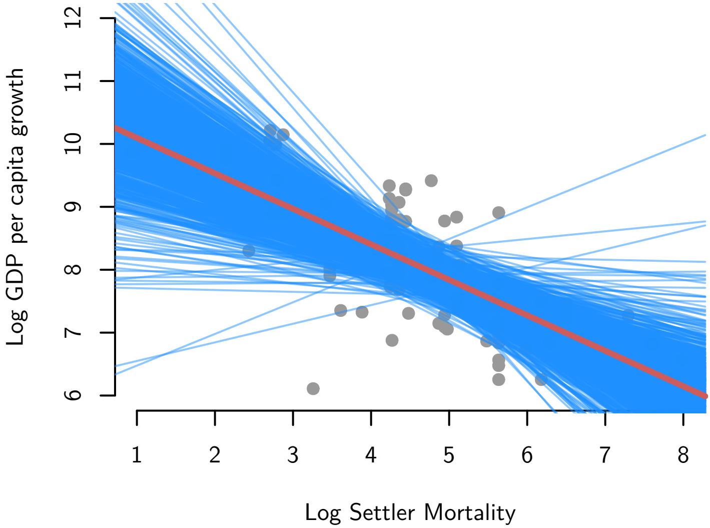

Sampling distribution of OLS

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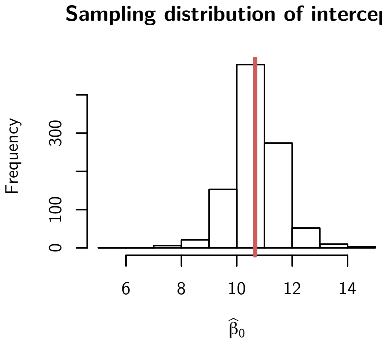
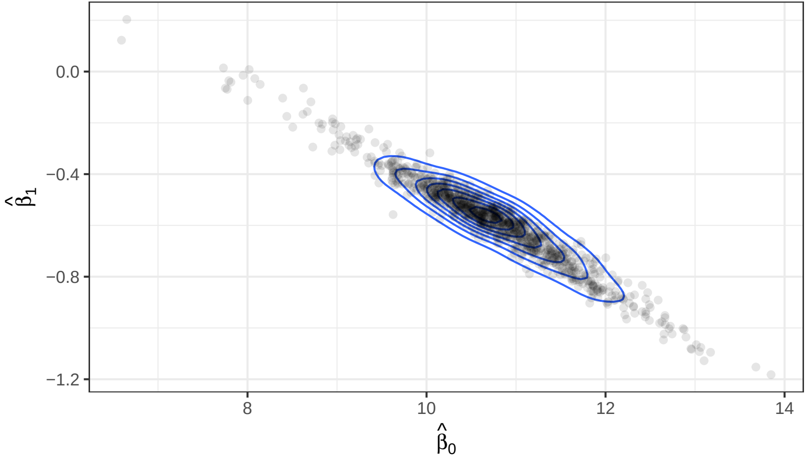
- You can see that the estimated slopes and intercepts vary from sample to sample, but that the "average" of the lines looks about right.



The Sampling Distribution is a Joint Distribution!

While both the intercept and the slope vary, they vary together.



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- We need fill in those ?s.

Our goal

- What is the sampling distribution of the OLS slope?

$$\widehat{\beta}_1 \sim ?(?, ?)$$

- We need fill in those ?s.
- We'll start with the mean of the sampling distribution. Is the estimator centered at the true value, β_1?

Classical Model: OLS Assumptions Preview

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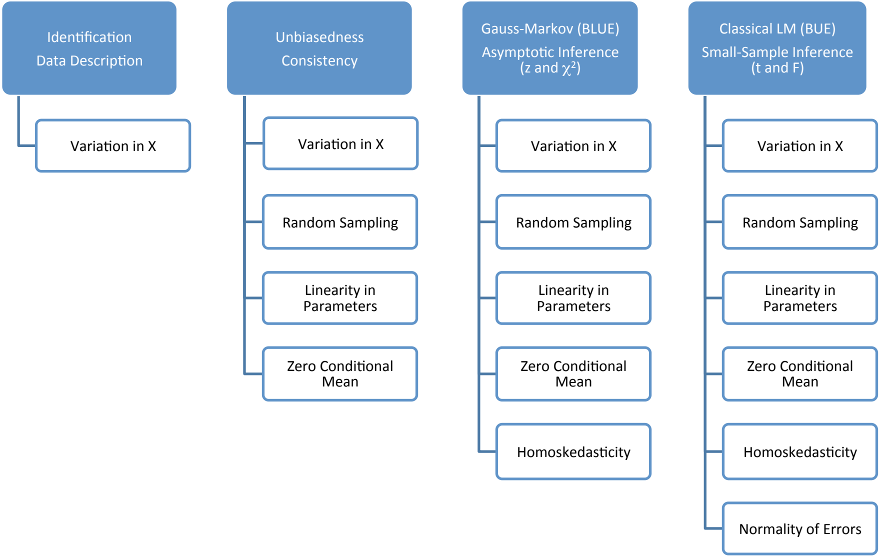

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The population regression model is linear in its parameters and correctly specified as:

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

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- β_0, β_1: Population parameters — fixed and unknown
- u_i: Unobserved random variable with $E[u_i] = 0$ — captures all other factors influencing Y_i other than X_i
- We assume this to be the structural model, i.e., the model describing the true process generating Y_i

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Assumption (II. Random Sampling)

The observed data:

$$(y_i,\ x_i) \text{ for } i = 1, ..., n$$

represent an i.i.d. random sample of size n following the population model.

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- Time series data (regressor values may exhibit persistence)
- Sample selection problems (sample not representative of the population)

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The observed data:

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Only assumption needed for using OLS as a pure data summary.

Stuck in a moment

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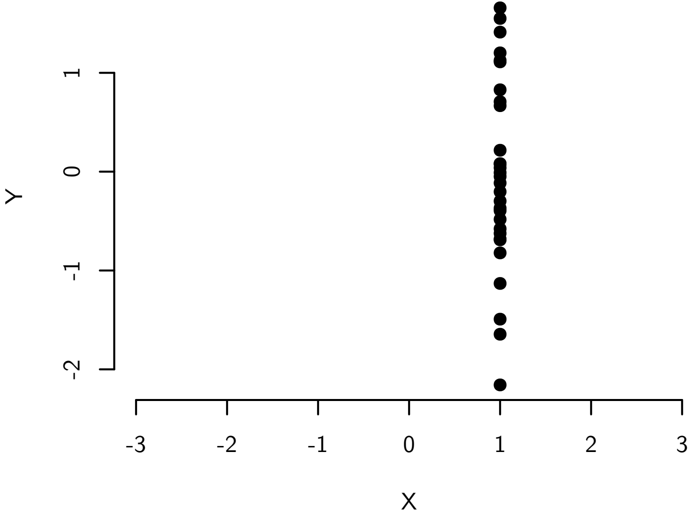
- Why does this matter? How would you draw the line of best fit through this scatterplot, which is a violation of this assumption?



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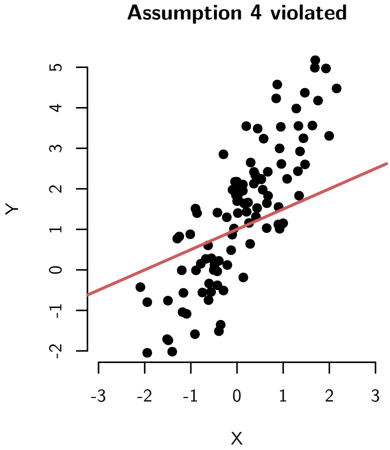

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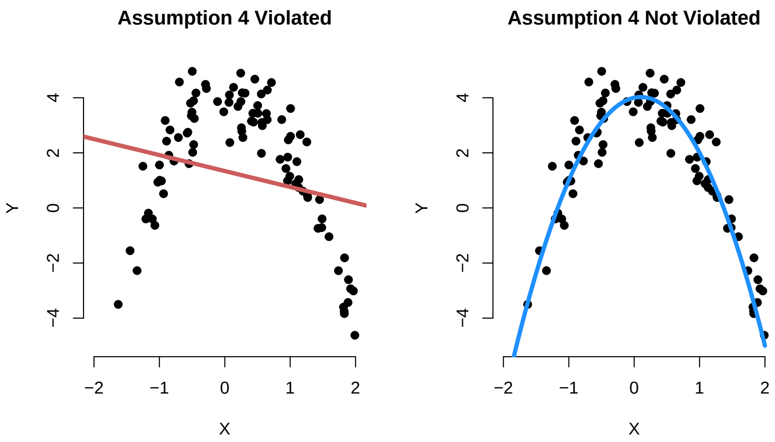

Unbiasedness

With Assumptions 1-4, we can show that the OLS estimator for the slope is unbiased, that is $E[\widehat{\beta}_1] = \beta_1$.

Let's prove it!

Lemma 3: Weighted Combinations of X_i

Lemma ($\sum_i W_i X_i = 1$)

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&= \frac{1}{\sum_{j=1}^{n}(X_j - \overline{X})^2} \left[\sum_{i=1}^{n} X_i(X_i - \overline{X}) - \sum_{i=1}^{n} \overline{X}(X_i - \overline{X}) \right]
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\hat{\beta}_1 - \beta_1 &= \sum_{i=1}^{n} W_i u_i
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Using iterated expectations we can show that it is also unconditionally biased $E[\hat{\beta}_1] = E[E[\hat{\beta}_1|X]] = E\beta_1] = \beta_1$.

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- Under A5 (zero conditional mean error) we have the slightly weaker property $Cov[X_i, u_i] = 0$ so as long as $V[X] > 0$, then we have,

$$\hat{\beta}_1 \xrightarrow{p} \beta_1$$

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- We even proved it to ourselves!

Next Time: The Classical Perspective Part 2: Variance.

Where We've Been and Where We're Going...

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- Last Week
  - hypothesis testing
  - what is regression
- This Week
  - mechanics and properties of simple linear regression
  - inference and measures of model fit
  - confidence intervals for regression
  - goodness of fit
- Next Week
  - mechanics with two regressors
  - omitted variables, multicollinearity
- Long Run
  - probability → inference → regression → causal inference

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- That is we know that the sampling distribution is centered on the true population slope, but we don't know the population variance.

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- This implies $\text{Var}[u] = \sigma_u^2$
  $\rightarrow$ all errors have an identical error variance ($\sigma_{u_i}^2 = \sigma_u^2$ for all i)

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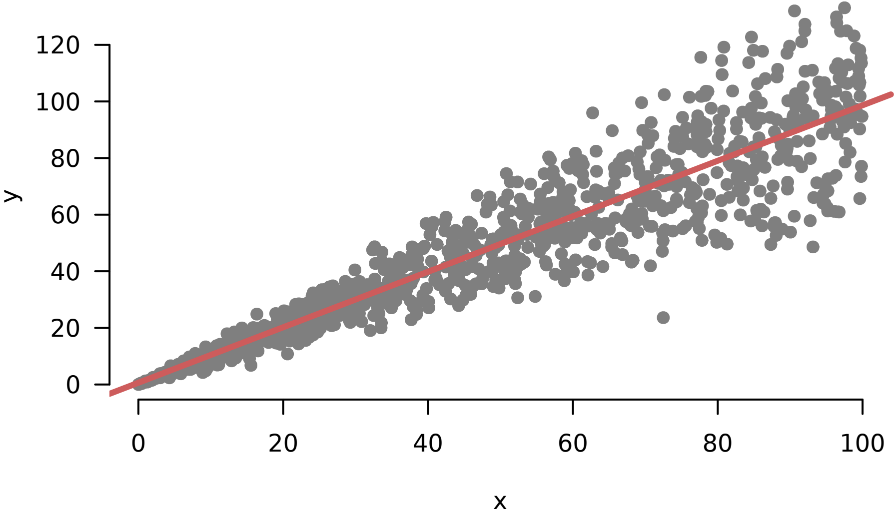
- Assumptions I–V are collectively known as the Gauss-Markov assumptions

Heteroskedasticity



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Variance of OLS Estimators

Theorem (Variance of OLS Estimators)

Given OLS Assumptions I–V (Gauss-Markov Assumptions):

$$V[\hat{\beta}_1 \mid X] = \frac{\sigma_u^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$V[\hat{\beta}_0 \mid X] = \sigma_u^2 \left\{ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \right\}$$

where $V[u \mid X] = \sigma_u^2$ (the error variance).

Understanding the sampling variance

$$V[\widehat{\beta}_1|X_1, \dots, X_n] = \frac{\sigma_u^2}{\sum_{i=1}^n (X_i - \overline{X})^2}$$

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  - As we increase n, the denominator gets large, while the numerator is fixed and so the sampling variance shrinks to 0.

Variance in X Reduces Standard Errors

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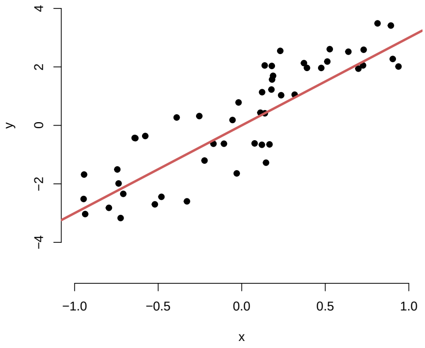

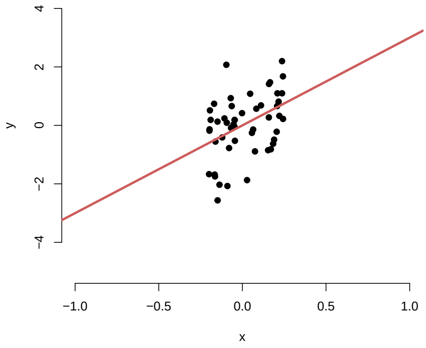

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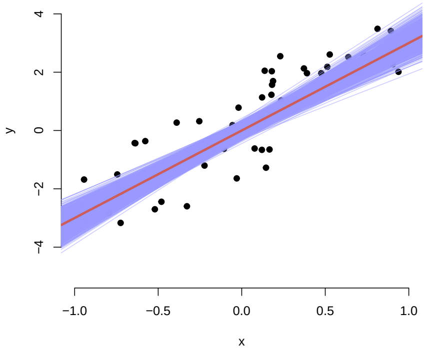

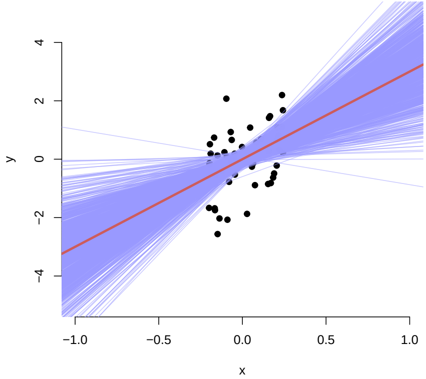

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Intuitively, the scatter of the residuals around the fitted regression line should reflect the unseen scatter about the true population regression line.

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- Thus, an unbiased estimator for the error variance is:

$$\hat{\sigma}_u^2 = \frac{n}{n-2}MSD(\hat{u}) = \frac{n}{n-2}\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2$$

We plug this estimate into the variance estimators for $\hat{\beta}_0$ and $\hat{\beta}_1$.

Where are we?

- Under Assumptions 1-5, we know that

$$\widehat{\beta}_1 \sim ?\left(\beta_1, \frac{\sigma_u^2}{\sum_{i=1}^n (X_i - \overline{X})^2}\right)$$

- Now we know the mean and sampling variance of the sampling distribution.
- How does this compare to other estimators for the population slope?

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- Identification / Data Description
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Given OLS Assumptions I–V, the OLS estimator is BLUE, i.e. the

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- Result fails to hold when the assumptions are violated!

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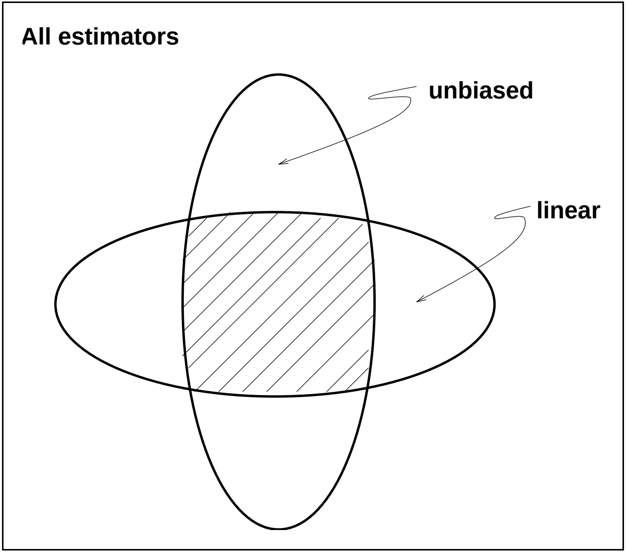

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- All of this depends on Normal errors!

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  - Zero Conditional Mean
  - Homoskedasticity
  - Normality of Errors

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- If the true CEF happens to be linear, the best linear predictor is it.

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- We will return in a few weeks to how you diagnose this approximation.

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- For now, just remember that regression is a linear approximation to the CEF!

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Next Time: Inference

Where We've Been and Where We're Going...

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- Last Week
  - hypothesis testing
  - what is regression
- This Week
  - mechanics and properties of simple linear regression
  - inference and measures of model fit
  - confidence intervals for regression
  - goodness of fit
- Next Week
  - mechanics with two regressors
  - omitted variables, multicollinearity
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

Where are we?

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- Under Assumptions 1-5 and in large samples, we know that

$$\widehat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma_u^2}{\sum_{i=1}^n (X_i - \overline{X})^2}\right)$$

- Under Assumptions 1-6 and in any sample, we know that

$$\frac{\widehat{\beta}_1 - \beta_1}{\widehat{SE}[\widehat{\beta}_1]} \sim t_{n-2}$$

Null and alternative hypotheses review

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- Notice these are statements about the population parameters, not the OLS estimates.

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- Thus, under the null, we know the distribution of T and can use that to formulate a rejection region and calculate p-values.
- By default, R shows you the test statistic for $\beta_1 = 0$ and uses the t distribution.

Rejection region

- Choose a level of the test, α, and find rejection regions that correspond to that value under the null distribution:

$$\mathbb{P}(-t_{\alpha/2,n-2} < T < t_{\alpha/2,n-2}) = 1 - \alpha$$

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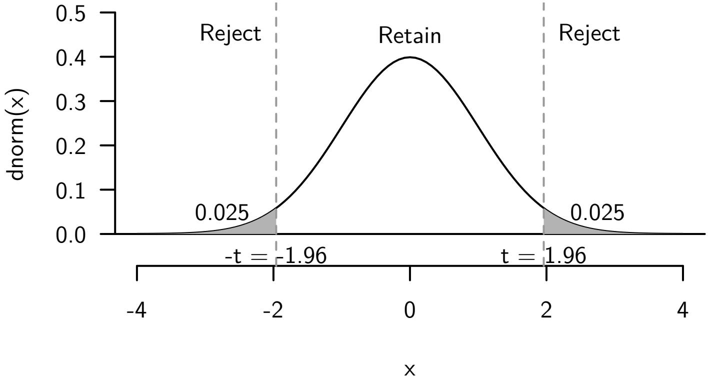
- This is exactly the same as with sample means and sample differences in means, except that the degrees of freedom on the t distribution have changed.



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- If the p-value is less than α we would reject the null at the α level.

Confidence intervals

- Very similar to the approach with sample means. By the sampling distribution of the OLS estimator, we know that we can find t-values such that:

$$\mathbb{P}\left(- t_{\alpha/2,n-2} \leq \frac{\widehat{\beta}_1 - \beta_1}{\widehat{SE}[\widehat{\beta}_1]} \leq t_{\alpha/2,n-2} \right) = 1 - \alpha$$

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- We can derive these for the intercept as well:

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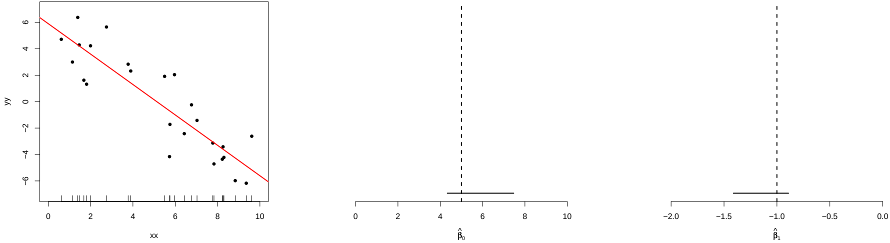
Returning to our simulation example we can simulate the sampling distributions of the 95 % confidence interval estimates for $\widehat{\beta}_1$ and $\widehat{\beta}_0$



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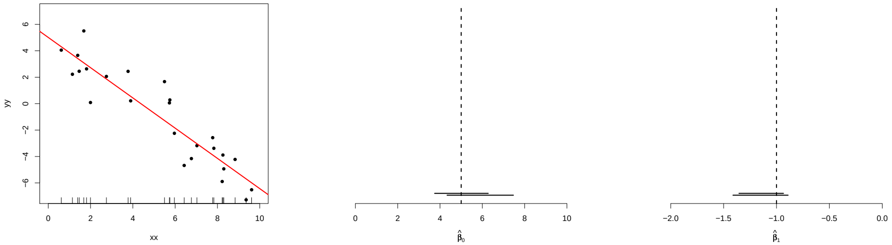

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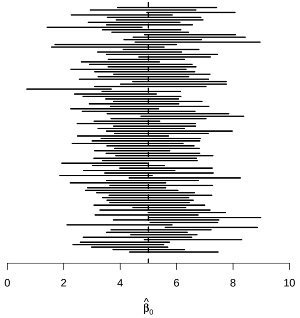

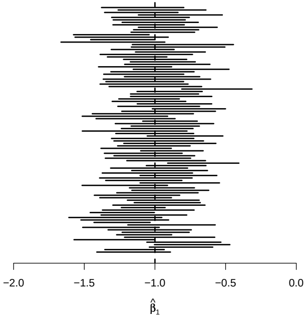

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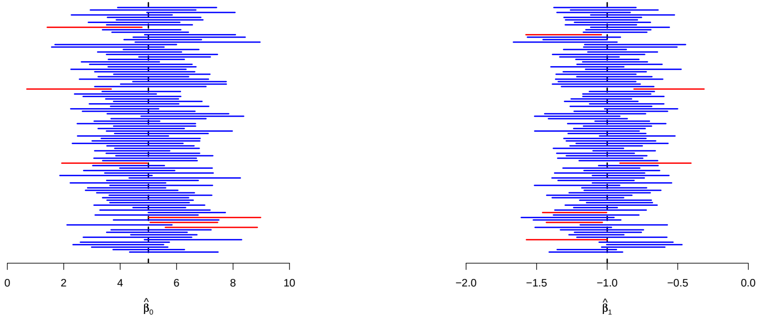

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- Once we have estimated our model, we have new prediction errors, which are just the sum of the squared residuals or SS_{res}:

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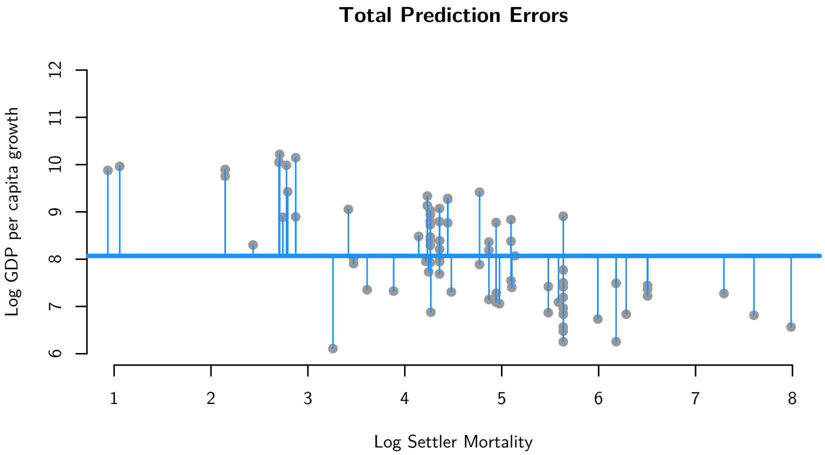
Sum of Squares



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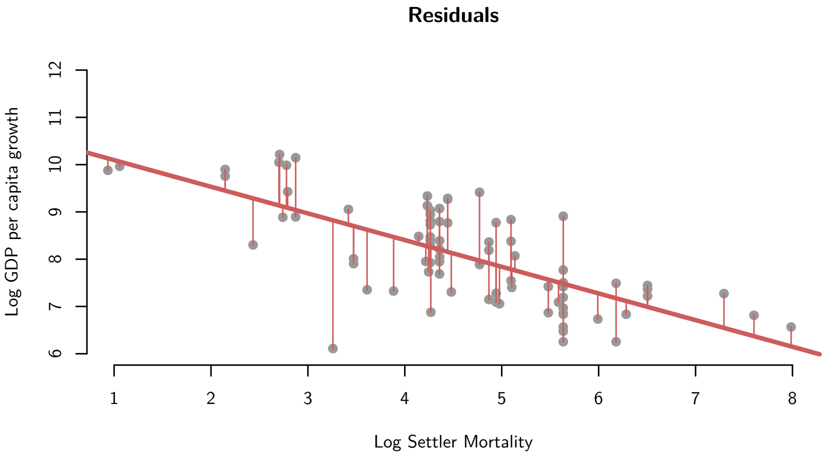

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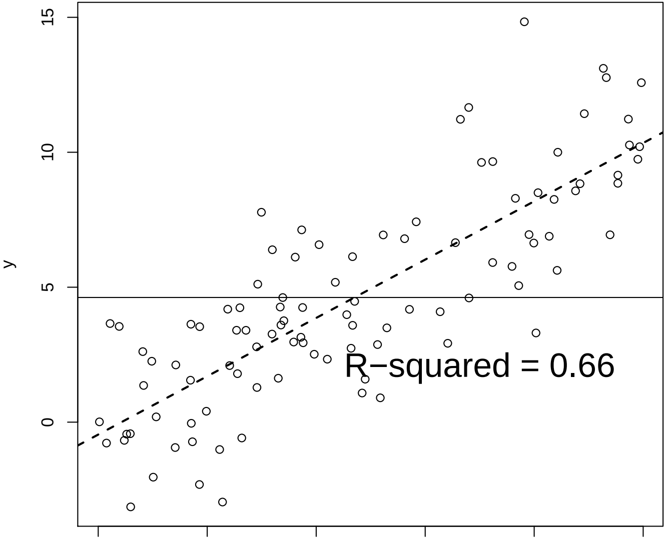
Is R-squared useful?



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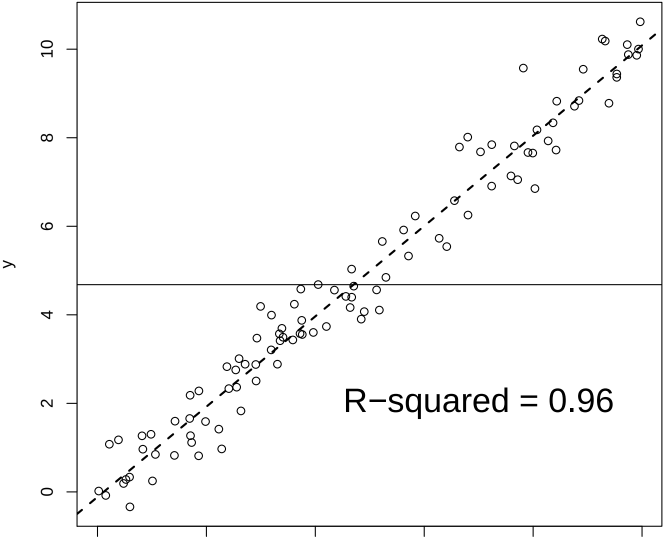

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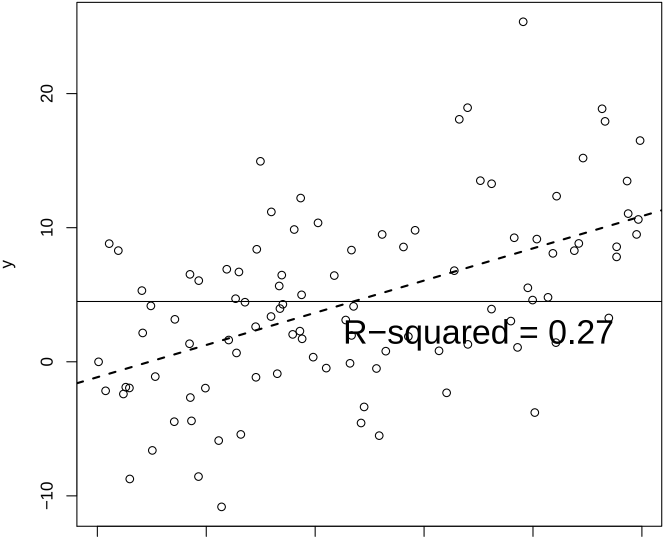

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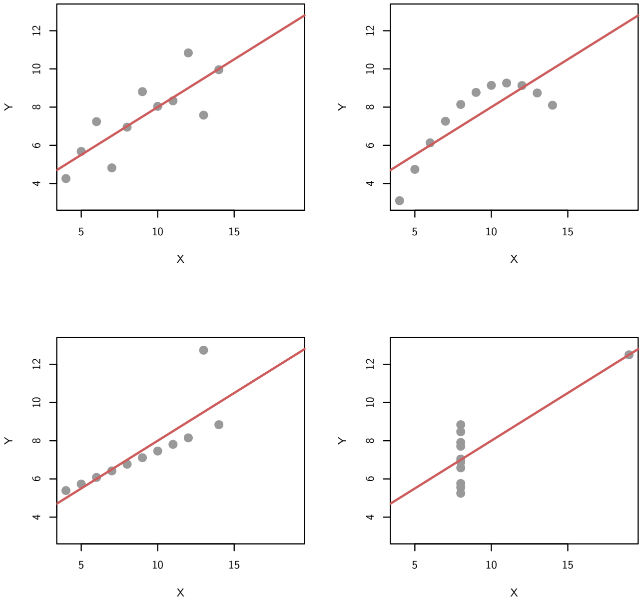
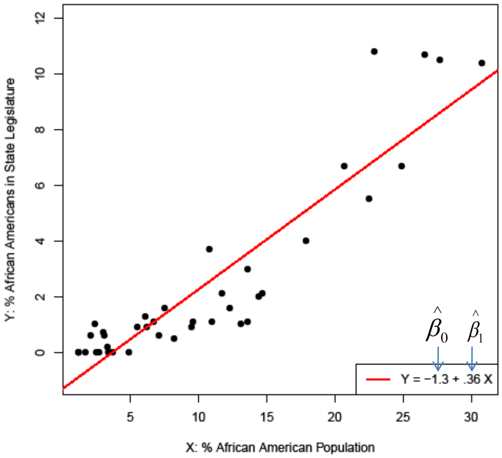


Interpreting a Regression

Let's have a quick chat about interpretation.



State Legislators and African American Population

Interpretations of increasing quality:

```
> summary(lm(beo ~ bpop, data = D))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.31489    0.32775  -4.012 0.000264 ***
bpop         0.35848    0.02519  14.232  < 2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.317 on 39 degrees of freedom
Multiple R-squared: 0.8385,Adjusted R-squared: 0.8344
F-statistic: 202.6 on 1 and 39 DF,  p-value: < 2.2e-16
```

"African American population is statistically significant ($p < 0.001$)"

(no effect size or direction)

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"Percent African American legislators increases with African American population ($p < 0.001$)"

(direction, but no effect size)

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"A one percentage point increase in the African American population causes a 0.35 percentage point increase in the fraction of African American state legislators ($p < 0.001$)."

(unwarranted causal language)

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"A one percentage point increase in the African American population is associated with a 0.35 percentage point increase in the fraction of African American state legislators ($p < 0.001$)."

(hints at causality)

State Legislators and African American Population

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"In states where an additional .01 proportion of the population is African American, we observe on average .035 proportion more African American state legislators ($p < 0.001$)."

(p value doesn't help people with uncertainty)

State Legislators and African American Population

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```

"In states where an additional .01 proportion of the population is African American, we observe on average .035 proportion more African American state legislators (between .03 and .04 with 95% confidence)."

(still not perfect, the best will be subject matter specific. is fairly clear it is non-causal, gives uncertainty.)

Ground Rules: Interpretation of the Slope

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1. Give a short, but precise interpretation of the association using interpretable language and units
2. If the association has a causal interpretation explain why, otherwise do not imply a causal interpretation.
3. Provide a meaningful sense of uncertainty
4. Indicate the practical significance of the finding for your argument.

Goal Check: Understand `lm()` Output

```
Call:
lm(formula = sr ~ pop15, data = LifeCycleSavings)

Residuals:
   Min     1Q Median     3Q    Max
-8.637 -2.374  0.349  2.022 11.155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.49660    2.27972   7.675 6.85e-10 ***
pop15       -0.22302    0.06291  -3.545 0.000887 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.03 on 48 degrees of freedom
Multiple R-squared:  0.2075,Adjusted R-squared:  0.191
F-statistic: 12.57 on 1 and 48 DF,  p-value: 0.0008866
```

We Covered

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- Hypothesis tests

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- Confidence intervals

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Next Time: Non-linearities

Where We've Been and Where We're Going...

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- Last Week
  - hypothesis testing
  - what is regression
- This Week
  - mechanics and properties of simple linear regression
  - inference and measures of model fit
  - confidence intervals for regression
  - goodness of fit
- Next Week
  - mechanics with two regressors
  - omitted variables, multicollinearity
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

1. Mechanics of OLS
2. Classical Perspective (Part 1, Unbiasedness)
   - Sampling Distributions
   - Classical Assumptions 1–4
3. Classical Perspective: Variance
   - Sampling Variance
   - Gauss-Markov
   - Large Samples
   - Small Samples
   - Agnostic Perspective
4. Inference
   - Hypothesis Tests
   - Confidence Intervals
   - Goodness of fit
   - Interpretation
5. Non-linearities
   - Log Transformations
   - Fun With Logs
   - LOESS

Non-linear CEFs

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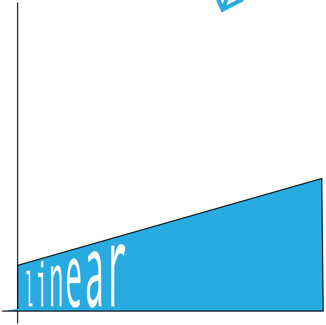
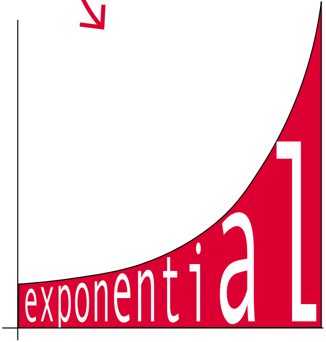
- Many of these non-linear transformations are made by creating multiple variables out of a single X and so will have to wait for future weeks.
- The function $\log(\cdot)$ is one common transformation that has only one parameter.
- This is particularly useful for positive and right-skewed variables.

Why does everyone keep logging stuff??

Logs linearize exponential growth.



grows by a fixed amount.



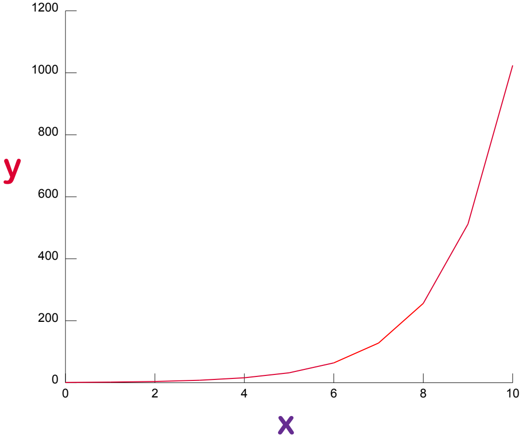
grows by a fixed percent.

How? Let's look.

First, here's a graph showing exponential growth.

We're going to use $y = 2^x$, but any other exponent will work

x	y=(2^x)
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024



What happens when we take the log of y?

$\log y = z \qquad e^z = y$

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x	y	z
0	1	
1	2	
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3	8	
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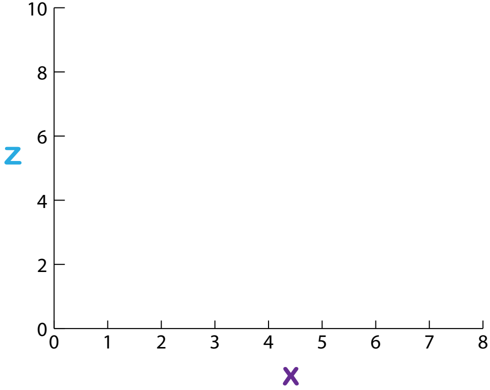
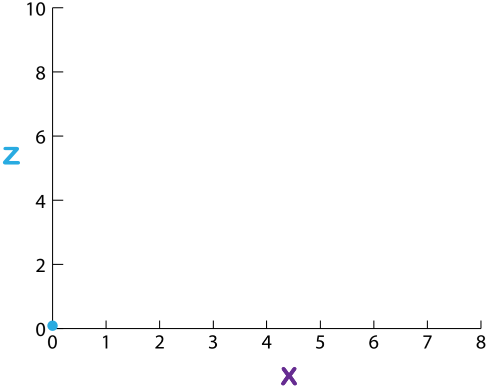


What happens when we take the log of y?

$\log 1 = 0 \qquad e^{0} = 1$

We're going to use $y = 2^{x}$, but any other exponent will work

x	y	z
0	1	0
1	2	
2	4	
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
10	1024	

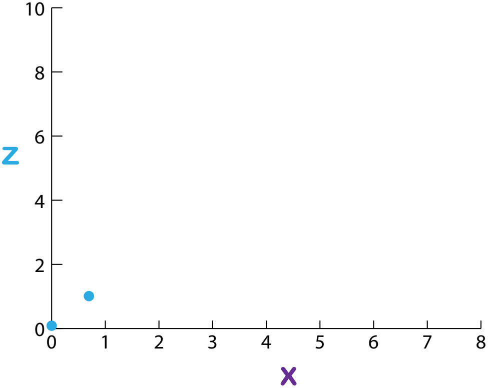


What happens when we take the log of y?

$$\log 2 = .69 \qquad e^{.69} = 2$$

We're going to use $y = 2^x$, but any other exponent will work

x	y	z
0	1	0
1	2	.69
2	4	
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
10	1024	

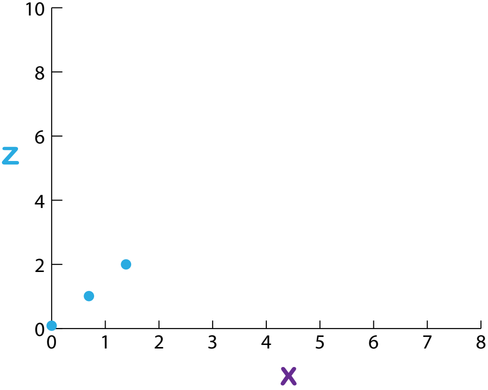


What happens when we take the log of y?

$$\log 4 = 1.39 \qquad e^{1.39} = 4$$

We're going to use $y = 2^x$, but any other exponent will work

x	y	z
0	1	0
1	2	.69
2	4	1.39
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
10	1024	

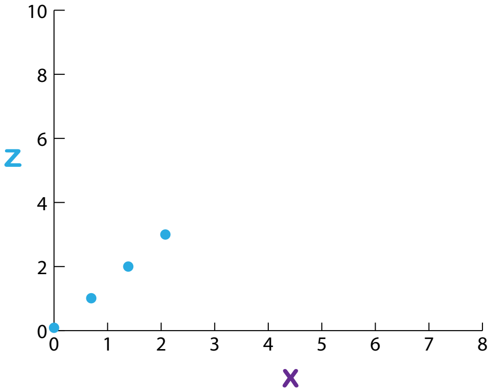


What happens when we take the log of y?

$$\log 8 = 2.08 \qquad e^{2.08} = 8$$

We're going to use $y = 2^x$, but any other exponent will work

x	y	z
0	1	0
1	2	.69
2	4	1.39
3	8	2.08
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
10	1024	



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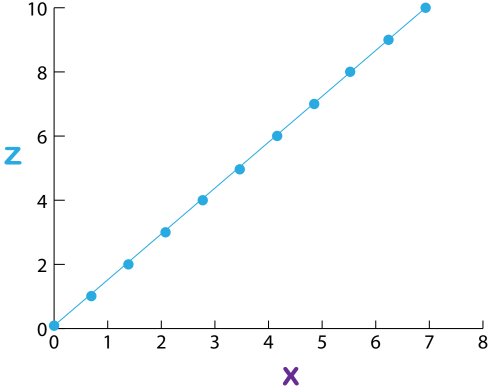
0	1	0
1	2	.69
2	4	1.39
3	8	2.08
4	16	2.77
5	32	3.47
6	64	4.16
7	128	4.85
8	256	5.55
9	512	6.24
10	1024	6.93



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- Note that these approximations work only for small increments.

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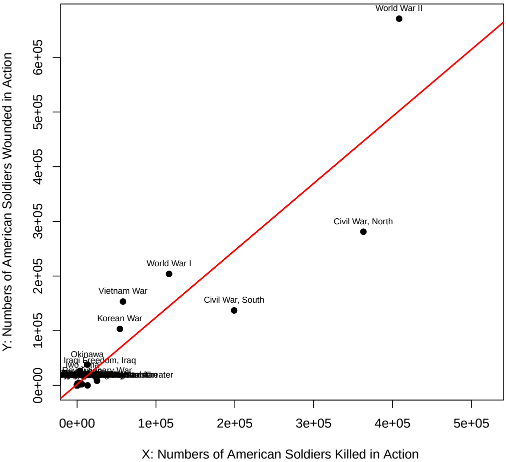
- Regress Y on $\log(X) \longrightarrow \beta_1$ approximates increase in Y associated with a percent increase in X.
- Note that these approximations work only for small increments.
- In particular, they do not work when X is a discrete random variable.

Example from the American War Library



$\hat{\beta}_1 = 1.23 \longrightarrow$

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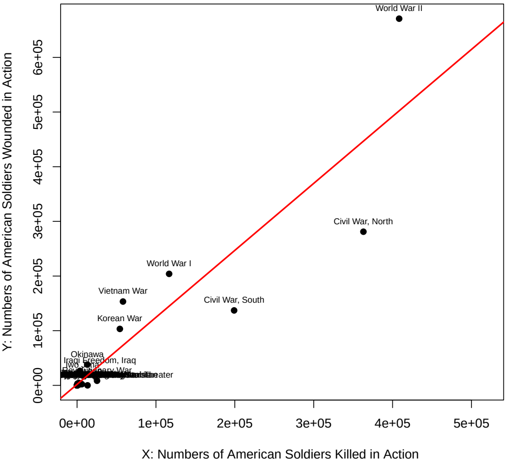
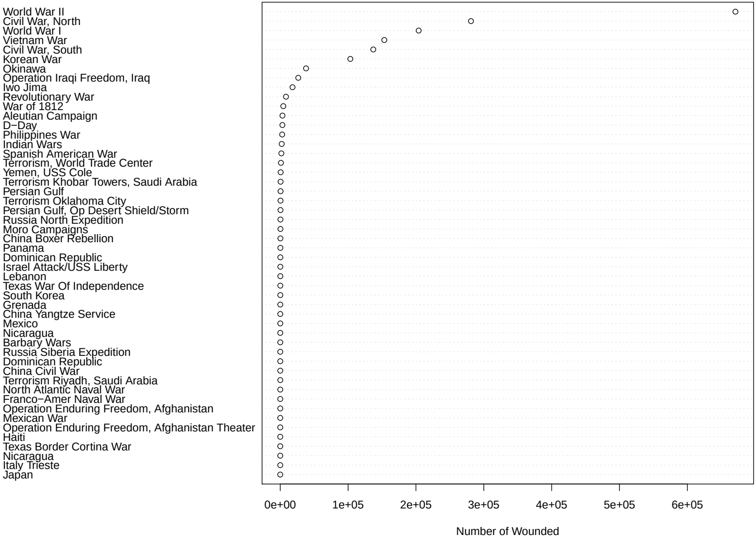
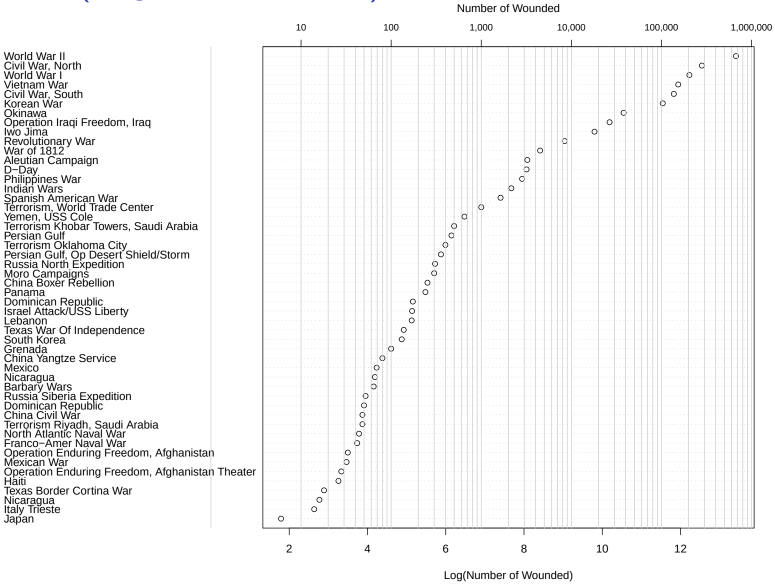


$\hat{\beta}_1 = 1.23 \longrightarrow$ One additional soldier killed predicts 1.23 additional soldiers wounded

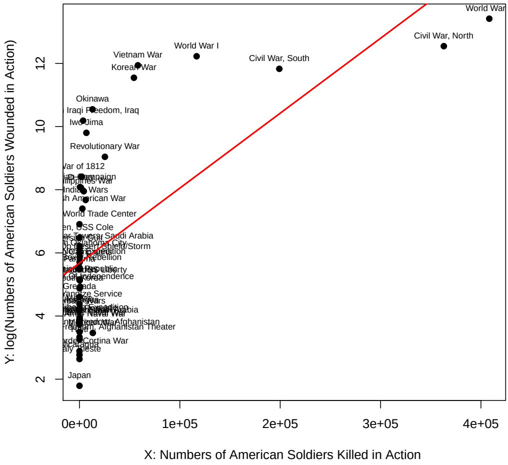
Wounded (Scale in Levels)



Wounded (Logarithmic Scale)



Regression: Log-Level



$\hat{\beta}_1 = 0.0000237 \longrightarrow$

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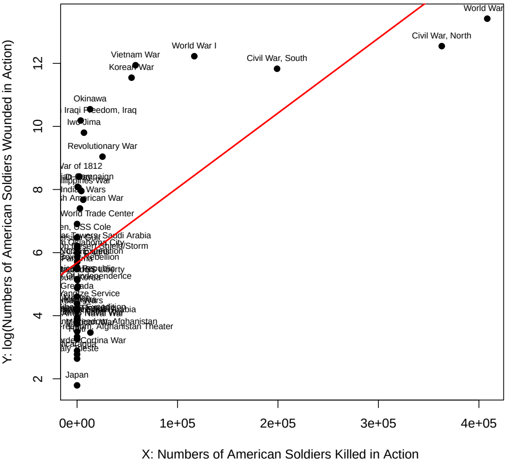
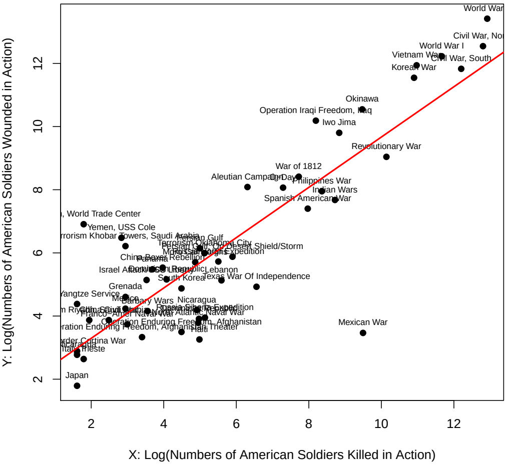


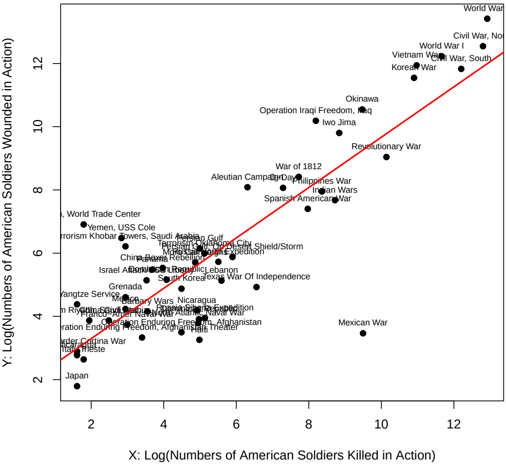
$\hat{\beta}_1 = 0.0000237 \longrightarrow$ One additional soldier killed predicts 0.0023 percent increase in the number of soldiers wounded

Regression: Log-Log



$\hat{\beta}_1 = 0.797 \longrightarrow$

Regression: Log-Log



$\hat{\beta}_1 = 0.797 \longrightarrow$ A percent increase in deaths predicts 0.797 percent increase in the wounded

Four Most Commonly Used Models

Model	Equation	β_1 Interpretation
Level-Level	$Y = \beta_0 + \beta_1 X$	$\Delta Y = \beta_1 \Delta X$
Log-Level	$log(Y) = \beta_0 + \beta_1 X$	$\%\Delta Y = 100\beta_1 \Delta X$
Level-Log	$Y = \beta_0 + \beta_1 log(X)$	$\Delta Y = (\beta_1/100)\%\Delta X$
Log-Log	$log(Y) = \beta_0 + \beta_1 log(X)$	$\%\Delta Y = \beta_1 \%\Delta X$

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$$\exp(x) \approx 1+x$$

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$$\begin{aligned} \log(1+x) &\approx x \\ \exp(x) &\approx 1+x \end{aligned}$$

This can be derived from a series expansion of the log function. Numerically, when $|x| \leq .1$, the approximation is within 0.001.

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Take two numbers $a > b > 0$. The percentage difference between a and b is

$$p = 100\left(\frac{a-b}{b}\right)$$

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Applying our approximation and multiplying by 100 we find,

$$p \approx 100\left(\log(a) - \log(b)\right)$$

Be Careful: Log-Level with binary X

Assume we have: $log(Y) = \beta_0 + \beta_1 X$ where X is binary with values 1 or 0. Assume $\beta_1 > .2$. What is the problem with saying that a one unit increase in X is associated with a $\beta_1 \cdot 100$ percent change in Y?

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$$100(Y_{X=1} - Y_{X=0})/Y_{X=0} \quad = \quad 100((Y_{X=1}/Y_{X=0}) - 1)$$

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$$
\begin{aligned}
100(Y_{X=1} - Y_{X=0})/Y_{X=0} &= 100((Y_{X=1}/Y_{X=0}) - 1) \\
&= 100((Y_{X=1}/Y_{X=0}) - 1) \\
&= 100(exp(\beta_1) - 1)
\end{aligned}
$$

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$$\begin{aligned} 100(Y_{X=1} - Y_{X=0})/Y_{X=0} &= 100((Y_{X=1}/Y_{X=0}) - 1) \\ &= 100((Y_{X=1}/Y_{X=0}) - 1) \\ &= 100(exp(\beta_1) - 1) \end{aligned}$$

Recall: $log(Y_{X=1}) - log(Y_{X=0}) = log(Y_{X=1}/Y_{X=0}) = \beta_1$.

Be Careful: Log-Level with binary X

Assume we have: $log(Y) = \beta_0 + \beta_1 X$ where X is binary with values 1 or 0. Assume $\beta_1 > .2$. What is the problem with saying that a one unit increase in X is associated with a $\beta_1 \cdot 100$ percent change in Y?

Log approximation is inaccurate for large changes like going from $X = 0$ to $X = 1$. Instead the percent change in Y when X goes from 0 to 1 needs to be computed using:

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- What are we characterizing? The geometric mean.

Geometric Mean

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\begin{aligned}
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The geometric mean is a robust measure of central tendency.

Application

THE INTERGENERATIONAL ELASTICITY OF WHAT? THE CASE FOR REDEFINING THE WORKHORSE MEASURE OF ECONOMIC MOBILITY

*Pablo A. Mitnik**
*David B. Grusky**

Abstract

The intergenerational elasticity (IGE) has been assumed to refer to the expectation of children's income when in fact it pertains to the geometric mean of children's income. We show that mobility analyses based on the conventional IGE have been widely misinterpreted, are subject to selection bias, and cannot disentangle the different channels for transmitting economic status across generations. The solution to these problems—estimating the IGE of expected income or earnings—returns the field to what it has long meant to estimate. Under this approach, intergenerational persistence is found to be substantially higher, thus raising the possibility that the field's stock results are misleading.

Keywords

intergenerational economic mobility, elasticity of expected income, selection bias, gender, marriage and economic mobility

Core Idea

$$\text{Classic approach :} \quad \underbrace{E\left(\log\left(Y\right) \mid X\right)}_{\substack{\text{Mean of log} \\ \text{offspring income } Y \\ \text{given parent income } X}} = \beta_0 + \underbrace{\beta_1}_{\substack{\text{Intergenerational} \\ \text{elasticity} \\ \text{(IGE)}}} \underbrace{\log\left(X\right)}_{\substack{\text{Log parent} \\ \text{income}}}$$

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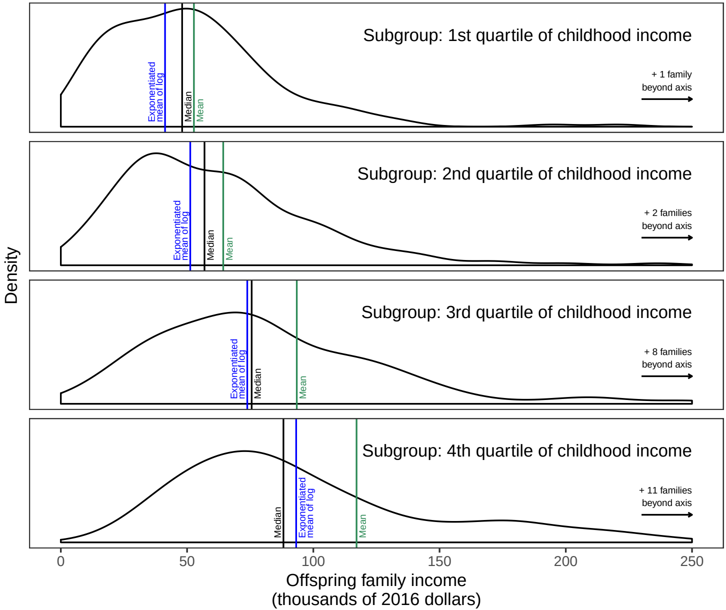
Geometric Mean is Closer to the Median Than the Mean



Our Response

COMMENT: SUMMARIZING INCOME MOBILITY WITH MULTIPLE SMOOTH QUANTILES INSTEAD OF PARAMETERIZED MEANS

*Ian Lundberg**
*Brandon M. Stewart**

*Department of Sociology and Office of Population Research, Princeton University, Princeton, NJ, USA
Corresponding Author: Ian Lundberg, ilundberg@princeton.edu
DOI: 10.1177/0081175020931126

Single-number summaries that capture the relationship of socioeconomic outcomes across generations are a cornerstone of economic mobility research. Studies often focus on the intergenerational elasticity (IGE) of income: the coefficient β_1 on parent log income in a model predicting offspring log income (e.g., Aaronson and Mazumder 2008; Björklund and Jäntti 1997; Solon 2004). A large β_1 is often interpreted as evidence that incomes persist to a substantial degree across generations.

Images from this section are from this paper or earlier drafts of it.

Two Implicit Choices

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(1) Summary Statistics for the Conditional Distribution

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(gets you down to one number per value of x)

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and

(2) Assume or Learn a Functional Form

(potentially simplifies the set of summary statistics to a single number)

Visualizing the MG Proposal

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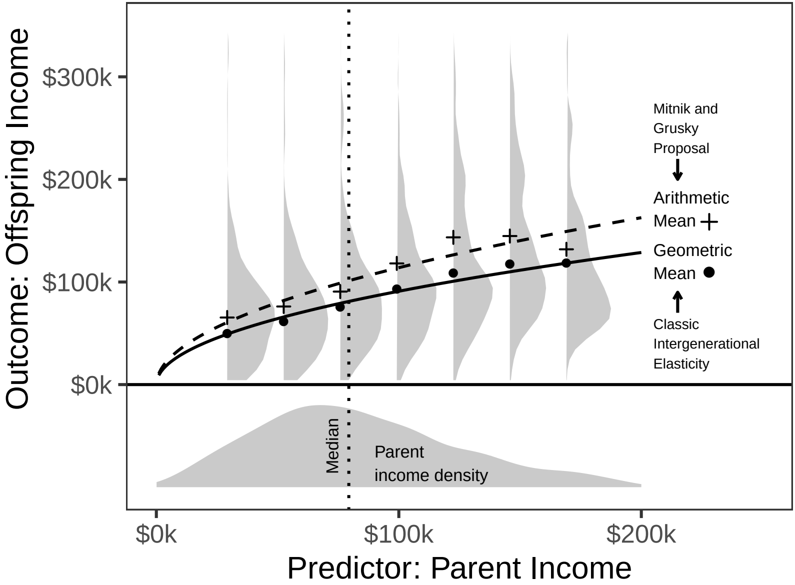

Single Summary Statistics Necessarily Mask Information

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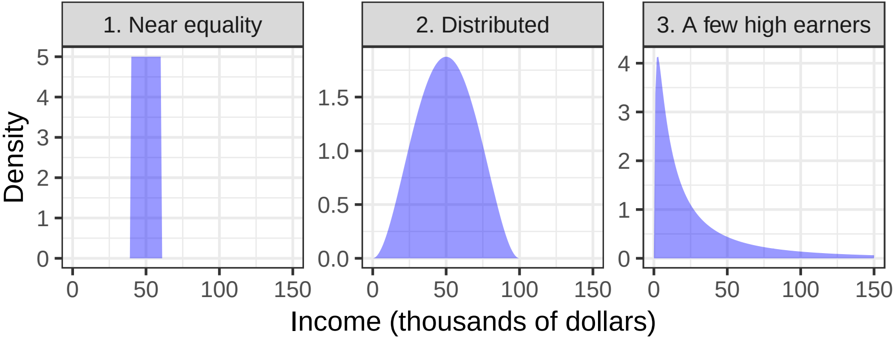

The Mean is a Normative Choice

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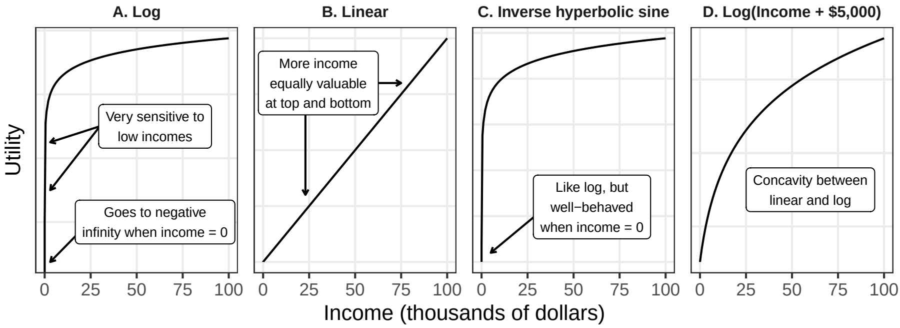

A New Proposal

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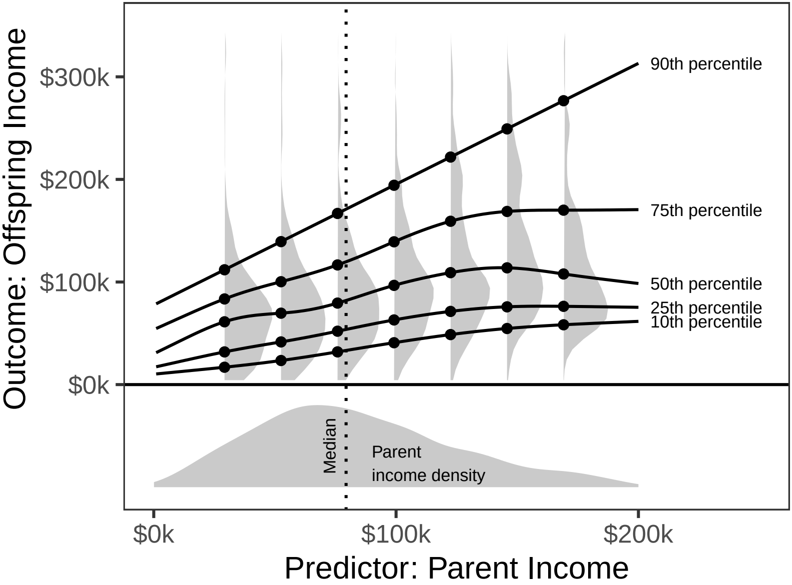

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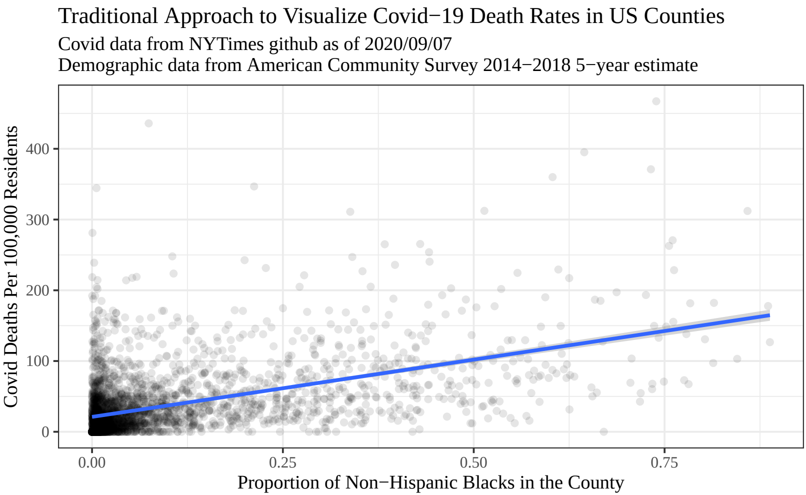
- If you are willing to commit to a quantity of interest, you can usually estimate it directly.
- At their best, single-number summaries are a way that the reader can calculate any approximation to a variety of quantities they are interested in. At their worst, they are a way for authors to abdicate responsibility for choosing a clear quantity of interest.

Broader Implications (Lee, Lundberg and Stewart)



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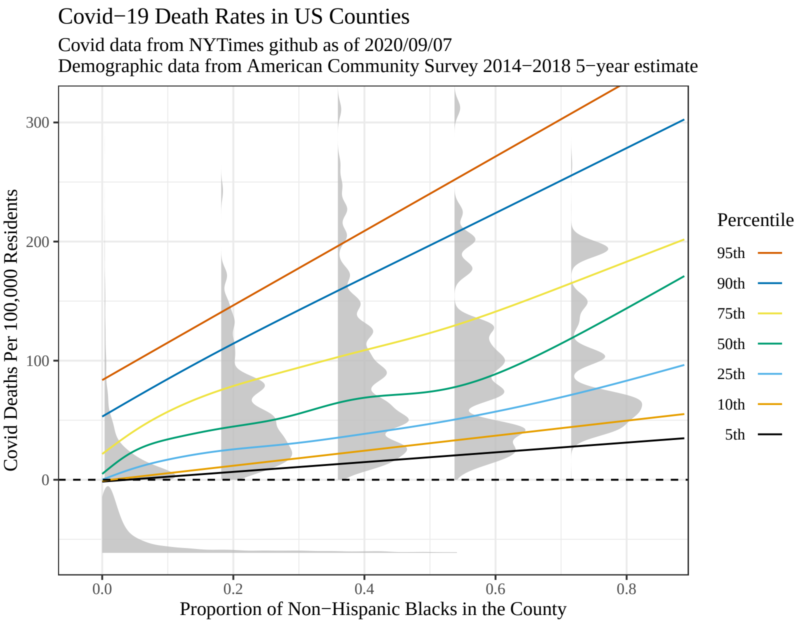
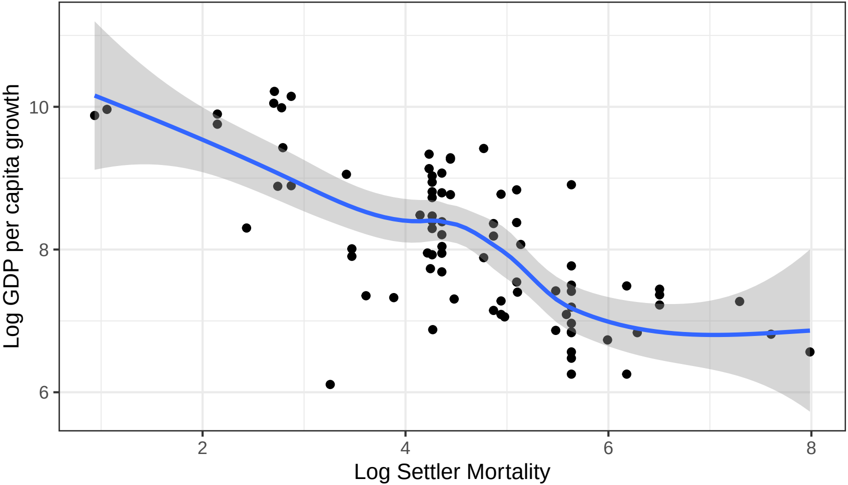


1. Mechanics of OLS
2. Classical Perspective (Part 1, Unbiasedness)
   - Sampling Distributions
   - Classical Assumptions 1–4
3. Classical Perspective: Variance
   - Sampling Variance
   - Gauss-Markov
   - Large Samples
   - Small Samples
   - Agnostic Perspective
4. Inference
   - Hypothesis Tests
   - Confidence Intervals
   - Goodness of fit
   - Interpretation
5. Non-linearities
   - Log Transformations
   - Fun With Logs
   - LOESS

So what is `ggplot2` doing?



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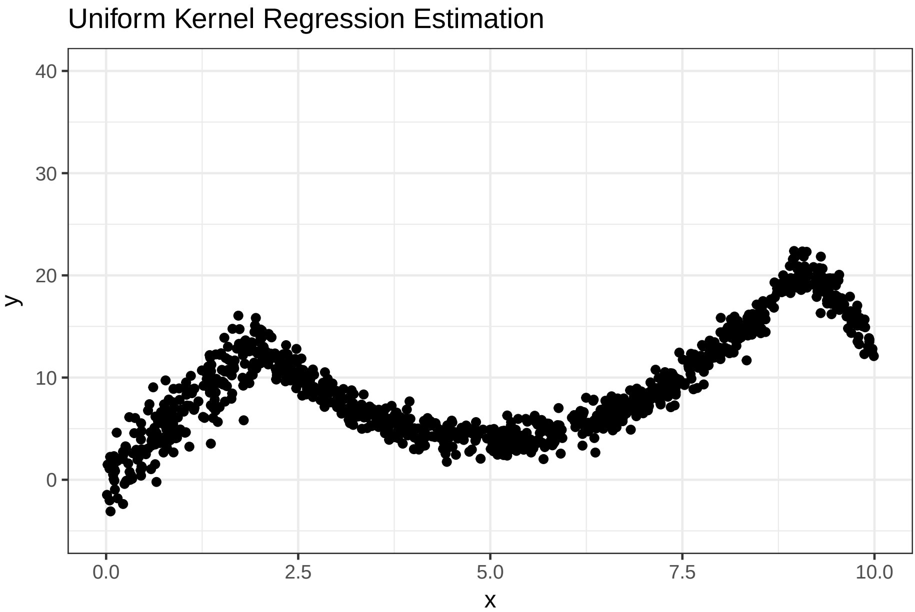
LOESS Example



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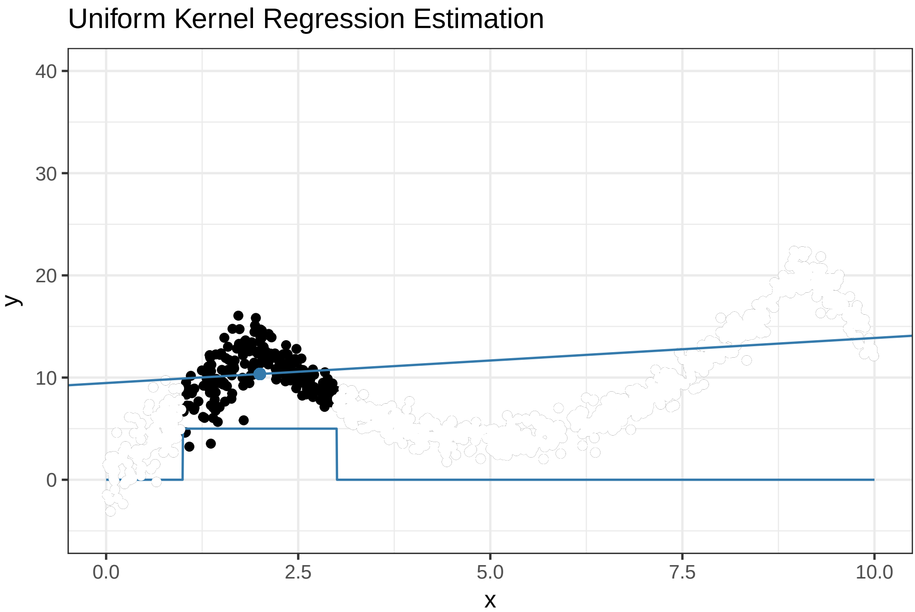

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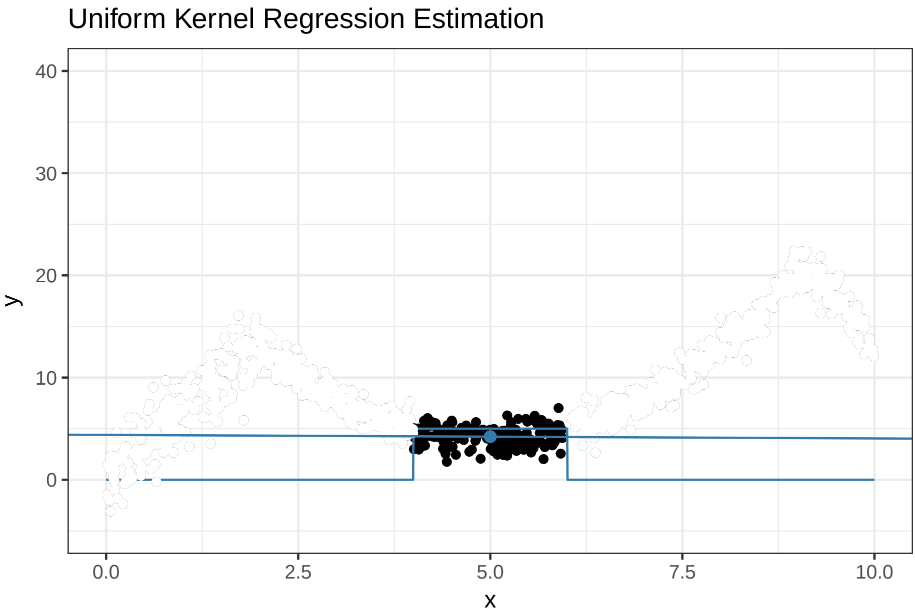

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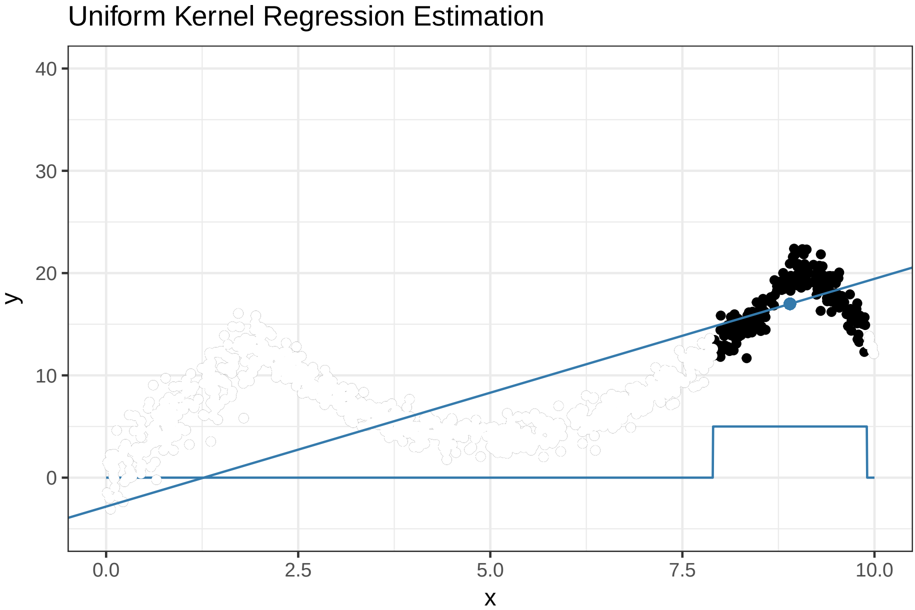

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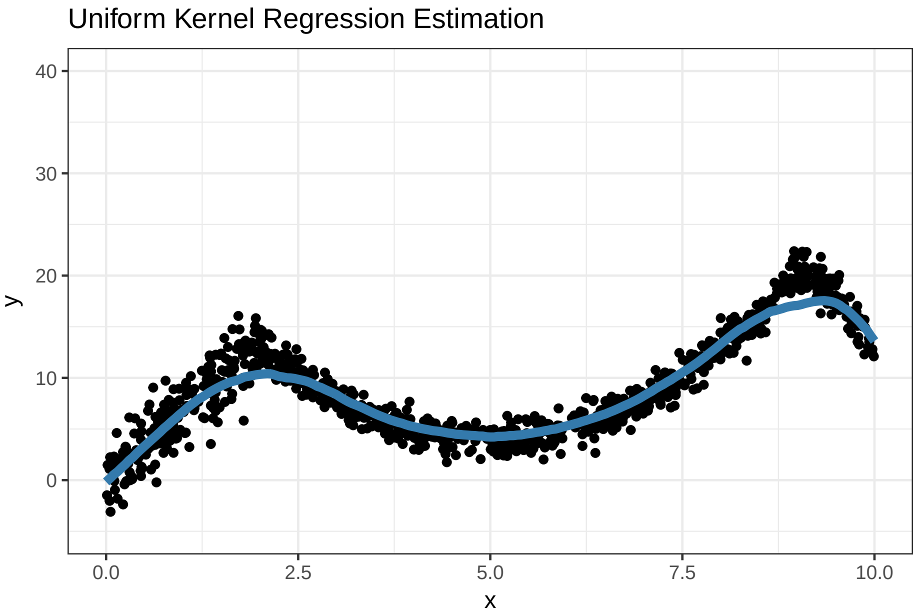

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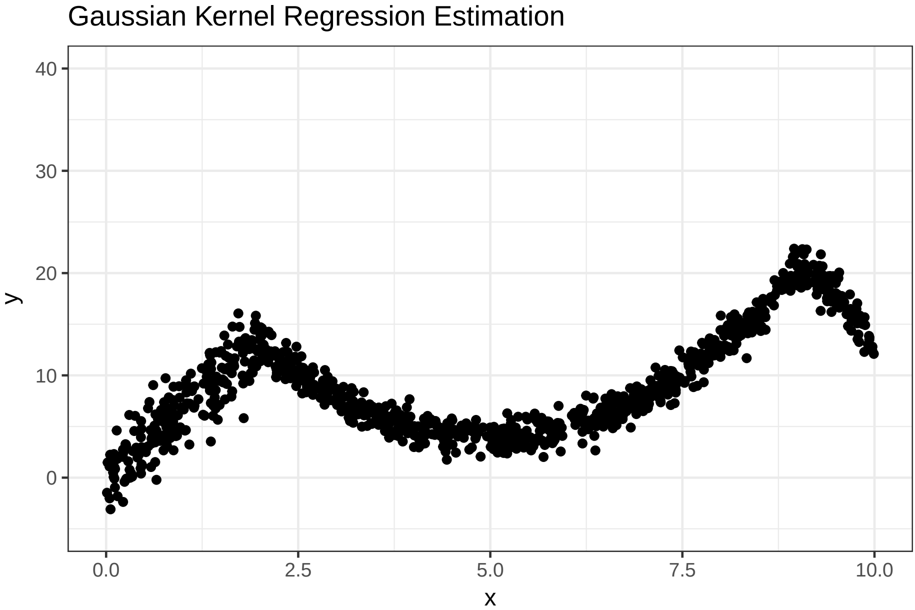

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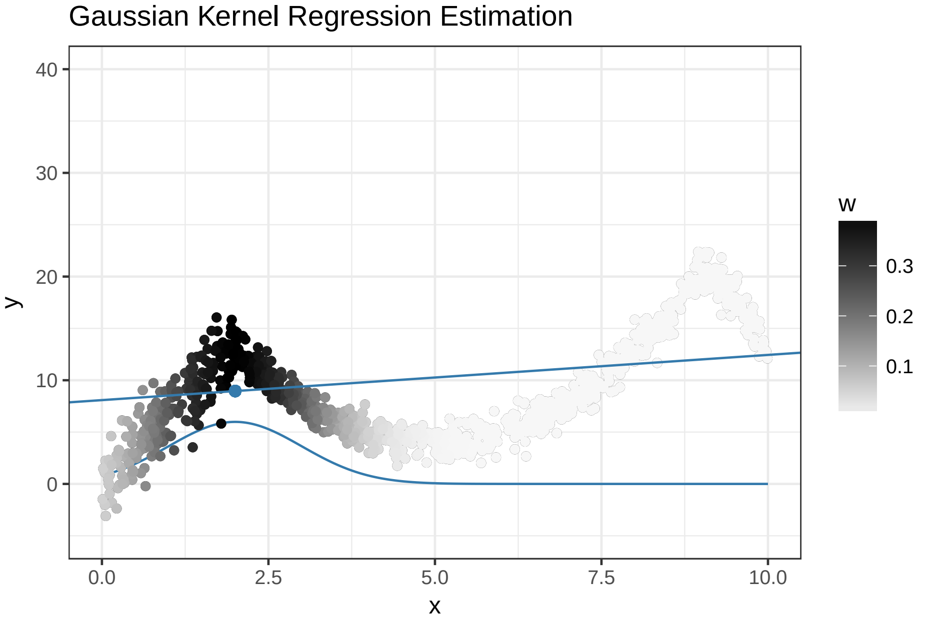

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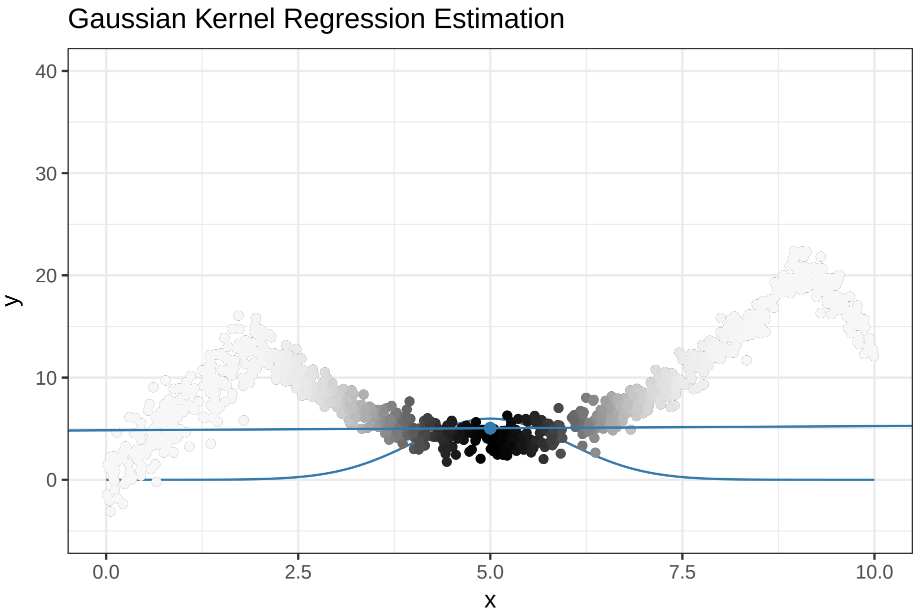

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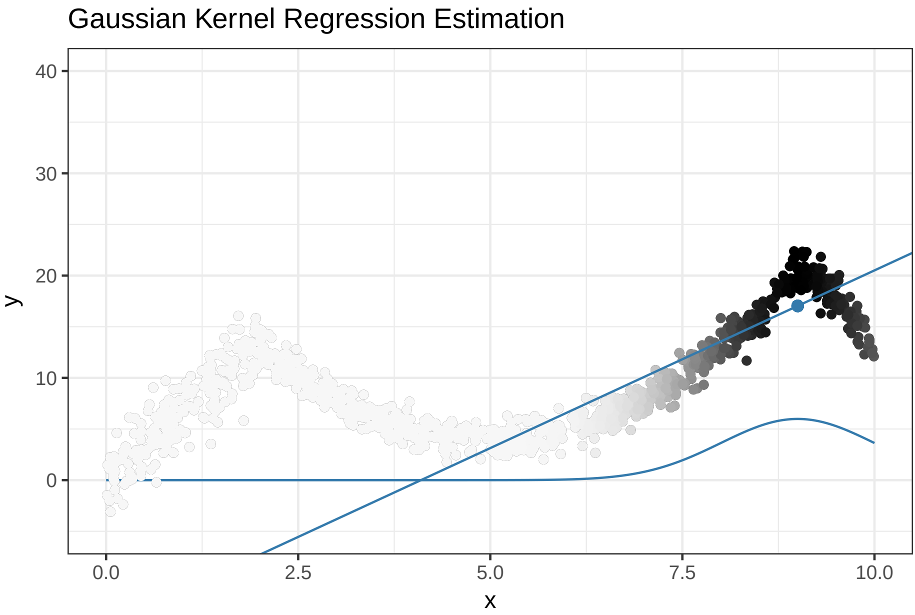

We Covered

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- Interpretation with logged independent and dependent variables
- The geometric mean!

This Week in Review

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- Classical regression assumptions!
- Inference!
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Next week: Linear Regression with Two Variables!