

Week 2: Random Variables

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Princeton

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¹These slides are heavily influenced by Adam Glynn, Justin Grimmer and Jens Hainmueller. Many illustrations by Shay O'Brien.

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 - ▶ welcome and outline of course
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Questions?

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- Notation guide

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- Using the slides (links, what's contained in a single deck etc.)

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- Any logistical hiccups?

1 Random Variables and Distributions

- What is a Random Variable?
- Discrete Distributions
- Continuous Distributions

2 Characteristics of Distributions

- Central Tendency
- Measures of Dispersion

3 Conditional Distributions

4 Fun with Averages

5 Fun with Sensitive Questions

6 Appendix: Why the Mean?

7 Joint Distributions

- Discrete Random Variable
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We will do this by introducing a random variable X to be Barack Obama's position on the 2008 New Hampshire primary ballot.

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- To do this we need to understand **random variables**

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- $X(\{heads, heads\}) = 2$
- $X(\{heads, tails\}) = 1$
- $X(\{tails, heads\}) = 1$
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- Sometimes the sample space is already numeric so its more obvious (e.g. how long until the train arrives)

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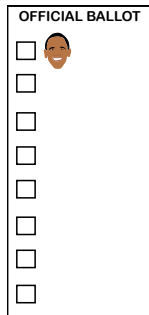
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yep. seriously. let's do an example!

NH Ballot Order Example

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$X = \left\{ \begin{array}{l} 1 \end{array} \right.$$



A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

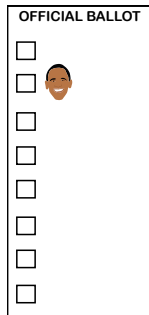
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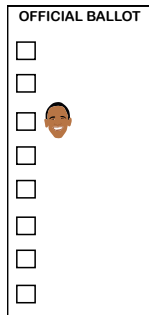
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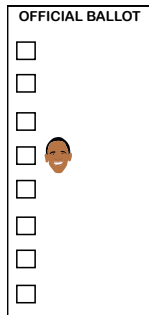
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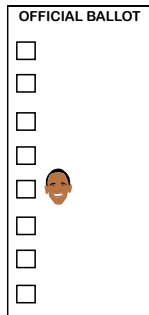
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$$X = \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right.$$



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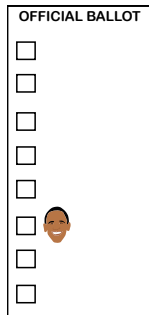
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$$X = \left\{ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \right.$$



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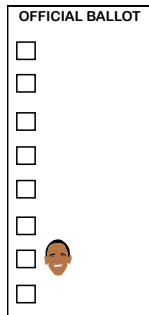
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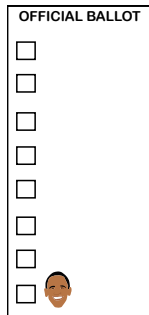
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- A probability mass function (pmf) and a cumulative distribution function (cdf) are two common ways to define the probability distribution for a discrete RV.
- Probability mass functions provide a compact way to represent information about **how likely** various outcomes are.

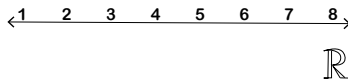
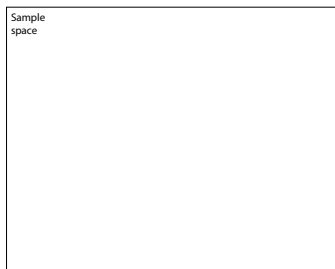
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The probabilities associated with each realization of the r.v. come from the underlying experiment and sample space.

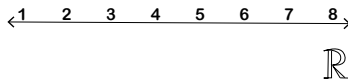
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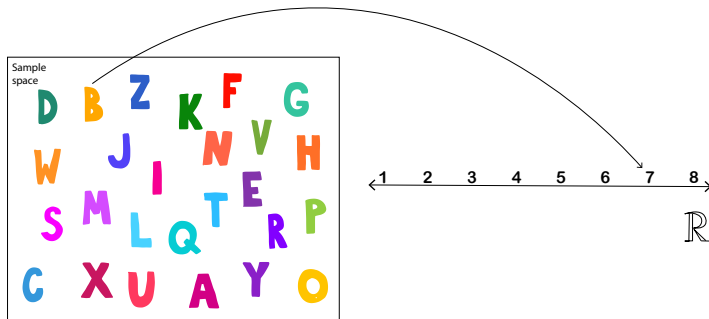
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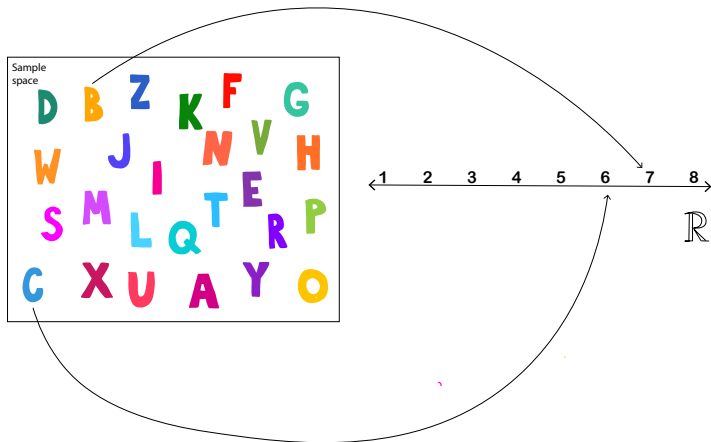
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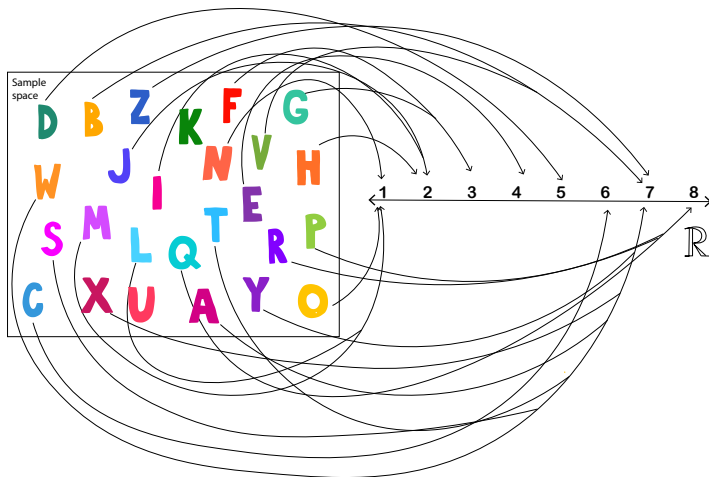
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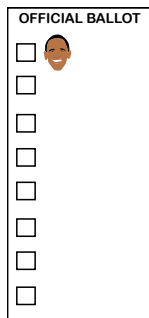


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$$f(x) = \left\{ \begin{array}{l} 4/26 \quad x = 1 \end{array} \right.$$



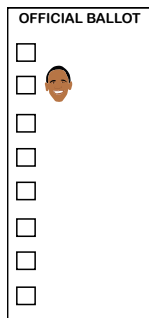
A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

Example: New Hampshire

Candidates:

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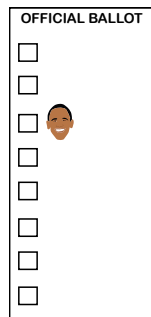
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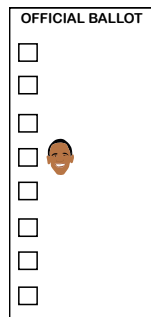
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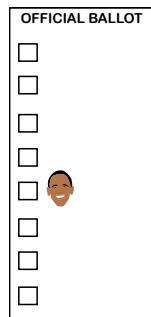
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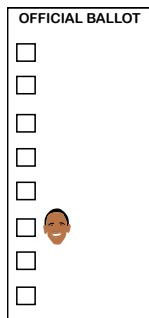
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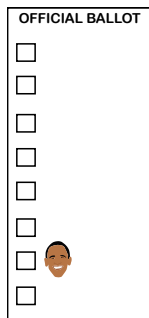
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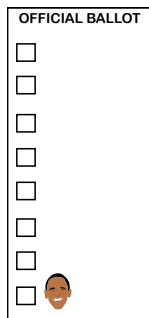
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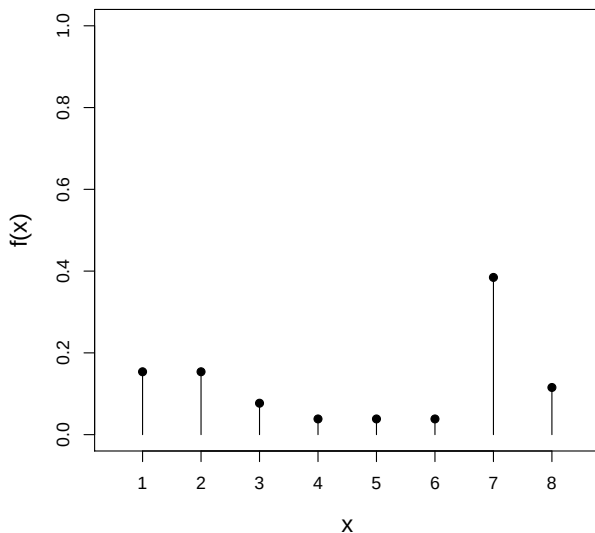
A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,**P,Q,R**,S,T,U,V,W,X,Y,Z

Discrete Probability Mass Functions

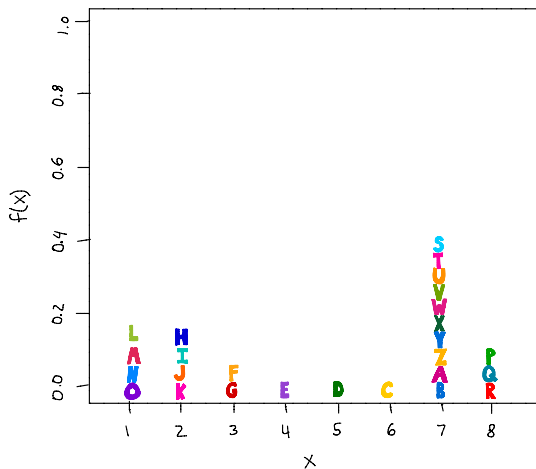
Discrete Probability Mass Functions

A probability mass function $f(x)$ of a random variable X is a non-negative function that gives the probability that $X = x$ and $\sum_x f(x) = 1$.

NH Obama Ballot Position PMF Plot



NH Obama Ballot Position PMF Plot

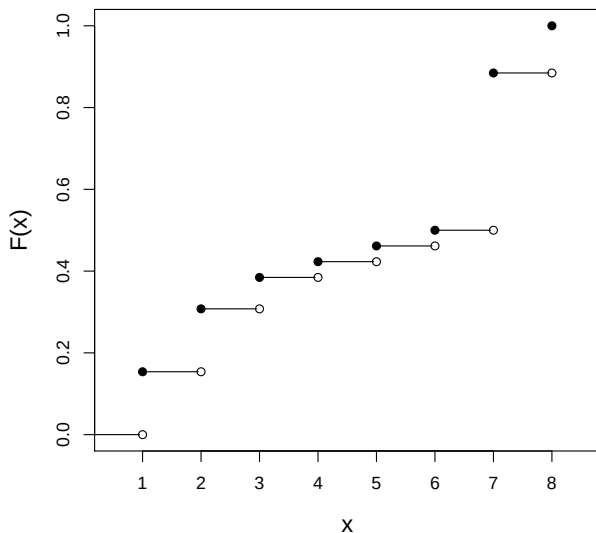


Discrete Cumulative Distribution Function

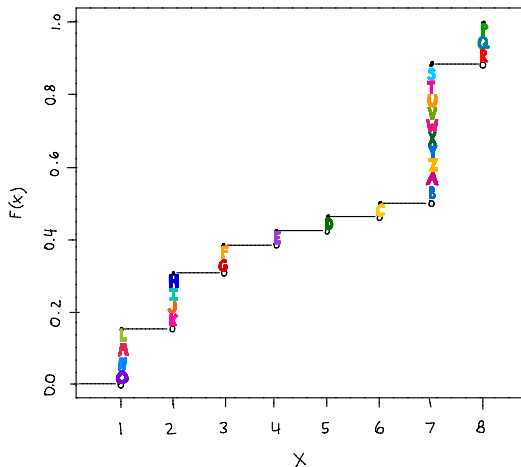
Discrete Cumulative Distribution Function

A cumulative distribution function $F(x)$ of a random variable X is a non-decreasing function that gives the probability that $X \leq x$.

NH Obama Ballot Position CDF Plot



NH Obama Ballot Position CDF Plot



Some Important Discrete Distributions

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- Let X be a binary variable with $P(X = 1) = p$ and, thus, $P(X = 0) = 1 - p$, where $p \in [0, 1]$. Then we say that X follows a **Bernoulli distribution** with the following pmf:

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- We can summarize these distributions with one number (e.g. the probability of variables being 1)

Empirical Distributions

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An **empirical mass function** $\hat{f}(x)$ of a variable X is a non-negative function that gives the frequency of the value x from data on X .

An **empirical cumulative distribution function** $\hat{F}(x)$ of a variable X is a non-decreasing function that gives the frequency of values of X less than x .

Example: Assessing Racial Prejudice

- We often want to ask **sensitive** questions which a survey respondent is unlikely to honestly answer
- A **list experiment** asks respondents how many items on a list they agree with
 - ▶ for example, what proportion of people would be upset by a black family moving in next door to them (Kuklinski et al 1997).
 - ▶ randomly split survey into two halves
 - ▶ first half ask how many of the following items upset you:
 1. the federal government increasing the tax on gasoline
 2. professional athletes getting million-dollar salaries
 3. large corporations polluting the environment.
 - ▶ second half, add a fourth item
 4. a black family moving in next door
 - ▶ use the answers to infer the proportion upset by the fourth item.
- To do this we need to understand **random variables**

Racial Prejudice Example (Kuklinski et al, 1997)

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$X = \#$ of angering items on the **baseline** list for Southerners:

x	0	1	2	3
$f(x)$	?	?	?	?
$\hat{f}(x)$	0.02	0.27	0.43	0.28
$\hat{F}(x)$	0.02	0.29	0.72	1.00

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Y = # of angering items on the **treatment** list for Southerners:

y	0	1	2	3	4
$f(y)$	?	?	?	?	?
$\hat{f}(y)$	0.02	0.20	0.40	0.28	0.10
$\hat{F}(y)$	0.02	0.22	0.62	0.90	1.00

Continuous Distributions

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- This is often a useful approximation when a variable takes on many values.
- A probability density function (pdf) and a cumulative distribution function (cdf) are two common ways to define the distribution for a continuous RV.

Example: Age in the Racial Prejudice Example

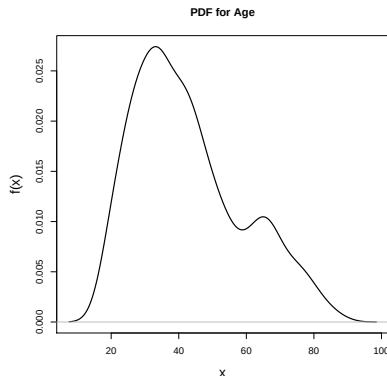
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Let X be the age of a randomly selected individual from the Kuklinski et al. (1997) data set.

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The probability distribution for this variable is well approximated by a **probability density function**.

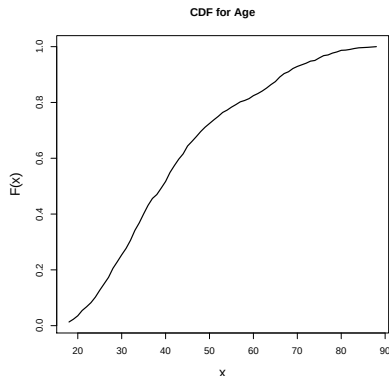


Continuous Cumulative Distribution Functions

Continuous Cumulative Distribution Functions

A **cumulative distribution function** $F(x)$ of a random variable X is a non-decreasing function that gives the probability that $X \leq x$. For a continuous RV, the cdf is continuous.

$$F(x) = \int_{-\infty}^x f(z) dz$$

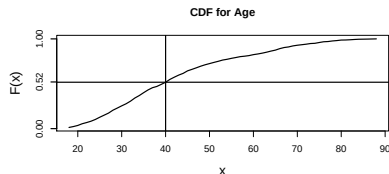
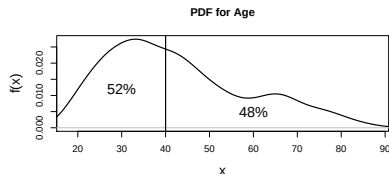


From PDFs to CDFs

From PDFs to CDFs

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(z) dz$$

$$.52 = P(X \leq 40) = \int_{-\infty}^{40} f(z) dz$$

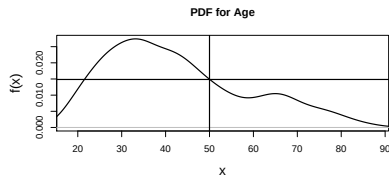
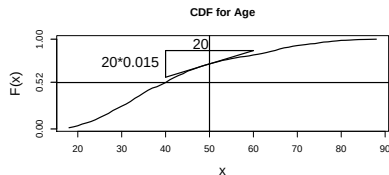


From CDFs to PDFs

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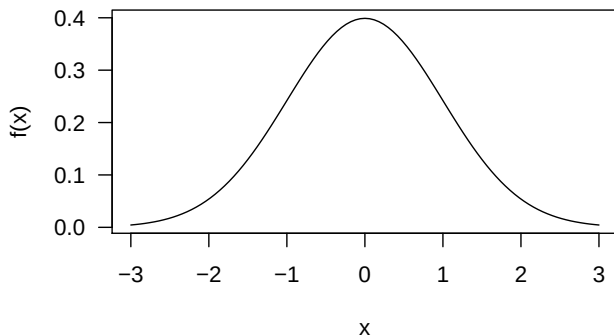
$$f(x) = \frac{dF(x)}{dx}$$

$$.015 = \frac{dF(50)}{dx}$$



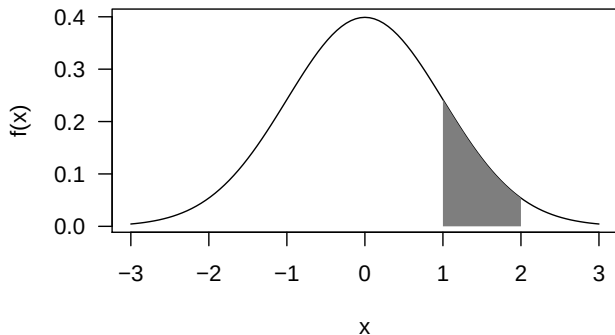
Subtleties of Continuous Densities

Remember- the height of the curve is not the probability of x occurring.



Subtleties of Continuous Densities

Remember- the height of the curve is not the probability of x occurring. To get the probability that X will fall in some region, you need the area under the curve.



1 Random Variables and Distributions

- What is a Random Variable?
- Discrete Distributions
- Continuous Distributions

2 Characteristics of Distributions

- Central Tendency
- Measures of Dispersion

3 Conditional Distributions

4 Fun with Averages

5 Fun with Sensitive Questions

6 Appendix: Why the Mean?

7 Joint Distributions

- Discrete Random Variable
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8 Conditional Expectation

9 Properties

- Independence
- Covariance and Correlation
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The expected value of a random variable X is denoted by $E[X]$ and is a measure of **central tendency** of X . Roughly speaking, an expected value is like a weighted average of all of the **values** weighted by **probability of occurrence**.

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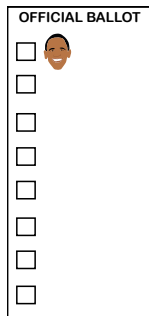
What did we expect for Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
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- Bill Richardson

4/26 × 1

+



A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

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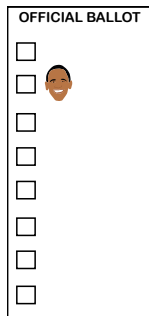
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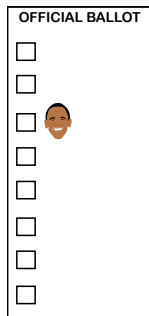


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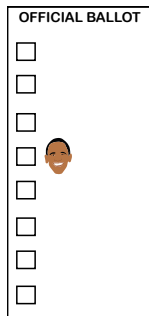


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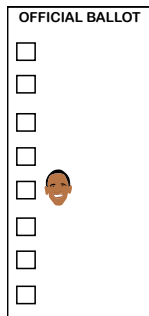


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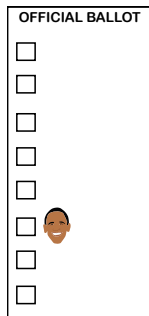


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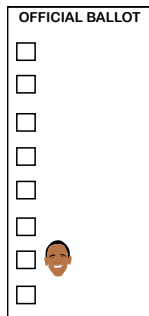


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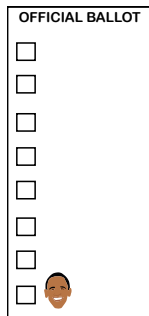


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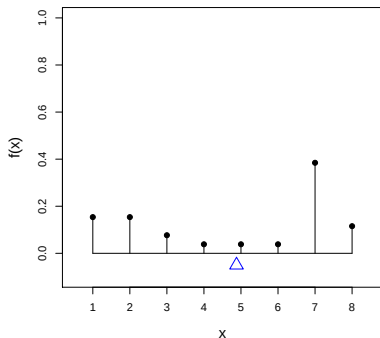
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		4.88

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

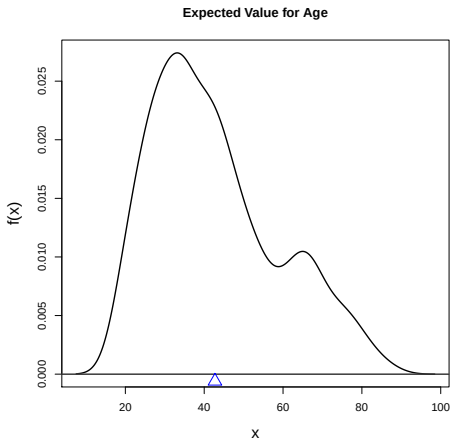
Interpreting Discrete Expected Value

The expected value for a discrete random variable is the balance point of the mass function.



Interpreting Continuous Expected Value

The expected value for a continuous random variable is the balance point of the density function.



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- We have some intuition about balance points.
- It has some useful and convenient properties.

► Appendix

Population Mean as an Expected Value

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$$\bar{x} = \sum_{\text{all } x_i} x_i f(x_i), \text{ where } f(x_i) = \frac{1}{N}$$

Property 1 of Expected Value: Additivity

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Together properties 1 and 2 are **linearity** (and this is sometimes presented as Linearity of Expectations).

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We will come back to this later. But it means that we can calculate the expected value of $g(X)$ **without** explicitly knowing the distribution of $g(X)$!

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- $E[XY] \neq E[X]E[Y]$ unless X and Y are independent

Racial Prejudice Example

$X = \#$ of angering items on the **baseline** list for Southerners:

x	0	1	2	3	Sum
$\hat{f}(x)$	0.02	0.27	0.43	0.28	1.00
$x \cdot \hat{f}(x)$	0.00	0.27	0.86	0.84	1.97

$Y = \#$ of angering items on the **treatment** list for Southerners:

y	0	1	2	3	4	Sum
$\hat{f}(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$y \cdot \hat{f}(y)$	0.00	0.20	0.80	0.84	0.40	2.24

Identifying the Percent Angry

Assume that $Y = X + A$, where for a randomly sampled respondent,

- Y = the number of total angering items
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- $A = 1$ if angered by a black family moving in next door
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- What properties and assumptions were necessary?

Variance

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The variance is a special case of this, and the variance of a random variable X (a measure of its dispersion) is given by

$$V[X] = E[(X - E[X])^2]$$

It is the expectation of the squared distances from the mean.

For a discrete random variable X

$$V[X] = \sum_{\text{all } x} (x - E[X])^2 f_X(x)$$

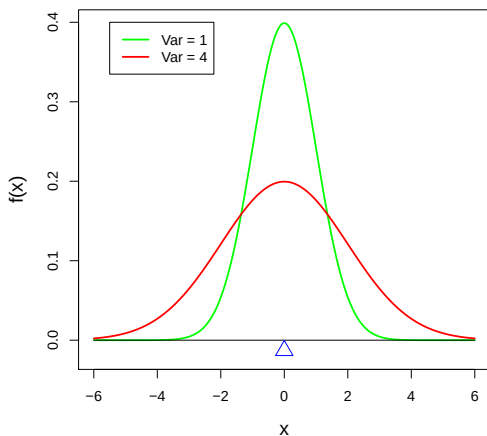
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For a continuous random variable X

$$V[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx$$

Variance Measures the Spread of a Distribution



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- It is a reasonable measure for the “spread” of a distribution.
- The Normal distribution (bell shaped with thin tails) is completely determined by its expected value (location) and variance (spread).
- The square root of the variance is the standard deviation.
- The variance and standard deviation have some useful properties.

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Suppose we have k independent random variables $X_1, \dots, X_k$. If $V[X_i]$ exists for all $i = 1, \dots, k$, then

$$V \left[\sum_{i=1}^k X_i \right] = V[X_1] + \dots + V[X_k]$$

NB: Technically independence is sufficient but not necessary.

What was the variance of Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$4/26 \times (1 - 4.88)^2$$

+

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

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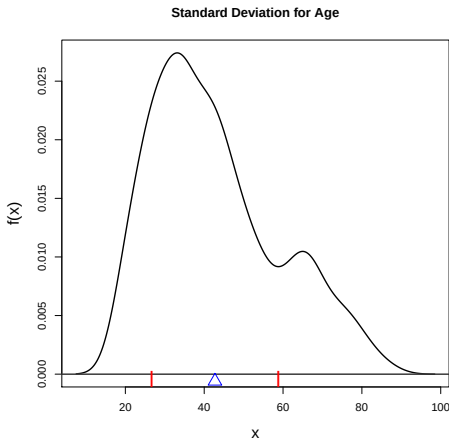
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Does variance matter for fairness?

Interpreting Continuous Standard Deviation

The standard deviation for a continuous random variable is a measure of the spread of the pdf.



Do we lose anything when we use the list experiment?

$Y = \#$ of angering items on the **treatment** list for Southerners:

y	0	1	2	3	4	Sum
$\hat{f}(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$(y - 2.24)^2 \cdot \hat{f}(y)$	0.15	0.31	0.02	0.16	0.31	0.95

More on this next week when we talk about estimator properties!

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- What is a Random Variable?
- Discrete Distributions
- Continuous Distributions

2 Characteristics of Distributions

- Central Tendency
- Measures of Dispersion

3 Conditional Distributions

4 Fun with Averages

5 Fun with Sensitive Questions

6 Appendix: Why the Mean?

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$$f(\text{●●}, \text{⊙}) = \pi_{\text{●●} \text{⊙}}$$

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$f(y, x)$ y	x		$f(y)$
	0	1	
0	π_{00}	π_{01}	$\pi_{00} + \pi_{01}$
1	π_{10}	π_{11}	$\pi_{00} + \pi_{01}$
2	π_{20}	π_{21}	$\pi_{00} + \pi_{01}$
3	π_{30}	π_{31}	$\pi_{00} + \pi_{01}$
4	π_{40}	π_{41}	$\pi_{00} + \pi_{01}$
$f(x)$	$\sum_{y=0}^4 \pi_{y0}$	$\sum_{y=0}^4 \pi_{y1}$	

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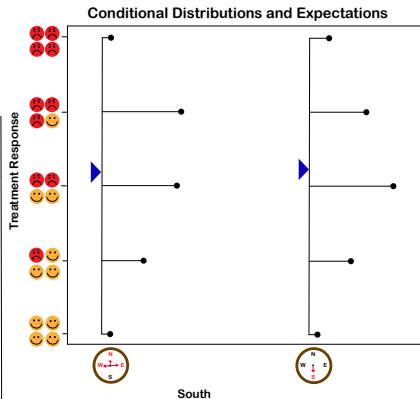
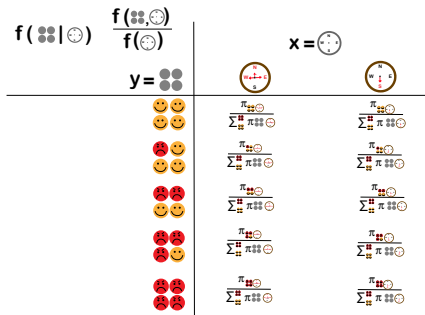
$f(\text{☉☉}, \text{⌚})$	$X = \text{⌚}$		$f(\text{☉☉})$
y			
☺☺	$\pi_{\text{☉☉} \text{⌚} \text{☺☺}}$	$\pi_{\text{☉☉} \text{⌚} \text{☺☺}}$	$\pi_{\text{☉☉} \text{⌚} \text{☺☺}} + \pi_{\text{☉☉} \text{⌚} \text{☺☺}}$
☹☺	$\pi_{\text{☉☉} \text{⌚} \text{☹☺}}$	$\pi_{\text{☉☉} \text{⌚} \text{☹☺}}$	$\pi_{\text{☉☉} \text{⌚} \text{☹☺}} + \pi_{\text{☉☉} \text{⌚} \text{☹☺}}$
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☹☹	$\pi_{\text{☉☉} \text{⌚} \text{☹☹}}$	$\pi_{\text{☉☉} \text{⌚} \text{☹☹}}$	$\pi_{\text{☉☉} \text{⌚} \text{☹☹}} + \pi_{\text{☉☉} \text{⌚} \text{☹☹}}$
$f(\text{⌚})$	$\sum_{\text{☉☉}} \pi_{\text{☉☉} \text{⌚} \text{☉☉}}$	$\sum_{\text{☉☉}} \pi_{\text{☉☉} \text{⌚} \text{☉☉}}$	

Discrete Conditional Distribution

Given the joint distribution, we can imagine what the conditional distribution and the conditional expectations would look like.

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(More on conditional expectations on Wednesday)

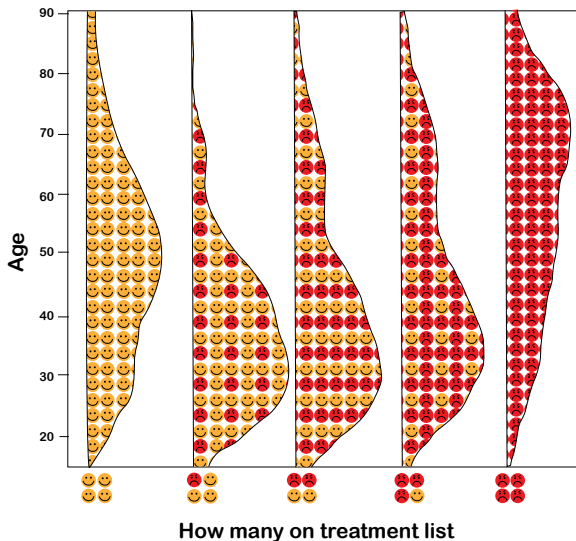
Example: Conditional Distribution with “Continuous” Y

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Suppose we define $X =$ “number of angering items” and $Y =$ “age” for a randomly selected respondent receiving the treatment list.

Example: Conditional Distribution with “Continuous” Y

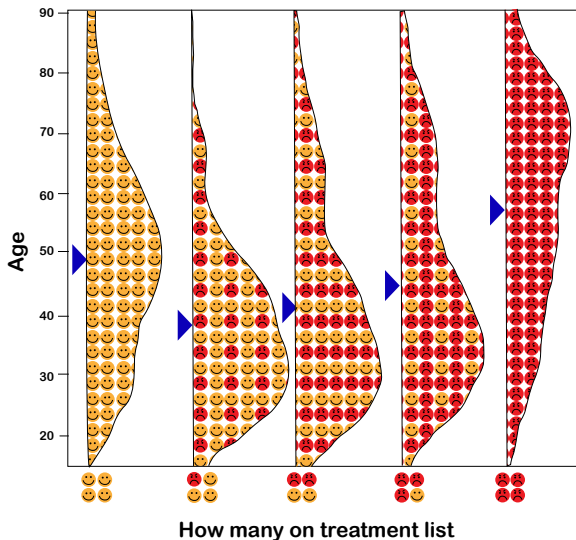
Suppose we define X = “number of angering items” and Y = “age” for a randomly selected respondent receiving the treatment list.



Conditional Expectation Function (CEF)

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The conditional expectations form a CEF: $E[Y|X = x] = h(x)$



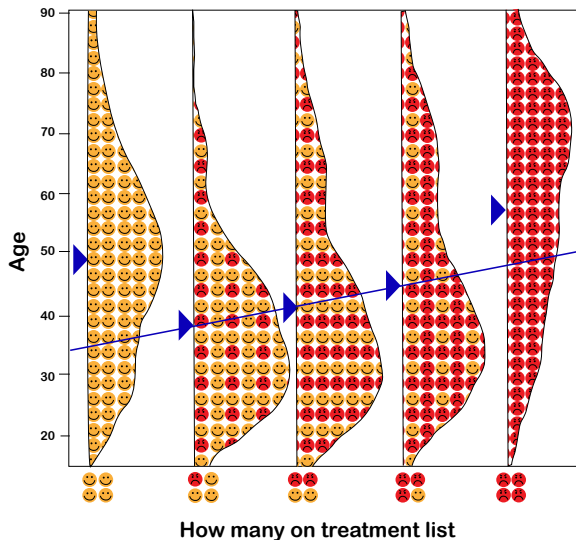
Linear CEF Assumption

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Often we will assume that the CEF is linear: $E[Y|X = x] = \beta_0 + \beta_1 x$

Linear CEF Assumption

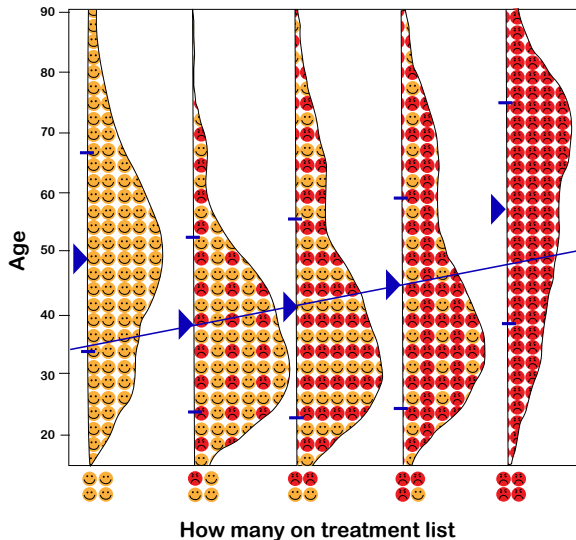
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Conditional Variance and Standard Deviation

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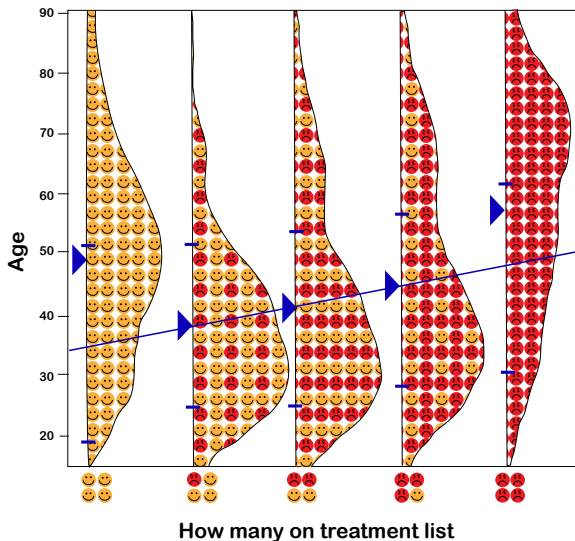
Similarly, we can assess the conditional standard deviation



Linear CEF and Constant Variance Assumptions

Linear CEF and Constant Variance Assumptions

Often, we assume that variance is the same for all values of x .



Preview: Interpreting the CEF

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Because the CEF is defined merely in terms of the larger population and not in terms of a causal effect (e.g., the causal effect of "number of angering items" on Age), we will utilize a descriptive interpretation of β_0 and β_1 .

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- β_1 is the expected difference in age between two individuals that have a one unit difference in the number of angering items.

Summary

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- Random variables and probability distributions provide useful infrastructure for everything we will do this year.
- Expected value and variance are two useful characteristics of the probability distributions associated with random variables.
- These concepts can be extended by conditioning on other variables.
- Next class we will cover joint distributions and conditional expectations in more depth.

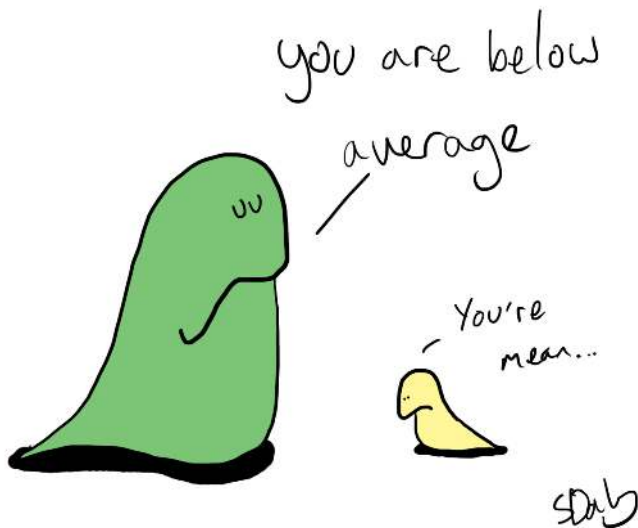
Fun with

Fun with Averages

Fun with Averages



Central Tendency



The Story of Averages



Measurements

MESURES de la POITRINE.	NOMBRE d'hommes.	NOMBRE PROPORTIONNEL.	PROBABILITÉ d'après L'OBSERVATION.	RANG dans LA TABLE.	RANG d'après le CALCUL.	PROBABILITÉ d'après LA TABLE.	NOMBRE d'OBSERVATIONS calculé.
Pouces.							
33	3	5	0,5000			0,5000	7
34	18	31	0,4995	52	50	0,4995	29
35	81	141	0,4964	42,5	42,5	0,4964	110
36	185	322	0,4823	33,5	34,5	0,4854	523
37	420	752	0,4501	26,0	26,5	0,4531	732
38	740	1305	0,3769	18,0	18,5	0,3799	1333
39	1073	1867	0,2464	10,5	10,5	0,2466	1858
			0,0597	2,5	2,5	0,0628	
40	1079	1882	0,1285	5,5	5,5	0,1359	1987
41	934	1628	0,2913	15	13,5	0,3034	1675
42	658	1148	0,4061	21	21,5	0,4130	1096
43	370	645	0,4706	30	29,5	0,4690	560
44	92	160	0,4866	55	57,5	0,4911	221
45	50	87	0,4955	41	43,5	0,4980	69
46	21	38	0,4991	49,5	53,5	0,4996	16
47	4	7	0,4998	56	61,8	0,4999	3
48	1	2	0,5000			0,5000	1
	5758	1,0000					1,0000

Social Physics

The determination of the average man is not merely a matter of speculative curiosity; it may be of the most important service to the science of man and the social system. It ought necessarily to precede every other inquiry into social physics, since it is, as it were, the basis. The average man, indeed, is in a nation what the centre of gravity is in a body; it is by having that central point in view that we arrive at the apprehension of all the phenomena of equilibrium and motion

- Quetelet

The Military Takes to the Idea



The Problem with Averages



The Average Man



The Face of the Average Man



Fun with Sensitive Questions

Fun with Sensitive Questions



Graeme Blair
(slides that follow from Graeme)

Fun with Sensitive Questions

Cannot ask direct questions when there are **incentives to conceal sensitive responses**

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- ① Social pressure
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How to Address Incentives to Conceal

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Develop trust with respondents, ask directly

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Survey experimental methods

- ① **Endorsement experiment** Evaluation bias
- ② **List experiment** Aggregation

How to Address Incentives to Conceal

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- ③ **Randomized response** Random noise

Bias in Direct Questions on Vote Buying

Estimated rate of vote buying from direct survey item
2.4%

Gonzalez-Ocantos et al. 2011, *AJPS*

Question text: "they gave you a gift or did you a favor"

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Estimated rate of vote buying from direct survey item

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Estimate using list experiment

24.3%

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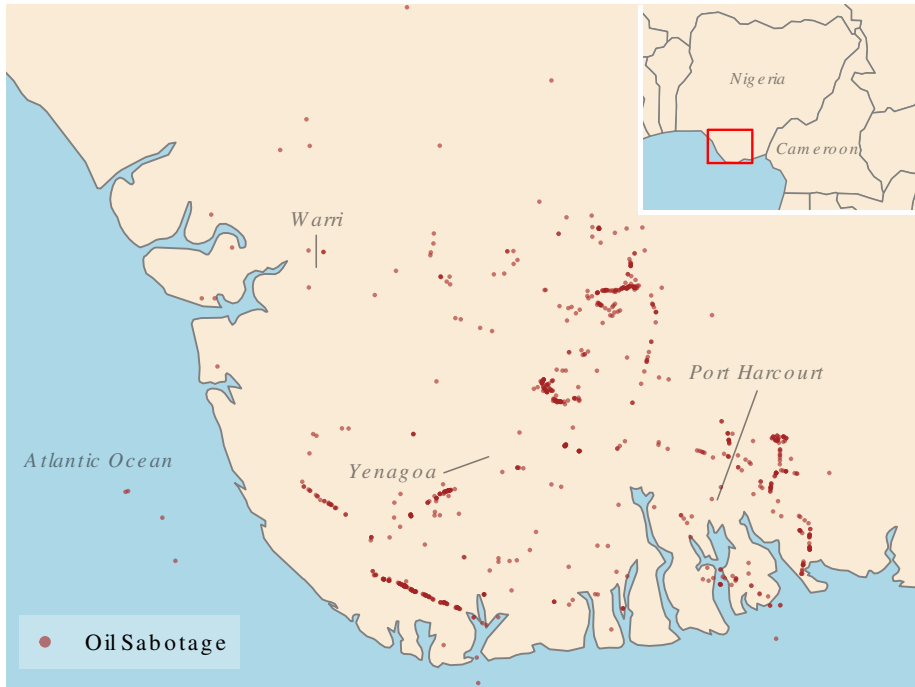
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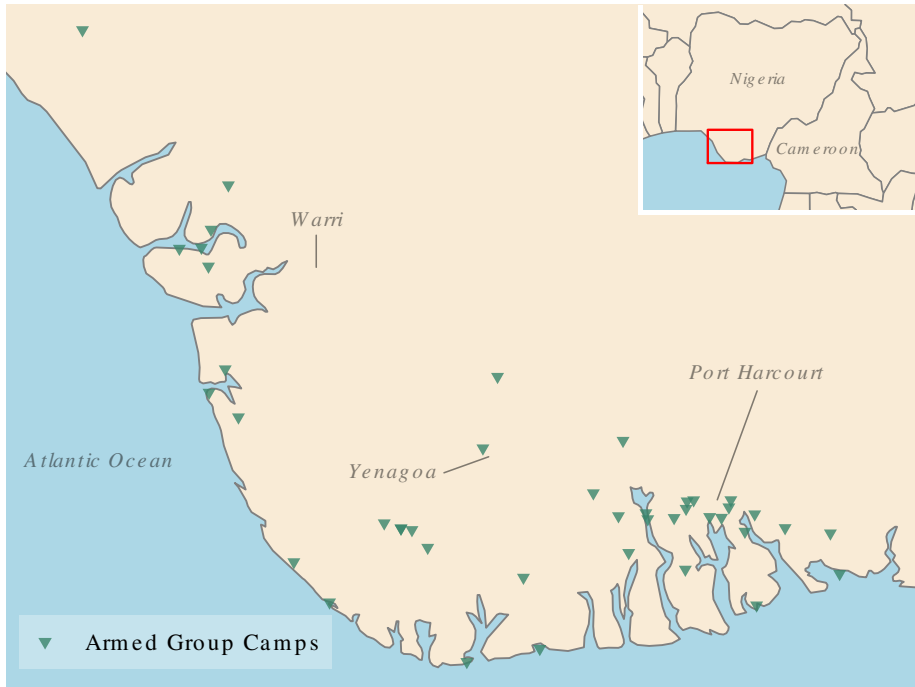
Survey

- Survey of 2,448 civilians in the Niger Delta

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Survey

- Survey of 2,448 civilians in the Niger Delta
- Random sample of 204 communities near and far from oil interruption sites and armed group camps
- Interviewed 12 people per community
Random walk pattern to select households; Kish grid within household

Funded by the International Growth Centre

Outcome

"Did you share information with **militants** about their enemies in the community, state counterinsurgency forces, or oil facility activities?"

Problems with using list or endorsement experiments

Too sensitive for list experiment

Often difficult to define "control" condition in endorsement experiment for behaviors

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Alternative: **Randomized response technique**

Randomized response technique

How? Introducing random noise

Randomized response technique

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- Roll the dice in private

Randomized response technique

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- Roll the dice in private
- If you roll a 1, tell me "no"

Randomized response technique

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- Roll the dice in private
- If you roll a 1, tell me "no"
- If you roll a 6, tell me "yes"

Randomized response technique

How? Introducing random noise

- Roll the dice in private
- If you roll a 1, tell me "no"
- If you roll a 6, tell me "yes"
- Otherwise, answer: "Did you share information with armed groups"

Analysis of the randomized response technique

- 1 Used fair dice, and actually rolled it.

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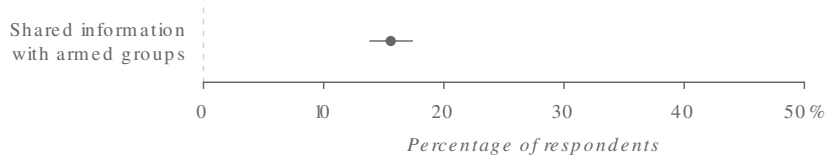
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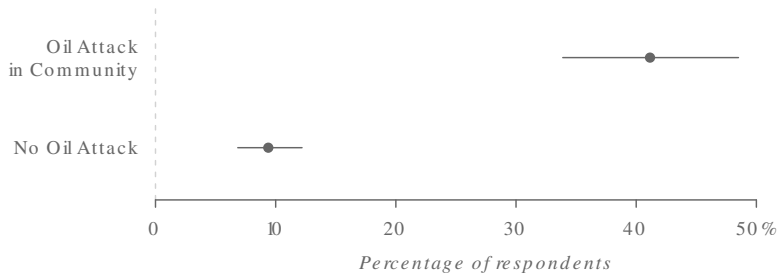
Proportion yes to sensitive item

$$= 3/2 \cdot (\text{Proportion answered yes} - 1/6)$$

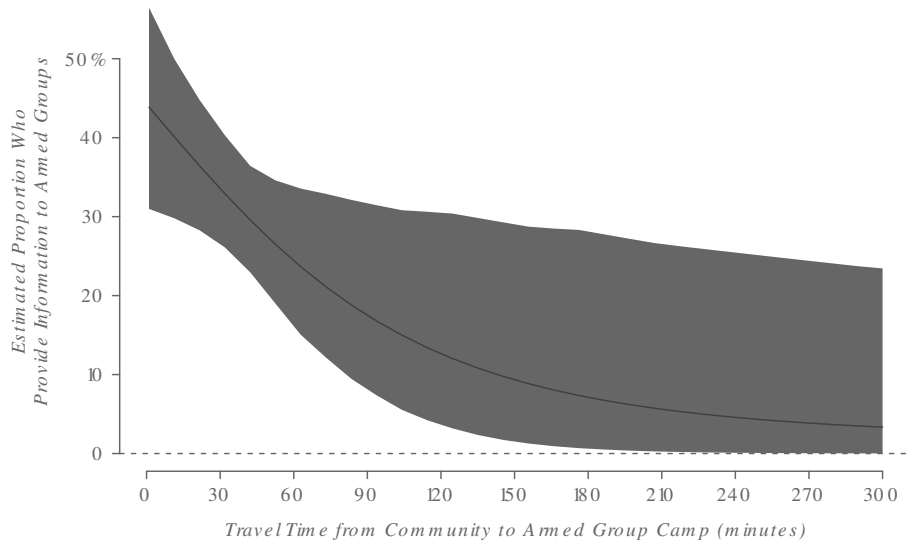
1. Civilians share information regularly with armed groups



2. Civilians near oil interruptions dominate collaboration



3. Civilians near armed group camps dominate collaboration



- `rr` package in R for randomized response

Blair with Yang-Yang Zhou and Kosuke Imai

- `list` package in R for list experiments

Blair with Kosuke Imai

- `endorse` package in R for endorsement experiments

Yuki Shiraito and Kosuke Imai

1 Random Variables and Distributions

- What is a Random Variable?
- Discrete Distributions
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Expected Value as Mean Square Error Minimizer

► Back

Suppose we want to pick a single number (c) that **summarizes** a random variable X . What we mean by **summarizes** determines the best choice of c .

Generally speaking we want a summary that is in the “center” of the data, i.e. that is as close as possible to all possible datapoints. Again though, the choice turns on what we mean by close.

Let's say we want to minimize:

- Mean Squared Error: $E(X - c)^2$

This leads to choosing the mean of X : μ

- Mean Absolute Error: $E[|X - c|]$

This leads to choosing the median of X : m

Let's prove the first result (see Blitzstein and Hwang 2014 Theorem 6.1.4 on pg 245 for this proof and the proof on mean absolute error).

Proof of Mean as Mean Square Error Minimizer

Let X be a random variable with mean μ . We want to show that the value of c that minimizes the mean squared error $E(X - c)^2$ is the mean, μ (Blitzstein and Hwang Theorem 6.1.4).

We will prove the following identity below:

$$E(X - c)^2 = \text{Var}(X) + (\mu - c)^2 \quad (1)$$

We are trying to choose c to minimize this term. The choice cannot affect $\text{Var}(X)$. Setting $c = \mu$ sets $(\mu - c)^2 = 0$ and any other choice makes $(\mu - c)^2 > 0$. Therefore (assuming the identity holds), $c = \mu$ minimizes Eq 1.

Now to prove the identity:

$$\text{Var}(X) = \text{Var}(X - c) \quad (\text{Prop 1 of Variance})$$

$$= E(X - c)^2 - (E[X - c])^2 \quad (\text{Defn of Variance})$$

$$= E(X - c)^2 - (\mu - c)^2 \quad (\text{Linearity of Exp})$$

$$\text{Var}(X) + (\mu - c)^2 = E(X - c)^2$$

References

- Kuklinski et al. 1997 “Racial prejudice and attitudes toward affirmative action” *American Journal of Political Science*
- Glynn 2013 “What can we learn with statistical truth serum? Design and analysis of the list experiment”
- All the Blair papers above.

Where We've Been and Where We're Going...

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 - ▶ in regression we think about how the distribution of one variable changes under different values of another variable
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- The **joint distribution** of two (or more) variables describes the pairs of observations that we are more or less likely to see.

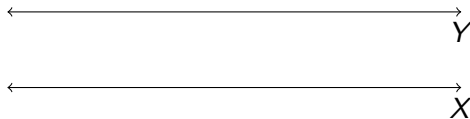
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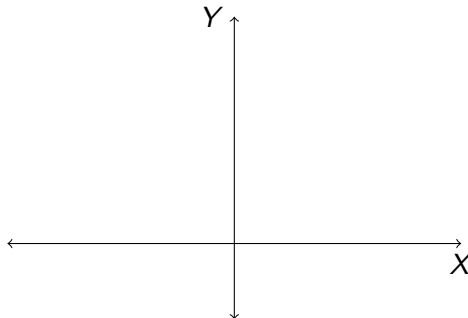
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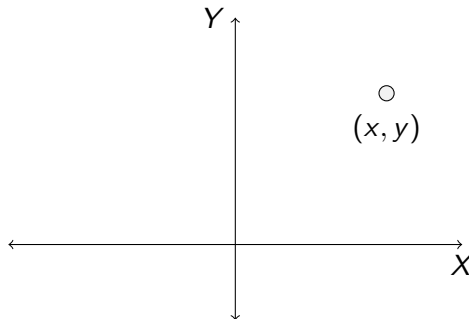
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- One realization of the r.v. is a point in that space



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- Often, we are interested in two random variables that are qualitatively different:

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- Often, we are interested in two random variables that are qualitatively different:
 - ▶ Y (response, outcome, dependent variable, etc.)
= the random variable we want to explain, or predict.

Understanding Joint Distributions

- Imagine we are throwing darts on a two-dimensional board: the joint distribution tells us where the darts are more likely to land.
- Distributions can be limited to a **subset** of the real line
 - ▶ e.g. two uniform random variables might be between 0 and 1
 - ▶ e.g. discrete random variables typically only include integers
- With two r.v.s. there are now **two dimensions** to deal with.
- Often, we are interested in two random variables that are qualitatively different:
 - ▶ Y (response, outcome, dependent variable, etc.)
= the random variable we want to explain, or predict.
 - ▶ X (predictor, explanatory/independent variable, covariate, etc.)
= the random variable with which we want to explain Y .

1 Random Variables and Distributions

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Joint Probability Mass Function

Definition

For two discrete random variables X and Y the **joint** PMF $P_{X,Y}(x,y)$ gives the probability that $X = x$ and $Y = y$ for all x and y :

$$P_{X,Y}(x,y) = \Pr(X = x \text{ and } Y = y)$$

Restrictions:

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Restrictions:

- $P_{X,Y}(x,y) \geq 0$ and $\sum_x \sum_y P_{X,Y}(x,y) = 1$.

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Should the U.S. allow more immigrants to come and live here?

		X: Education			
		less HS	HS	College	BA
Y: Support	oppose	0.07	0.22	0.18	0.15
	neutral	0.02	0.06	0.05	0.05
	favor	0.01	0.03	0.04	0.11

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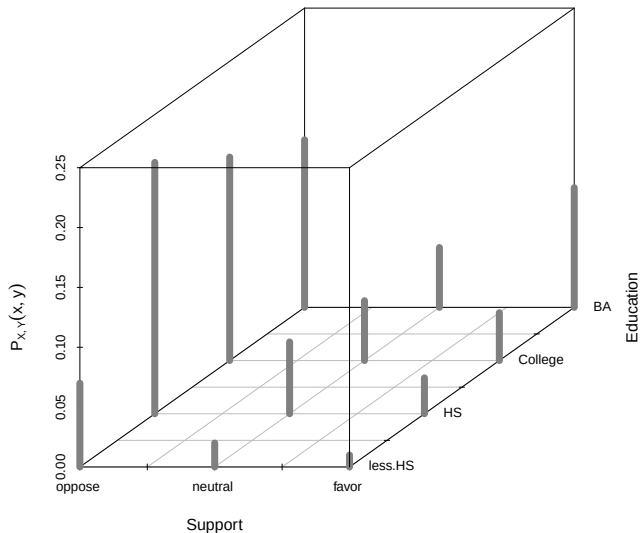
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With discrete r.v.s this is very similar to thinking about a cross-tab, with frequencies/ probabilities in the cells instead of raw numbers.

Joint Probability Mass Function



From Joint to Marginal PMF

Given the **joint** PMF $P_{X,Y}(x,y)$ can we recover the **marginal** PMF $P_Y(y)$ (distribution over a single variable)?

		X: Education				
		less HS	HS	College	BA	
Y: Support	oppose	0.07	0.21	0.17	0.14	
	neutral	0.02	0.06	0.05	0.05	
	favor	0.01	0.03	0.04	0.10	

From Joint to Marginal PMF

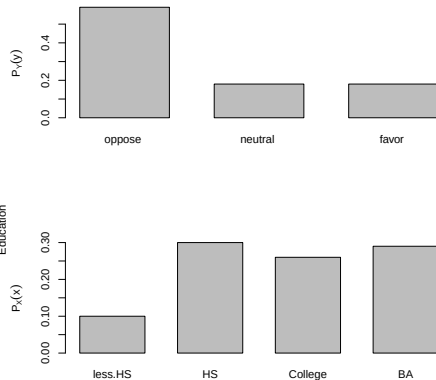
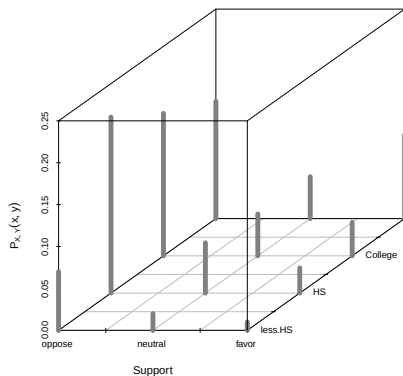
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		X: Education				$P_Y(y)$
		less HS	HS	College	BA	
Y: Support	oppose	0.07	0.21	0.17	0.14	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.10	0.19

To obtain $P_Y(y)$ we **marginalize** the joint probability function $P_{X,Y}(x,y)$ over X :

$$P_Y(y) = \sum_x P_{X,Y}(x,y) = \sum_x \Pr(X = x, Y = y)$$

Joint and Marginal Probability Mass Functions



Why Does Marginalization Work?

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Marginalizing **over** y to get $p(x)$ is then,

$$p(x_j) = \sum_{i=1}^N p(x_j|y_i)p(y_i)$$

A Table

	$Y = 0$	$Y = 1$	
$X = 0$	$p(0,0)$	$p(0, 1)$	$p_X(0)$
$X = 1$	$p(1,0)$	$p(1,1)$	$p_X(1)$
	$p_Y(0)$	$p_Y(1)$	

A Table

	Y = 0	Y = 1	
X = 0	0.01	0.05	?
X = 1	0.25	0.69	?
	0.26	0.74	

$$\begin{aligned}p_X(0) &= p(0|y=0)p(y=0) + p(0|y=1)p(y=1) \\&= \frac{0.01}{0.26} \times 0.26 + \frac{0.05}{0.74} \times 0.74 \\&= 0.06\end{aligned}$$

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$$\begin{aligned}p_X(1) &= p(1|y=0)p(y=0) + p(1|y=1)p(y=1) \\&= \frac{0.25}{0.26} \times 0.26 + \frac{0.69}{0.74} \times 0.74 \\&= 0.94\end{aligned}$$

Conditional PMF

Conditional PMF

Definition

The **conditional** PMF of Y given X , $P_{Y|X}(y|x)$, is the PMF of Y when X is known to be at a particular value $X = x$:

$$P_{Y|X}(y|x) = \frac{\Pr(X = x \text{ and } Y = y)}{\Pr(X = x)} = \frac{P_{X,Y}(x, y)}{P_X(x)}$$

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Key relationships:

- $P_{X,Y}(x, y) = P_{Y|X}(y|x)P_X(x)$ (multiplicative rule)
- $P_{Y|X}(y|x) = P_{X|Y}(x|y)P_Y(y)/P_X(x)$ (Bayes' rule)

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Conditional PMFs are just like ordinary PMFs, but refer to a universe where the “conditioning event” ($X = x$) is known to have occurred.

Conditional distributions are key in statistical modeling because they inform us how the distribution of Y varies across different levels of X .

From Joint to Conditional: $P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$

Table: Joint PMF $P_{X,Y}(x,y)$ and Marginal PMFs $P_X(x), P_Y(y)$

		Education				
	$P_{X,Y}(x,y)$	less HS	HS	College	BA	$P_Y(y)$
Support	oppose	0.07	0.22	0.18	0.15	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
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	$P_X(x)$	0.11	0.32	0.27	0.31	1.00

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	$P_X(x)$	0.11	0.32	0.27	0.31	1.00

Table: Conditional PMF $P_{Y|X}(y|x)$

		Education				
	$P_{Y X}(y x)$	less HS	HS	College	BA	
Support	oppose	0.70	0.70	0.65	0.48	0.62
	neutral	0.20	0.20	0.19	0.17	0.19
	favor	0.10	0.10	0.15	0.34	0.19
		1.00	1.00	1.00	1.00	1.00

Joint and Conditional Probability Mass Functions

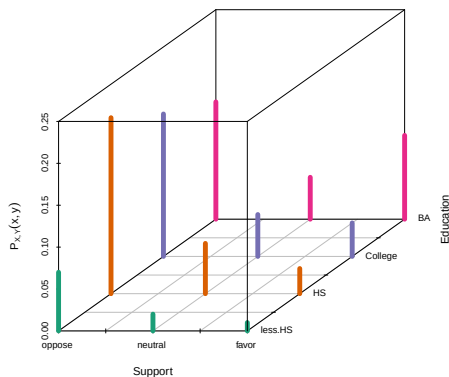


Figure: Joint

Joint and Conditional Probability Mass Functions

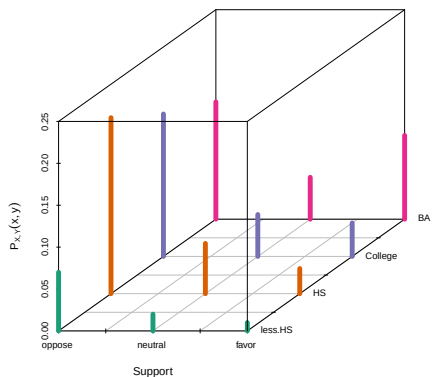


Figure: Joint

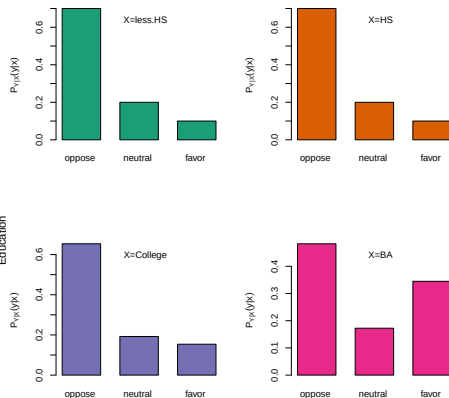


Figure: Conditional

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Joint Probability Density Function

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For two **continuous** random variables X and Y the **joint** PDF $f_{X,Y}(x,y)$ gives the density height where $X = x$ and $Y = y$ for all x and y .

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The multiplicative rule:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$$

where

- $f_{Y|X}(y|x)$: **Conditional** PDF of Y given $X = x$
- $f_X(x)$: **Marginal** PDF of X

Restrictions:

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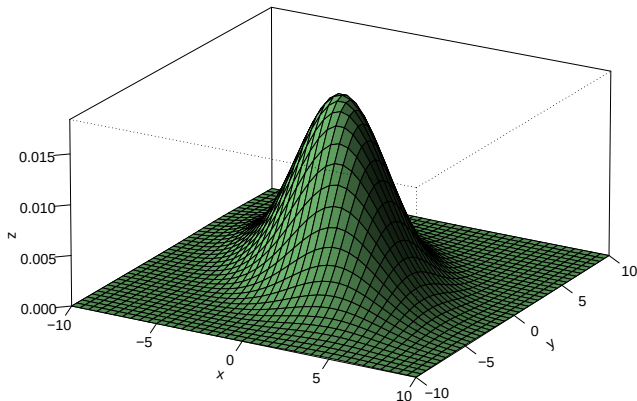
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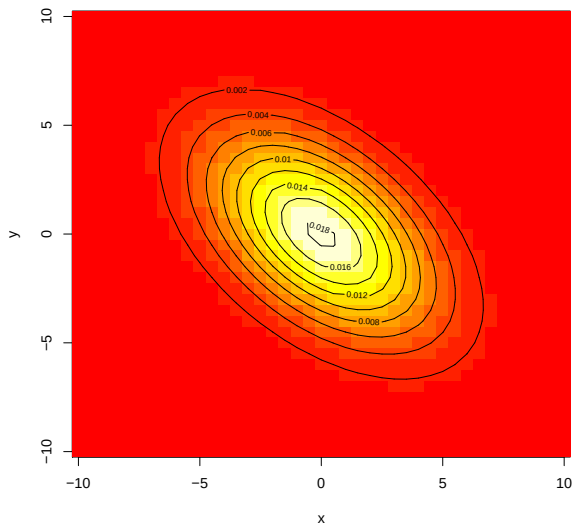
- $\int_x \int_y f_{X,Y}(x,y) dy dx = 1$

3D Plot of a Joint Probability Density Function

Bivariate Normal Distribution: $z = f_{X,Y}(x, y)$



Contour Plot of a Joint Probability Density Function



From Joint to Marginal PDF

How can we obtain $f_Y(y)$ from $f_{X,Y}(x,y)$?

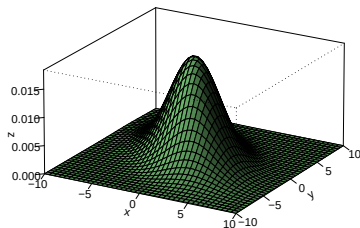
From Joint to Marginal PDF

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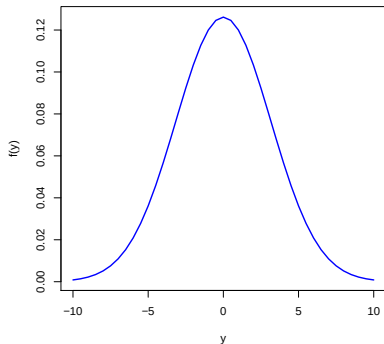
We marginalize the joint probability function $f_{X,Y}(x,y)$ over X :

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

Joint PDF $f(x,y)$



Marginal PDF $f(y)$



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Conditioning on X

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- Goal in statistical modeling is often to characterize the conditional distribution of the outcome variable $f_{Y|X}(y|x)$ across different levels of $X = x$.
- Typically, we summarize the conditional distributions with a few parameters such as the **conditional mean** of $E[Y|X = x]$ and the **conditional variance** $V[Y|X = x]$
- Moreover, we are often interested in estimating $E[Y|X]$, i.e. the **conditional expectation function** that describes how the conditional mean of Y varies across all possible values of X (we sometimes call this the **population regression function**)

Conditional Expectation

Definition (Conditional Expectation (Discrete))

Let Y and X be discrete random variables. The conditional expectation of Y given $X = x$ is defined as:

$$E[Y|X = x] = \sum_y y \Pr(Y = y|X = x) = \sum_y y P_{Y|X}(y|x)$$

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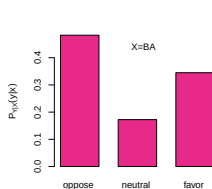
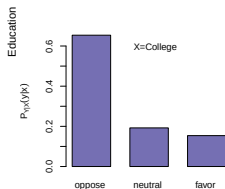
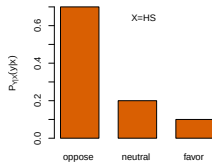
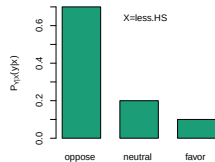
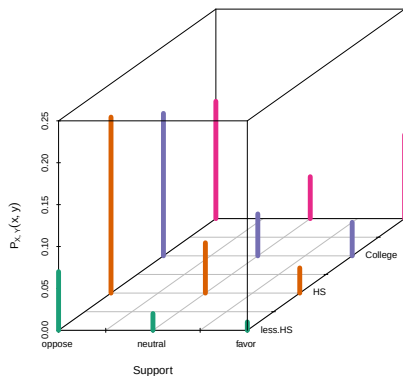
$$E[Y|X = x] = \sum_y y \Pr(Y = y|X = x) = \sum_y y P_{Y|X}(y|x)$$

Definition (Conditional Expectation (Continuous))

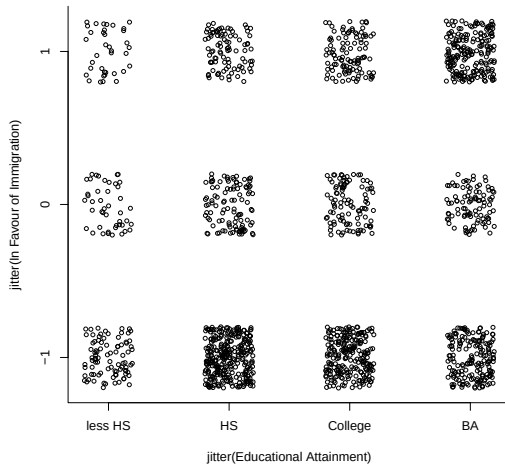
Let Y and X be continuous random variables. The conditional expectation of Y given $X = x$ is given by:

$$E[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy$$

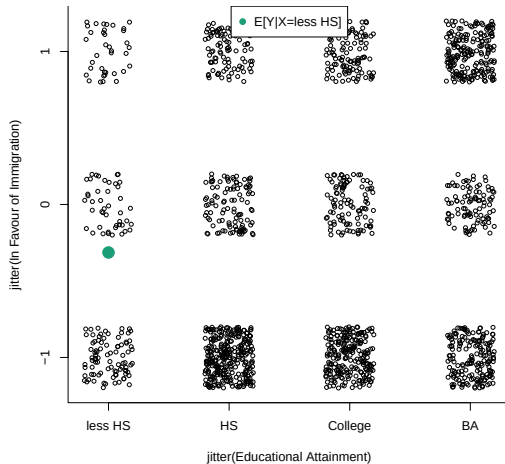
Joint and Conditional Probability Mass Functions



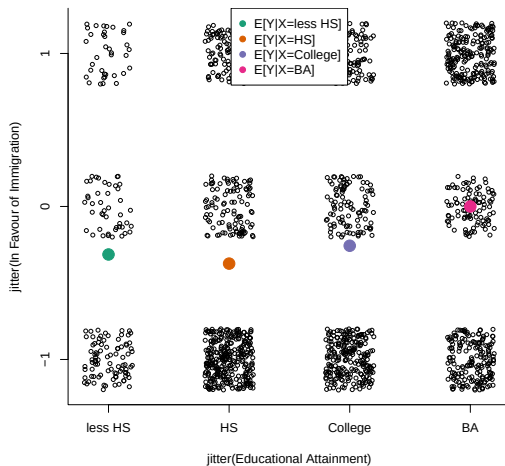
Conditional PMF $P_{Y|X}(y|x)$



Conditional Expectation $E[Y|X = 1]$



Conditional Expectation Function $E[Y|X]$



Law of Iterated Expectations

Theorem (Law of Iterated Expectations)

For two random variables X and Y ,

$$E[Y] = E[E[Y|X]] = \begin{cases} \sum E[Y|X=x] \cdot P_X(x) & (\text{discrete } X) \\ \int_{-\infty}^{\infty} E[Y|X=x] \cdot f_X(x) dx & (\text{continuous } X) \end{cases}$$

Note that the outer expectation is taken with respect to the distribution of X .

Law of Iterated Expectations

Theorem (Law of Iterated Expectations)

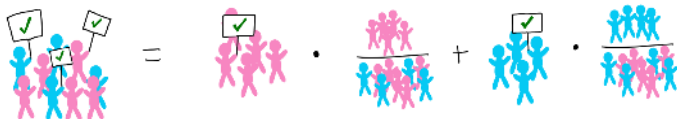
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Example: Y (support) and $X \in \{1, 0\}$ (gender). Then, the LIE tells us:

$$\underbrace{E[Y]}_{\text{Average Support}} = \underbrace{E[E[Y|X]]}_{\text{Average Support|Woman}} = \underbrace{E[Y|X=1]}_{\text{Average Support|Woman}} \cdot \underbrace{P_X(1)}_{\text{Pr(Woman)}} + \underbrace{E[Y|X=0]}_{\text{Average Support|Man}} \cdot \underbrace{P_X(0)}_{\text{Pr(Man)}}$$



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- ① $E[c(X)|X] = c(X)$ for any function $c(X)$.
 - Basically, any function of X is a constant with regard to the conditional expectation. If we know X , then we also know X^2 , for instance.

Properties of Conditional Expectation

Conditional expectations have some convenient properties

- ① $E[c(X)|X] = c(X)$ for any function $c(X)$.
 - ▶ Basically, any function of X is a constant with regard to the conditional expectation. If we know X , then we also know X^2 , for instance.
- ② If $E[Y^2] < \infty$ and $E[g(X)^2] < \infty$ for some function g , then
 $E[(Y - E[Y|X])^2|X] \leq E[(Y - g(X))^2|X]$ and
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The second property is quite important. It says that the conditional expectation is the function of X that **minimizes the squared prediction error** for Y across any possible function of X .

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Remember, the conditional distribution of $Y|X$ is basically like any other probability distribution, so we are going to want to summarize the **center and spread**.

Conditional Variance

Definition

The **conditional variance** of Y given $X = x$ is defined as:

$$V[Y|X = x] = \begin{cases} \sum_{\text{all } y} (y - E[Y|X = x])^2 P_{Y|X}(y|x) & (\text{discrete } Y) \\ \int_{-\infty}^{\infty} (y - E[Y|X = x])^2 f_{Y|X}(y|x) dy & (\text{continuous } Y) \end{cases}$$

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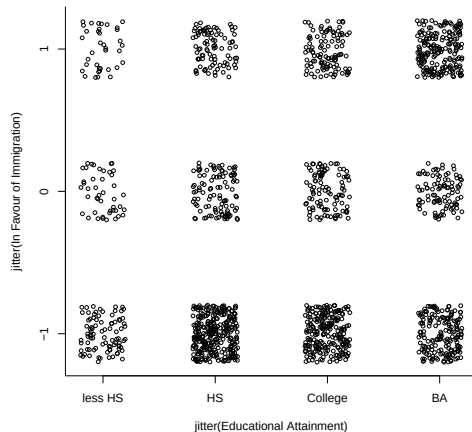
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Example: Y (support) and $X \in \{1, 0\}$ (gender). The LTV says that the total variance in support can be decomposed into two parts:

- 1 On average, how much support varies within gender groups (**within variance**)
- 2 How much average support varies between gender groups (**between variance**)

Conditional Variance Function $V[Y|X]$



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Let's look at this in pictures.

(If you want to know more: Blitzstein and Hwang pg 392-393 is great.)

Important Subtleties in Pictures



Sample space

Important Subtleties in Pictures



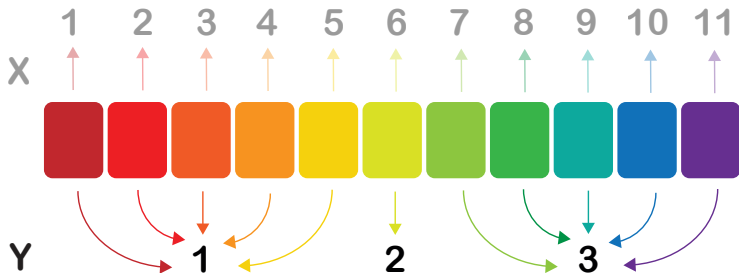
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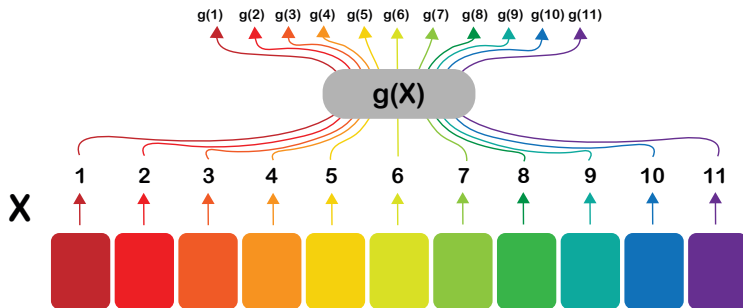
Random variable

Important Subtleties in Pictures



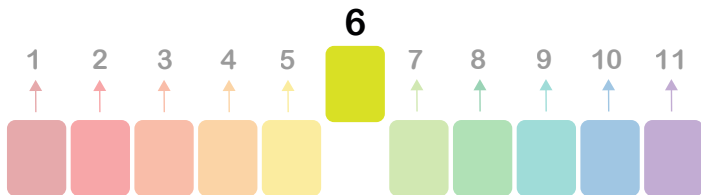
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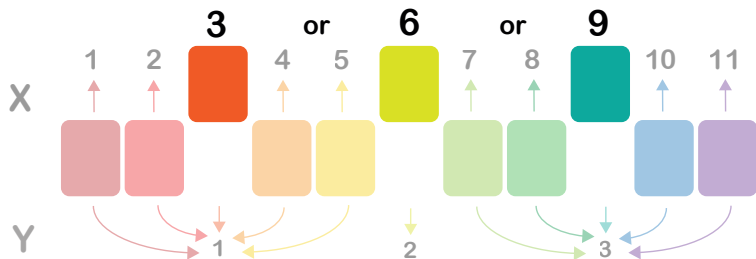
Function of a random variable is a random variable

Important Subtleties in Pictures



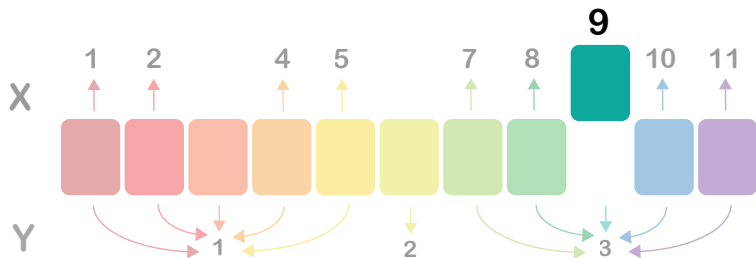
$$E[X]$$

Important Subtleties in Pictures



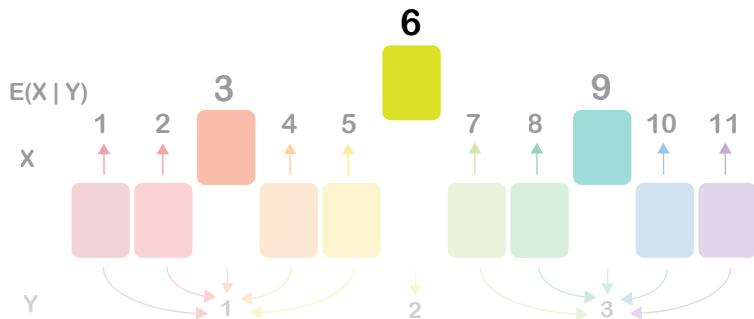
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Important Subtleties in Pictures



$$E[X|Y = 3]$$

Important Subtleties in Pictures



$$E[E(X|Y)] = E[X]$$

1 Random Variables and Distributions

- What is a Random Variable?
- Discrete Distributions
- Continuous Distributions

2 Characteristics of Distributions

- Central Tendency
- Measures of Dispersion

3 Conditional Distributions

4 Fun with Averages

5 Fun with Sensitive Questions

6 Appendix: Why the Mean?

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Independence

Definition (Independence of Random Variables)

Two random variables Y and X are independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for all x and y . We write this as $Y \perp\!\!\!\perp X$.

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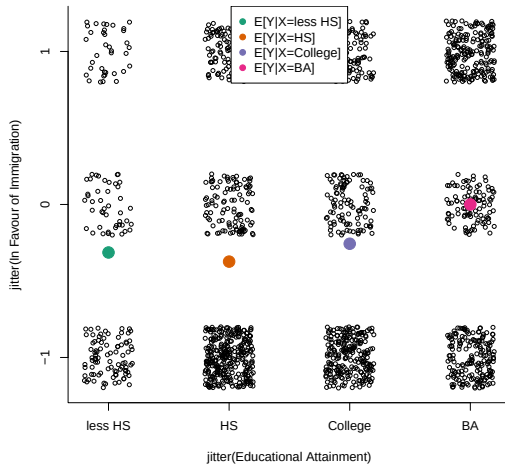
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Is $Y \perp\!\!\!\perp X$?



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We can prove the continuous case by following the same steps, with $\sum$ replaced by $\int$.

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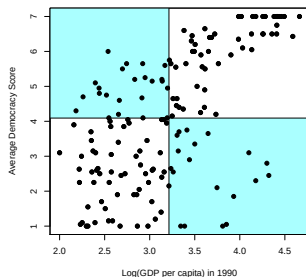
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- Points in upper right and lower left quadrants (relative to the means) add to the covariance.
- Points in the upper left and lower right quadrants subtract from the covariance.



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Therefore, $X \perp\!\!\!\perp Y \implies \text{Cov}[X, Y] = 0$, but not vice versa.

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Proof: Plug in to the definition of variance and expand (try it yourself!)

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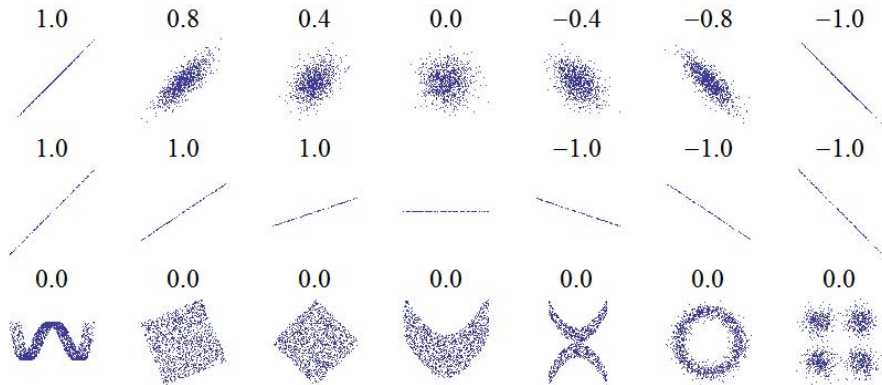
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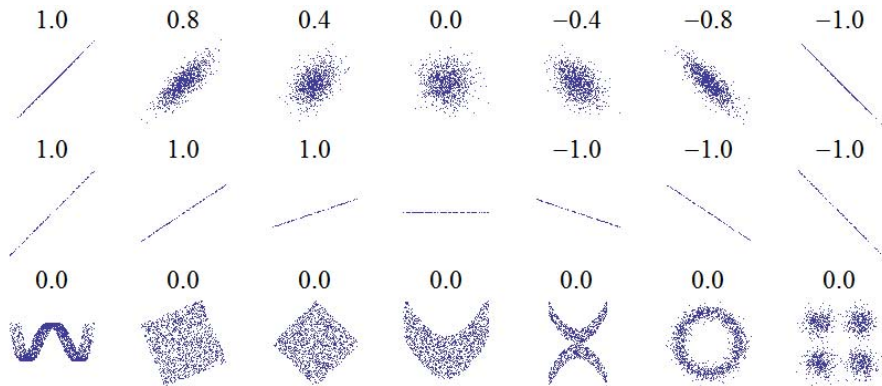
- $\text{Cor}[X, Y]$ is a standardized measure of linear association between X and Y .
- Always satisfies: $-1 \leq \text{Cor}[X, Y] \leq 1$.

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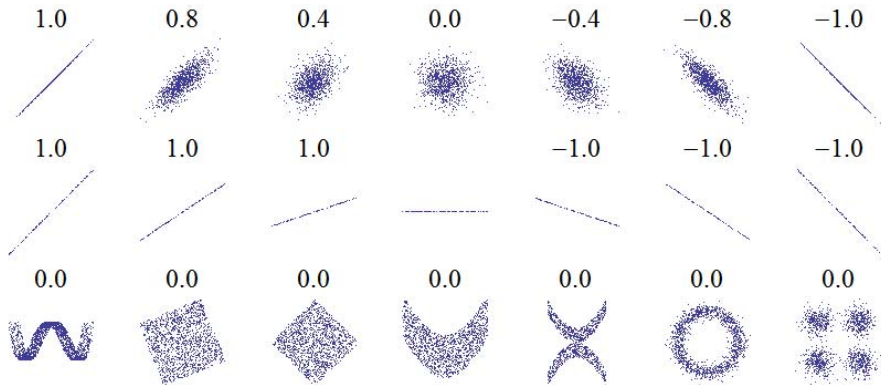


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- Like covariance, correlation measures the **linear** association between X and Y .

Conditional Independence

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Definition (Conditional Independence of Random Variables)

Random variables Y and X are conditionally independent given Z iff

$$f_{X,Y|Z}(x,y|z) = f_{Y|Z}(y|z) \cdot f_{X|Z}(x|z)$$

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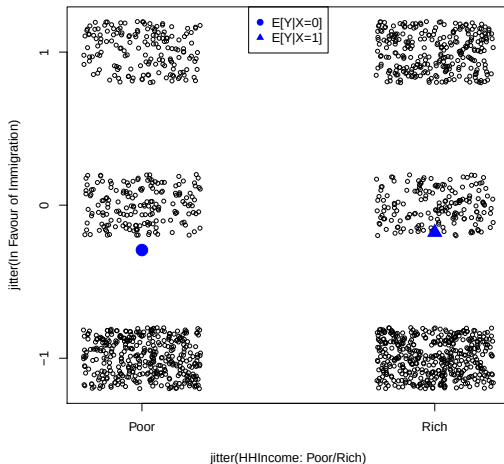
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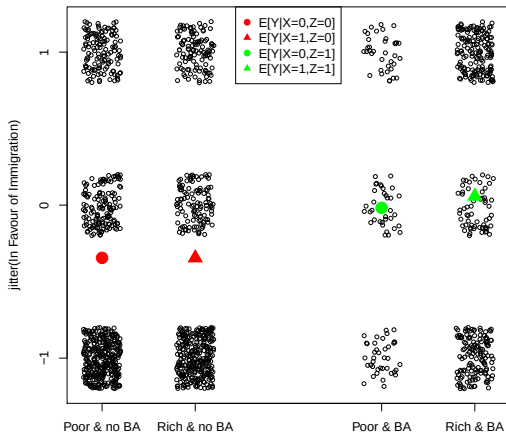
Is $Y \perp\!\!\!\perp X$?

Example: X = wealth, Y = support for immigration, Z = education.



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- Continuous Distributions

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- Central Tendency
- Measures of Dispersion

3 Conditional Distributions

4 Fun with Averages

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6 Appendix: Why the Mean?

7 Joint Distributions

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- Examples: Bernoulli, Binomial, Gamma, Normal, Poisson, t -distribution



Bernoulli Random Variable

Definition

Suppose X is a random variable, with $X \in \{0, 1\}$ and $P(X = 1) = \pi$. Then we will say that X is **Bernoulli** random variable,

$$p(X = x) = \pi^x(1 - \pi)^{1-x}$$

for $x \in \{0, 1\}$ and $p(X = x) = 0$ otherwise.

We will (equivalently) say that

$$X \sim \text{Bernoulli}(\pi)$$

$\sim$ means equality in distribution (not values!). Often $X \sim \text{Bernoulli}(\pi)$ would be read 'X is distributed Bernoulli with parameter π '

Bernoulli Random Variable Mean and Variance

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$$\begin{aligned} E[X] &= 1 \times P(X = 1) + 0 \times P(X = 0) \\ &= \pi + 0(1 - \pi) = \pi \end{aligned}$$

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$$E[X] = \pi$$

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Importantly, we can also just look this up!

Normal/Gaussian Random Variables

Definition

Suppose X is a random variable with $X \in \mathbb{R}$ and **density**

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Then X is a **normally** distributed random variable with parameters μ and σ^2 .

Equivalently, we'll write

$$X \sim \text{Normal}(\mu, \sigma^2)$$

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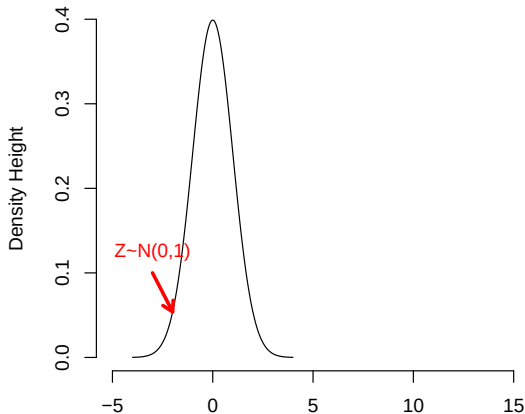
Proposition

Scale/Location. If $Z \sim N(0, 1)$, then $X = aZ + b$ is,

$$X \sim \text{Normal}(b, a^2)$$

Intuition

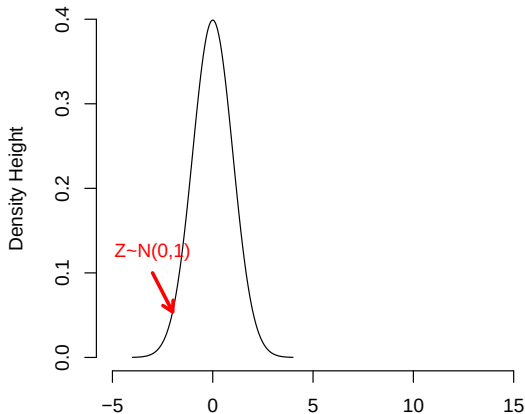
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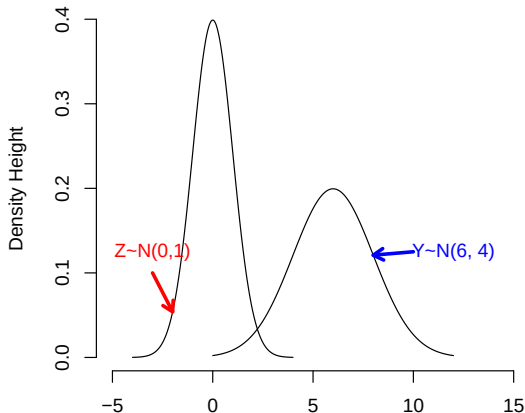


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$$Y \sim \text{Normal}(6, 4)$$



Proof: $Z \sim N(0, 1)$ and $Y = aZ + b$, then $Y \sim N(b, a^2)$

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Multivariate Normal

Definition

Suppose $\mathbf{X} = (X_1, X_2, \dots, X_N)$ is a vector of random variables. If $\mathbf{X}$ has pdf

$$f(\mathbf{x}) = (2\pi)^{-N/2} \det(\mathbf{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \mathbf{\Sigma}(\mathbf{x} - \boldsymbol{\mu})\right)$$

Then we will say $\mathbf{X}$ has a **Multivariate Normal** Distribution,

$$\mathbf{X} \sim \text{Multivariate Normal}(\boldsymbol{\mu}, \mathbf{\Sigma})$$

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$$\begin{aligned} f(x_1, x_2) &= (2\pi)^{-2/2} 1^{-1/2} \exp \left(-\frac{1}{2} \left((\mathbf{x} - \mathbf{0})' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{x} - \mathbf{0}) \right) \right) \\ &= \frac{1}{2\pi} \exp \left(-\frac{1}{2} (x_1^2 + x_2^2) \right) \\ &= \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x_1^2}{2} \right) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x_2^2}{2} \right) \end{aligned}$$

Multivariate Normal Distribution

Consider the (bivariate) special case where $\boldsymbol{\mu} = (0, 0)$ and

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$\rightsquigarrow$ product of univariate standard normally distributed random variables

Properties of the Multivariate Normal Distribution

Suppose $\mathbf{X} = (X_1, X_2, \dots, X_N)$

$$\begin{aligned} E[\mathbf{X}] &= \boldsymbol{\mu} \\ \text{cov}(\mathbf{X}) &= \boldsymbol{\Sigma} \end{aligned}$$

So that,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \dots & \text{cov}(X_1, X_N) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \dots & \text{cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \dots & \text{var}(X_N) \end{pmatrix}$$

One Step Deeper: Exponential Family

Nearly every distribution we will discuss is in the exponential family. An exponential family distribution has the density of the following form:

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Many other examples, including: Normal, Bernoulli/binomial, Gamma, multinomial, exponential, negative binomial, beta, uniform, chi-squared, etc.

This slide and the following based on material from Teppei Yamamoto

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- **Joint** and **conditional** distributions capture the relationship between random variables.
- There is a common set of famous distributions such as the **Normal** distribution.

Where We've Been and Where We're Going . . .

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New reading:

- Aronow and Miller Chapter 3.1-3.2.6, 3.4.1
- Optional: Fox Chapter 3: Examining Data

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- Use documents with known categories to estimate function
- Then apply model to new data, classify those observations

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Goal: For each document $\mathbf{x}_i$, we want to infer most likely **category**

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This is called a Naïve Bayes classifier.

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Simple intuition about Naïve Bayes:

- Learn what documents in class j look like

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- 1) Learn $\hat{p}(C)$ and $\hat{p}(\mathbf{x}_i|C_k)$ on **labeled data**
- 2) Use this to identify most likely C_k for each document i in **unlabeled data**

Simple intuition about Naïve Bayes:

- Learn what documents in class j look like
- Find class k that document i is most similar to

Example: Building a Spam Filter

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Scoring the algorithm is easy.

$$p(C_k|\mathbf{x}_i) \propto p(C_k) \prod_{j=1}^J p(x_{i,j}|C_k)^{x_{ij}}$$

which is simply the probability of the class multiplied by the product of the probabilities for the words that are observed in the test document.

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- More on estimators next week!